Determination of the Mode Coefficients

Consider a waveguide structure whose infinite dimension is along the z direction. Defining the TE and TM eigenfunctions as

$$
\begin{aligned}
\psi^{\mathrm{TM}} &= \sum_n \sum_m A_{nm}^{\mathrm{TM}} \psi_{nm}^{\mathrm{TM}} \qquad (\mathbf{A} = \hat{z}\psi^{\mathrm{TM}}) \\
\psi^{\mathrm{TE}} &= \sum_n \sum_m A_{nm}^{\mathrm{TE}} \psi_{nm}^{\mathrm{TE}} \qquad (\mathbf{F} = \hat{z}\psi^{\mathrm{TE}})
\end{aligned}
\tag{1}
$$

the electric and magnetic fields in the guide are given by

$$
\begin{aligned}
\mathbf{E} &= -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A} - \frac{1}{\varepsilon}\nabla\times\mathbf{F} = \mathbf{E}^{+} + \mathbf{E}^{-} \\
\mathbf{H} &= -j\omega\mathbf{F} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{F} + \frac{1}{\mu}\nabla\times\mathbf{A} = \mathbf{E}^{+} + \mathbf{E}^{-}
\end{aligned}
\tag{2}
$$

These can be more conveniently rewritten as

$$
\begin{aligned}
\mathbf{E}^{+} &= \sum_n \sum_m C_{nm}^{+}[\mathbf{e}_{nm} + \hat{z}e_{z_{nm}}]e^{-j\beta_{z_{nm}}z} = \sum_n \sum_m C_{nm}^{+}\mathbf{E}_{nm}^{+} \\
\mathbf{H}^{+} &= \sum_n \sum_m C_{nm}^{+}[\mathbf{h}_{nm} + \hat{z}h_{z_{nm}}]e^{-j\beta_{z_{nm}}z} = \sum_n \sum_m C_{nm}^{+}\mathbf{H}_{nm}^{+}
\end{aligned}
\tag{3}
$$

for the modal fields propagating in the $+z$ direction and as

$$
\begin{aligned}
\mathbf{E}^{-} &= \sum_n \sum_m C_{nm}^{-}[\mathbf{e}_{nm} - \hat{z}e_{z_{nm}}]e^{j\beta_{z_{nm}}z} = \sum_n \sum_m C_{nm}^{-}\mathbf{E}_{nm}^{-} \\
\mathbf{H}^{-} &= \sum_n \sum_m C_{nm}^{-}[-\mathbf{h}_{nm} + \hat{z}h_{z_{nm}}]e^{j\beta_{z_{nm}}z} = \sum_n \sum_m C_{nm}^{-}\mathbf{H}_{nm}^{-}
\end{aligned}
\tag{4}
$$

for the modal fields propagating in the $-z$ direction. The specific form of the modal fields $(\mathbf{E}_{nm}^{\pm}, \mathbf{H}_{nm}^{\pm})$ can be extracted from (2) once the eigenfunctions ψ^{TE} and ψ^{TM} have been determined. Note that the form of $(\mathbf{E}_{nm}^{\pm}, \mathbf{H}_{nm}^{\pm})$ was chosen so that this pair satisfies Maxwell's equations. Thus, we can write

$$
\begin{aligned}
\nabla\times\mathbf{E}_{nm}^{-} &= -j\omega\mu\mathbf{H}_{nm}^{-} \qquad & \nabla\times\mathbf{E}_{nm}^{+} &= -j\omega\mu\mathbf{H}_{nm}^{+} \\
\nabla\times\mathbf{H}_{nm}^{-} &= j\omega\mu\mathbf{E}_{nm}^{-} \qquad & \nabla\times\mathbf{H}_{nm}^{+} &= j\omega\mu\mathbf{E}_{nm}^{+}
\end{aligned}
\tag{5}
$$

We are interested in finding the waveguide coefficients C_{nm} for a given electric current distribution $\mathbf{J}(\mathbf{r})$ placed within the waveguide. In the following, we present a rather standard procedure for accomplishing this.

We begin with Maxwell's equations

$$
\begin{aligned}
\nabla\times\mathbf{E} &= -j\omega\mu\mathbf{H} \\
\nabla\times\mathbf{H} &= +j\omega\varepsilon\mathbf{E} + \mathbf{J}
\end{aligned}
\tag{6}
$$

When these are combined with (5) we obtain that

$$
\nabla\cdot(\mathbf{E}\times\mathbf{H}_{nm}^{-} - \mathbf{E}_{nm}^{-}\times\mathbf{H}) = \mathbf{J}\cdot\mathbf{E}_{nm}^{-} \tag{7}
$$

