Solutions of Maxwell's equations in unbounded space

Which solution among

$$f(x) = Ae^{-jkx} + Be^{-jkx} \tag{1}$$
$$f(x) = A' \cos kx + B' \sin kx \tag{2}$$

should we choose?

Both are equivalent forms of the same fundamental expressions since one can be obtained from the other!

However, the boundary conditions to be imposed give us an avenue for selecting the most convenient form.

In free space, the only boundary condition to use is that the field decays as it moves away from the source.

Since $k = \beta - j\alpha$ ($\alpha > 0$), if $\alpha \neq 0$, then decay is guaranteed.

For $x > 0$,

$$f(x) = Ae^{-jkx} = Ae^{-j\beta x}e^{-\alpha x} \to 0 \quad \text{as } x \to \infty$$

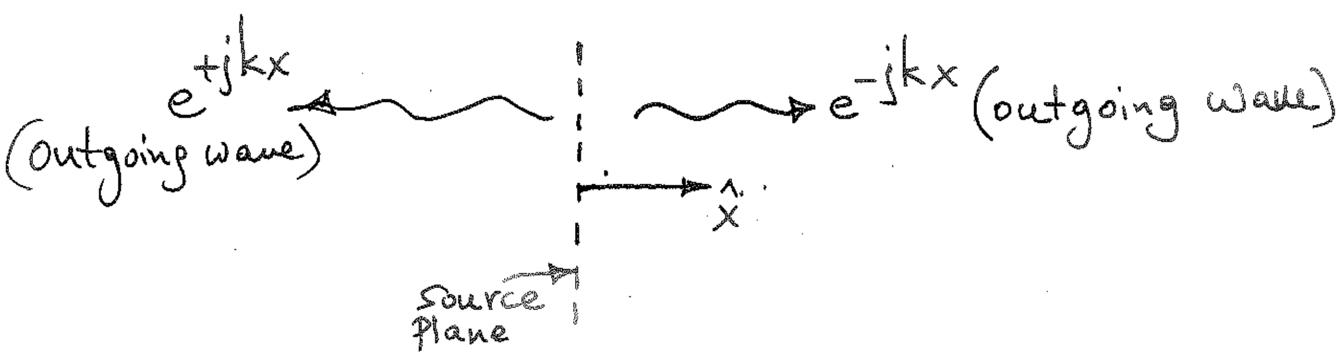
Must therefore set $B = 0$ since $Be^{+jkx} \to \infty$ as $x \to \infty$.

For $x < 0$,

$$f(x) = Be^{+jkx}$$

Thus, solution is

$$f(x) = \begin{cases} Ae^{-jkx} & x > 0 \\ Be^{+jkx} & x < 0 \end{cases}$$



In deriving the above solution, the boundary condition that was imposed was

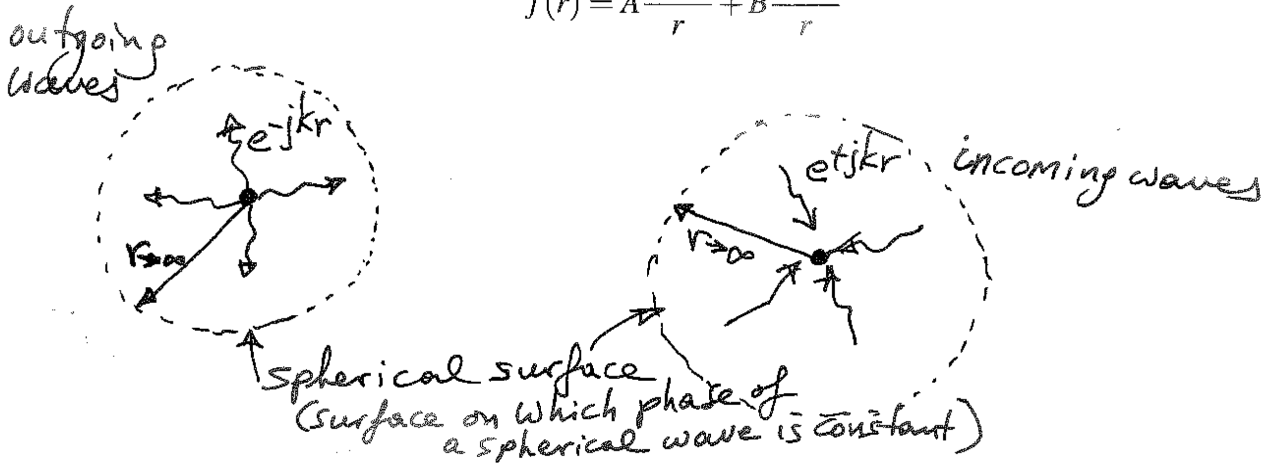
$$\boxed{\frac{df(x)}{dx} + jk\, f(x) = 0}$$

