Time harmonic fields

(plane, spherical and cylindrical waves; solution of wave equation or radiation condition; wave velocity; Poynting theorem)

Hereon we will concentrate on time harmonic fields (i.e., sinusoidal) of the form

$$\cos(\omega t - kx),\ \sin(\omega t - ky),\ \cos(\omega t - kr) \text{ or } \sin(\omega t - k\rho) \text{ (cylindrical), etc.}$$

If the field is not in this form, we can invoke Fourier Series to rewrite the given field as

$$E(t,\ x=0) = \sum (A_n \cos n\omega_0 t + B_n \sin n\omega_0 t) = \sum c_n e^{jn\omega t}$$

where $\omega_0 =$ the fundamental frequency. Thus, any time-domain field can be represented as a superposition of *harmonics*, and for our analysis we can consider each harmonic component separately. That is, we can instead deal with phasor quantities as done in circuits.

In general, we can express **E** as

$$\mathbf{E}(x,y,z;t) = \text{Re (or Im)} \left\{ \mathbf{E}(x,y,z)\, e^{j\omega t} \right\}$$

Note that we are adopting the $e^{+j\omega t}$ convention (although in many physics books the $e^{-j\omega t}$ convention is used). This representation should be compared to

$$V(t) = \text{Re}\left\{ V_{\text{phasor}} e^{j\omega t} \right\}$$

Here, the phasor will be the field components, viz.

$$E_{\substack{x\\y\\z}}(x,y,z,t) = \text{Re (or Im)} \left\{ E_{\substack{x\\y\\z}}(x,y,z)\, e^{j\omega t} \right\}$$

with

$$E_{\substack{x\\y\\z}}(x,y,z) = \left| E_{\substack{x\\y\\z}} \right| e^{j\phi_{\substack{x\\y\\z}}} = E_{\substack{x0\\y0\\z0}} e^{j\phi_{\substack{x\\y\\z}}}$$

Thus, the general form (hereon Re{ } and $e^{j\omega t}$ conventions will be adopted) will be

$$\mathbf{E}(x,y,z,t) \text{ or } \mathcal{E}(x,y,z,t) = \text{Re}\left\{ \mathbf{E}(x,y,z)\, e^{j\omega t} \right\}$$

$$= \hat{x} E_{x0} \cos(\omega t + \phi_x) + \hat{y} E_{y0} \cos(\omega t + \phi_y) + \hat{z} E_{z0} \cos(\omega t + \phi_z)$$

(see p. 15 of Harrington). The coefficients E_{x0}, E_{y0}, E_{z0} represent peak values of the harmonic field. Harrington instead considers E_{x0}, E_{y0}, E_{z0} to be RMS values. This difference will simply impact the formulas for the time-average power/energy.

Example: (see p. 22 of Harrington)

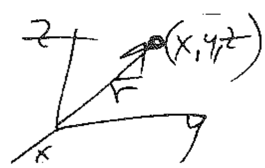
$\frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2\,dv$ average power dissipated in V (no $\frac{1}{2}$ in Harrington)

$\frac{1}{4}\iiint_V \varepsilon_0\varepsilon_r|\mathbf{E}|^2\,dv$ average electric energy stored in V ($\frac{1}{2}$ instead of $\frac{1}{4}$ in Harrington)

We can readily rewrite Maxwell's equations in phasor form by noting that

$$\frac{\partial \mathbf{E}}{\partial t} = j\omega\,\mathbf{E}(x,y,z,t) = j\omega\,\mathbf{E}(\mathbf{r})$$

where $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is the spherical or position vector.



Basically, we set

$$\frac{\partial}{\partial t} \longrightarrow j\omega$$

when dealing with sinusoidal/phasor quantities.

$\nabla\times\mathbf{E} = -j\omega\mathbf{H}$ $\nabla\times\mathbf{H} = \mathbf{J} - j\omega\varepsilon\mathbf{E}$ $\nabla\cdot\mathbf{J} = -j\omega\rho$	$\nabla\times\mathbf{E} = -\mathbf{M} - j\omega\mu\mathbf{H}$ $(M \to \text{V/m}^2)$ $\nabla\times\mathbf{H} = \mathbf{J} + j\omega\varepsilon\mathbf{E}$ $(J \to \text{A/m}^2)$ $\nabla\cdot\mathbf{J} = -j\omega\rho$ $\nabla\cdot\mathbf{M} = -j\omega\rho_m$ $(\rho_m = \text{magn. charges})$
$\nabla\cdot(\mu\mathbf{H}) = 0$ $\nabla\cdot(\varepsilon\mathbf{E}) = \rho$	$\nabla\cdot\mu\mathbf{H} = \rho_m$ $(\rho_m \to \text{Wb/m}^3)$ $\nabla\cdot\varepsilon\mathbf{E} = \rho$ $(\rho \to \text{C/m}^3)$
$\mathbf{D} = \varepsilon E$ $\mathbf{B} = \mu\mathbf{H}$	$\mathbf{D} = \varepsilon\mathbf{E}$ $(D \to \text{C/m}^2)$ $\mathbf{B} = \mu\mathbf{H}$ $(B \to \text{Wb/m}^2)$