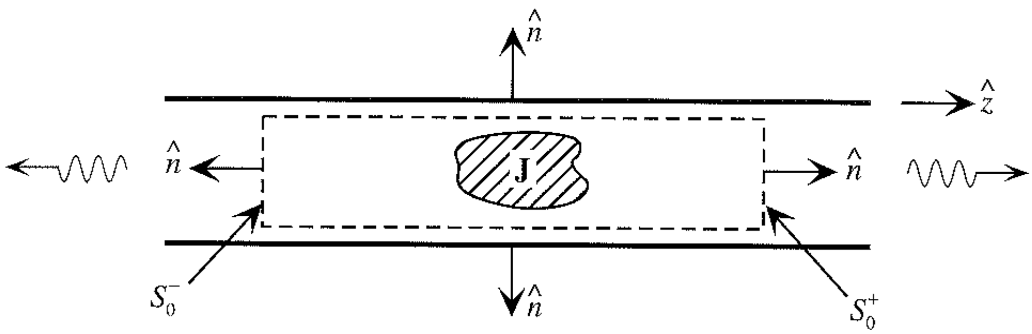
(refer to Poynting's theorem for a proof of this identity), and upon application of the divergence theorem we have

$$\iiint_V \nabla \cdot (\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H})\, dv = \iiint_V \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}_{nm}^-\, dv$$

or

$$\oiint_S [\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H}] \cdot \hat{n}\, ds = \iiint_V \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}_{nm}^-\, dv \tag{8}$$

The latter surface integral involves integration over the waveguide walls and the surfaces S_0^+ and S_0^- to the left and right of the source as illustrated in the figure below.



Since $\hat{n} \times \mathbf{E}_{nm}^-$ and $\hat{n} \times \mathbf{E}$ satisfy the boundary conditions on the waveguide's walls, it follows that

$$\oiint_S [\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H}] \cdot \hat{n}\, ds = \iint_{S_0^+ + S_0^-} [\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H}] \cdot \hat{n}\, ds \tag{9}$$

For the last integral we have (by virtue of mode orthogonality, the infinite sums reduce to a single summand):

$$\begin{aligned}\iint_{S_0^+ + S_0^-} &[\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H}] \cdot ds = \\ &\iint_{S_0^+} C_{nm}^+ [(\mathbf{e}_{nm} + \hat{z} e_{z_{nm}}) \times (-\mathbf{h}_{nm} + \hat{z} h_{z_{nm}}) - (\mathbf{e}_{nm} - \hat{z} e_{z_{nm}}) \times (\mathbf{h}_{nm} + \hat{z} h_{z_{nm}})] \cdot \hat{z}\, ds \\ &+ \iint_{S_0^-} C_{nm}^- (-\hat{z}) \cdot [(\mathbf{e}_{nm} - \hat{z} e_{z_{nm}}) \times (-\mathbf{h}_{nm} + \hat{z} h_{z_{nm}}) - (\mathbf{e}_{nm} - \hat{z} e_{z_{nm}}) \times (-\mathbf{h}_{nm} + \hat{z} h_{z_{nm}})]\, ds\end{aligned}$$

From this expansion of the fields under the integrals, it is readily seen that the integrand associated with the surface S_0^- vanishes. On making some additional cancellations in the other integral over S_0^+, we find that

$$\oiint_S [\mathbf{E} \times \mathbf{H}_{nm}^- - \mathbf{E}_{nm}^- \times \mathbf{H}] \cdot \hat{n}\, ds = -2C_{nm}^+ \iint_{S_0^+} (\mathbf{e}_{nm} \times \mathbf{h}_{nm}) \cdot \hat{z}\, dx\, dy \tag{10}$$

From (8), we then conclude that

$$\boxed{C_{nm}^+ = -\frac{1}{P_{nm}} \iiint_V (\mathbf{E}_{nm}^- \cdot \mathbf{J})\, dv} = -\frac{1}{P_{nm}} \iiint (\mathbf{e}_{nm} - \hat{z} e_{z_{nm}}) \cdot \mathbf{J} e^{j\beta_{z_{nm}} z}\, dv \tag{11}$$

with

$$P_{nm} = 2 \iint_{S_0^+ \text{ or } S_0^-} \mathbf{e}_{nm} \times \mathbf{h}_{nm} \cdot \hat{z}\, dx\, dy$$

To find an expression for C_{nm}^-, we repeat the process with (E_{nm}^-, H_{nm}^-) replaced by (E_{nm}^+, H_{nm}^+). We obtain

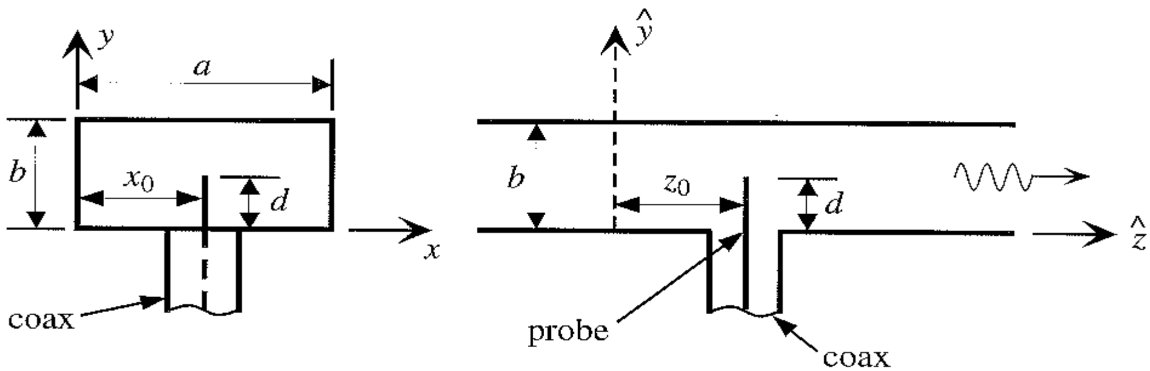
$$\boxed{C_{nm}^- = -\frac{1}{P_{nm}} \iiint_V \mathbf{E}_{nm}^+ \cdot \mathbf{J}\, dv}$$