This is a form of the radiation condition for 1D outgoing waves (away from origin). For plane waves traveling from infinity toward the origin, the appropriate radiation condition is

$$\frac{df(x)}{dx} - jk\, f(x) = 0$$

For 3D (spherical coordinates), we already know that

$$f(r) = A\frac{e^{-jkr}}{r} + B\frac{e^{+jkr}}{r}$$



To choose outgoing waves, the appropriate boundary condition is

$$\lim_{r\to\infty}\left[r\left(\frac{df}{dr} + jk\,f(r)\right)\right] = 0$$

In general, the radiation condition for vector fields is

$$\lim_{r\to\infty}\left[r(\nabla\times\mathbf{E} \pm jk_0\hat{r}\times\mathbf{E})\right] = 0$$

(in which the plus sign corresponds to outgoing waves and the minus sign to incoming waves) or

$$\lim_{r\to\infty} r[\pm jk\mathbf{E} + \hat{r}\times(\nabla\times\mathbf{E})] = 0$$

If we were to choose outgoing plane waves as our solution, then in general

$$\begin{aligned}\mathbf{E}(x,y,z) &= \hat{e}\,X(x)\,Y(y)\,Z(z)\\ &= \hat{e}E_0 e^{-jk_x x}\cdot e^{-jk_y y}\cdot e^{-jk_z z}\\ &= \hat{e}E_0 e^{-j\Phi(x,y,z)}\end{aligned}$$

where

$$\Phi(x,y,z) = k_x x + k_y y + k_z z$$

is the phase of the wave.

Let us stop for the moment and interpret the physical meaning/characteristics of this wave!

$$\mathcal{E}(x,y,z;t) = \hat{e}E_0\cos[\omega t - \Phi(x,y,z)]$$

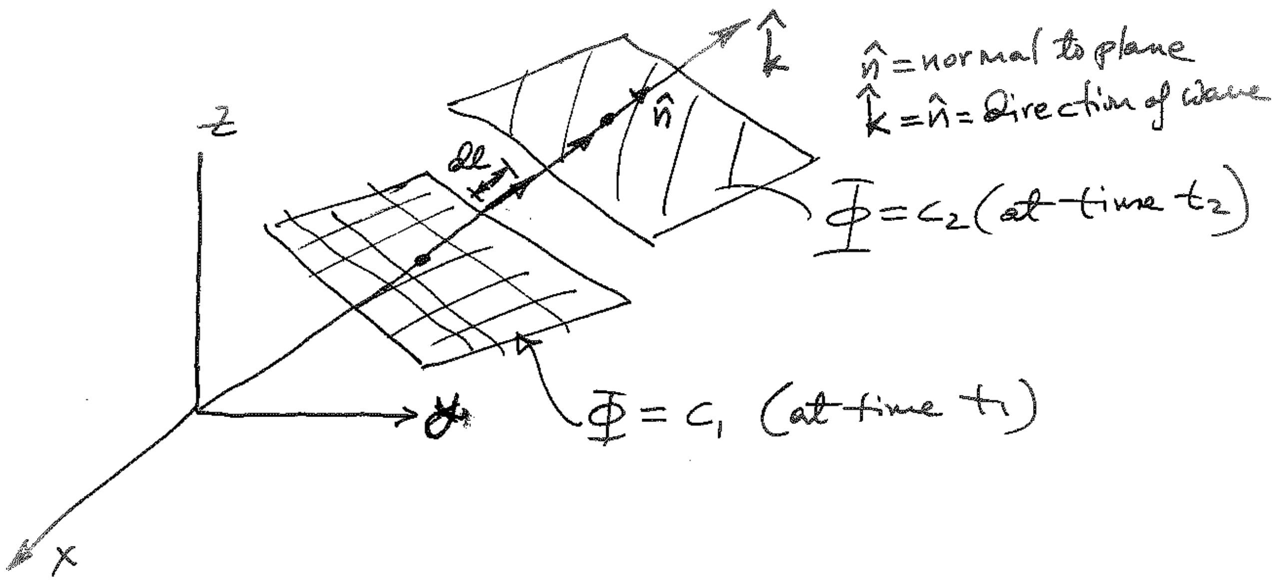
Setting $\omega t - \Phi = \text{const.}$ gives

$$\boxed{-\Phi = \text{const.}}$$

since ωt is a constant at some specific time $t = t_0$. Thus, we have

$$k_x x + k_y y + k_z z = \text{const.} \qquad \text{(at a given time)}$$

This is the equation of a plane and at different instants we graph this plane as shown below:



These planes are referred to as *phase fronts*, and are the surfaces on which information traveling through space arrive at the "same time." Note the following statements:

- information arrives on phase front at the same time (phase $\leftrightarrow$ time delay)
- planar phase fronts $\rightarrow$ wave is called a *plane wave*
- normal $\hat{n}$ to the phase front is the direction that information or the wave travels through space.