When magnetic currents are present, the above expressions generalize to

$$\boxed{\begin{aligned} C_{nm}^{+} &= -\frac{1}{P_{nm}}\iiint_V (\mathbf{E}_{nm}^{-}\cdot\mathbf{J}-\mathbf{H}_{nm}^{-}\cdot\mathbf{M})\,dv \\ C_{nm}^{-} &= -\frac{1}{P_{nm}}\iiint_V (\mathbf{E}_{nm}^{+}\cdot\mathbf{J}-\mathbf{H}_{nm}^{+}\cdot\mathbf{M})\,dv \end{aligned}}$$

Example

Consider a rectangular waveguide whose cross sections are shown below ($b < a$).



The fields in this guide can be represented as ($\beta = \omega\sqrt{\mu\varepsilon}$, $\beta_{z_{mn}} = \sqrt{\beta^2 - (n\pi/a)^2 - (m\pi/a)^2}$)

$$\mathbf{H}^{+} = \sum_n\sum_m A_{nm}\left\{\frac{\beta_{z_{nm}}}{\omega\mu\varepsilon}\left[\hat{x}\frac{n\pi}{a}\sin\frac{n\pi}{a}x\cos\frac{m\pi}{b}y + \hat{y}\frac{m\pi}{b}\cos\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\right]\right.$$
$$\left. + \hat{z}\frac{\beta^2-\beta_{z_{nm}}^2}{j\omega\mu\varepsilon}\cos\frac{n\pi}{a}x\cos\frac{m\pi}{a}y\right\}e^{-j\beta_{z_{nm}}z}$$

$$\mathbf{E}^{+} = \sum_n\sum_m \frac{A_{nm}}{\varepsilon}\left[\hat{x}\frac{m\pi}{b}\cos\frac{n\pi}{a}x\sin\frac{n\pi}{b}y - \hat{y}\frac{n\pi}{a}\sin\frac{n\pi}{a}x\cos\frac{m\pi}{b}y\right]e^{-j\beta_{z_{nm}}z}$$

Consequently,

$$\mathbf{e}_{nm} = \hat{x}\frac{m\pi}{b}\cos\frac{n\pi}{a}x\sin\frac{n\pi}{b}y - \hat{y}\frac{n\pi}{a}\sin\frac{n\pi}{a}x\cos\frac{m\pi}{b}y$$
$$\mathbf{h}_{nm} = \left(\hat{x}\frac{n\pi}{a}\sin\frac{n\pi}{a}x\cos\frac{m\pi}{b}y + \hat{y}\frac{m\pi}{b}\cos\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\right)\frac{\beta_{z_{nm}}}{\omega\mu}$$

and for the lowest order TE_{10} mode, we have

$$\mathbf{e}_{10} = -\hat{y}\frac{\pi}{a}\sin\frac{\pi}{a}x$$
$$\mathbf{h}_{10} = +\hat{x}\left(\frac{\pi}{a}\sin\frac{\pi}{a}x\right)\frac{\beta_{z_{nm}}}{\omega\mu}$$

By comparison with (3), $C_{nm}^{+} = A_{nm}/\varepsilon$.

Assuming that the probe current distribution is given by

$$I(y) = I_0\sin\beta(d-y) \qquad 0 < y < d$$

we find that

$$C_{nm}^{+} = -\frac{I_0}{P_{nm}} \int_0^d \sin[\beta(d-y)](\mathbf{e}_{nm} \cdot \hat{y}) e^{j\beta_{znm} z_0}\, dy$$
$$= \left(\frac{n\pi}{a}\right) \frac{I_0}{P_{nm}} \sin\left(\frac{n\pi}{a} x_0\right) \int_0^d \sin[\beta(d-y)] \cos\left(\frac{m\pi}{b} y\right) dy$$

For example,

$$C_{10}^{+} = \left(\frac{\pi}{a}\right) \frac{I_0}{P_{10}} \sin\left(\frac{\pi}{a} x_0\right) \frac{[1-\cos\beta_0 d]}{\beta_0}$$

with

$$P_{10} = 2 \int_0^a \int_0^b \left(\frac{\beta_{z10}}{\omega\mu}\right) \left(\frac{\pi}{a}\right)^2 \sin^2 \frac{\pi x}{a}\, dx\, dy = ab \left(\frac{\beta_{z10}}{\omega\mu}\right) \left(\frac{\pi}{a}\right)^2$$

mode admittance.