For plane waves

$$\hat{n} = \frac{\nabla\Phi}{|\nabla\Phi|} = \hat{k} = \frac{k_x\hat{x} + k_y\hat{y} + k_z\hat{z}}{\sqrt{k_x^2 + k_y^2 + k_z^2}}$$

The velocity of the wave is found as before for 1D waves. We can derive the velocity expression by noting that the complete derivative of $\omega t - \Phi$ must be zero since $\omega t - \Phi =$ constant defines the phase fronts (or information fronts). More specifically, we have

$$d(\omega t - \Phi) = 0 \quad \Longrightarrow$$

$$\boxed{\omega\, dt - d\Phi = 0}$$

$$\begin{aligned} d\Phi &= \frac{\partial\Phi}{\partial x}\, dx + \frac{\partial\Phi}{\partial y}\, dy + \frac{\partial\Phi}{\partial z}\, dz \\ &= \nabla\Phi \cdot [\hat{x}\, dx + \hat{y}\, dy + \hat{z}\, dz] \\ &= \nabla\Phi \cdot \mathbf{d\ell} \\ &= (\nabla\Phi \cdot \hat{n})\, d\ell = \frac{\partial\Phi}{\partial n}\, d\ell \end{aligned}$$

The velocity υ_x along x is found from

$$\omega\, dt - (\nabla\Phi \cdot \hat{x})\, dx = 0 \qquad \Longrightarrow$$

$$\boxed{\frac{dx}{dt} = \upsilon_x = \frac{\omega}{\nabla\Phi \cdot \hat{x}} = \frac{\omega}{k_x}}$$

Similarly

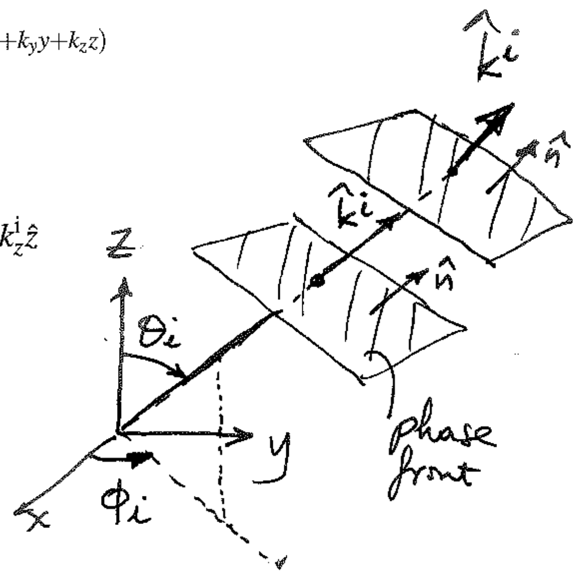
$$\frac{dy}{dt} = \upsilon_y = \frac{\omega}{\nabla\Phi \cdot \hat{y}} = \frac{\omega}{k_y}$$

etc.

We can rewrite $\mathbf{E} = \hat{e}e^{-j\Phi(x,y,z)}$ as

$$\mathbf{E} = \hat{e}E_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{e}E_0 e^{-j(k_x x + k_y y + k_z z)}$$

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$$



Then, if we set

$$\mathbf{k}_0 = \mathbf{k}^{\mathrm{i}} = k_x^{\mathrm{i}}\hat{x} + k_y^{\mathrm{i}}\hat{y} + k_z^{\mathrm{i}}\hat{z}$$

with

$$k_x^{\mathrm{i}} = k_0 \cos\phi_{\mathrm{i}} \sin\theta_{\mathrm{i}}$$

$$k_y^{\mathrm{i}} = k_0 \sin\phi_{\mathrm{i}} \sin\theta_{\mathrm{i}}$$

$$k_z^{\mathrm{i}} = k_0 \cos\theta_{\mathrm{i}}$$

we get

$$\mathbf{E} = \hat{e}E_0 e^{-jk_0(\hat{x}\cos\phi_{\mathrm{i}}\sin\theta_{\mathrm{i}} + \hat{y}\cos\phi_{\mathrm{i}}\sin\theta_{\mathrm{i}} + \hat{z}\cos\theta_{\mathrm{i}})}$$

This is a plane wave traveling along the spherical direction $(\phi_{\mathrm{i}}, \theta_{\mathrm{i}})$. We will routinely use this formula to write waves traveling along a given direction. *So remember it* and *understand its parameters.*

Usually,

$$\hat{e} = \left\{\hat{\phi}_{\mathrm{i}}E_0 \text{ or } \hat{\theta}_{\mathrm{i}}E_0\right\}$$

No $\hat{r}$ component appears in the polarization of plane waves since $\nabla \cdot \mathbf{E} \neq 0$ for an outgoing wave.