Magnetic Current

For mathematical convenience we may also assume the presence of magnetic currents as added sources. That is, we accept the possibility that the south and north poles of a magnet can exist in isolation. If magnetic charges exist (caused by electron spins) than

$$\iint \mathbf{B}\cdot ds = \iiint_V \rho_m\,dv \quad\Leftrightarrow\quad \nabla\cdot\mathbf{B} = \rho_m \quad\text{or}\quad \nabla\cdot\mu\mathbf{H} = \rho_m$$

The corresponding continuity equation for magnetic charges is

$$\nabla\cdot\mathbf{M} = -j\omega\rho_m$$

and the modified Maxwell's equation is

$$\nabla\times\mathbf{E} = -\mathbf{M} - j\omega\mu\mathbf{H}$$

such that

$$\nabla\cdot(\nabla\times\mathbf{E}) = 0 = -\nabla\cdot\mathbf{M} - j\omega\nabla\cdot\mu\mathbf{H}$$

satisfies the continuity relation. In practice, magnetic currents are useful for modeling and representing jumps in the electric field.

Complex permittivity

- Goal is to incorporate σ into ε

Taking a look at Ampère's Law, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + \underbrace{\sigma \mathbf{E}}_{\mathbf{J}_\mathrm{c}} + j\omega\varepsilon\mathbf{E}$$

We see that we can rewrite this as

$$\begin{aligned}
\nabla \times \mathbf{H} &= \mathbf{J}_\mathrm{i} + (\sigma + j\omega\varepsilon)\mathbf{E} \\
&\qquad\downarrow \\
&\quad j\omega\left(\frac{\sigma}{j\omega} + \varepsilon\right)\mathbf{E} \\
&\qquad\downarrow \\
&\text{new } \varepsilon \to \dot{\varepsilon} \\
\nabla \times \mathbf{H} &= \mathbf{J}_\mathrm{i} + j\omega\dot{\varepsilon}\mathbf{E} \\
\dot{\varepsilon} &= \varepsilon + \frac{\sigma}{j\omega} = \varepsilon' - j\varepsilon'' = \varepsilon'(1 - j\tan\delta_\mathrm{e}) \\
\tan\delta_\mathrm{e} &= \text{loss tangent} = \frac{\varepsilon'}{\varepsilon''} \\
\tan\delta_\mathrm{e} &= 0 \quad \leftarrow \text{lossless media (non-conducting)} \\
\tan\delta_\mathrm{e} &\neq 0 \quad \leftarrow \text{lossy media } (\sigma \neq 0)
\end{aligned}$$

Hereon, we will simply write Ampère's Law as

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + j\omega\varepsilon\mathbf{E}$$

and drop the "$\cdot$" over ε. Thus, whenever we write ε in a time harmonic/phasor equation, we will imply that it has real and imaginary parts

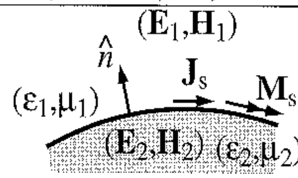
$$\begin{aligned}
\varepsilon &= \varepsilon' - j\varepsilon'' && (\varepsilon', \varepsilon'') \text{ are given} \\
&= \varepsilon'(1 - j\tan\delta_\mathrm{e}) && (\varepsilon', \delta_\mathrm{e}) \text{ are given} \\
&= \varepsilon_\mathrm{real} + \frac{\sigma}{j\omega} && (\varepsilon_\mathrm{real}, \sigma) \text{ given}
\end{aligned}$$

(Any of these combinations can be given.) Similarly,

$$\begin{aligned}
\varepsilon_\mathrm{r} = \frac{\varepsilon}{\varepsilon_0} &= \varepsilon_\mathrm{r}' - j\varepsilon_\mathrm{r}'' \qquad \text{(complex relative permittivity)} \\
&= (\varepsilon_\mathrm{r})_\mathrm{real} + \frac{\sigma}{j\omega\varepsilon_0}
\end{aligned}$$

Likewise for magnetic material

$$\begin{aligned}
\mu &= \mu' - j\mu'' \\
&= \mu'(1 - j\tan\delta_\mathrm{m}) \qquad \delta_\mathrm{m}: \text{ magnetic loss tangent} \\
&= \mu_\mathrm{real} + \frac{\sigma_\mathrm{m}}{j\omega}
\end{aligned}$$

Final Equations for Time Harmonic Fields	Boundary Conditions
$\nabla \times \mathbf{E} = -\mathbf{M}_\mathrm{i} - j\omega\mu\mathbf{H}$ $\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + j\omega\varepsilon\mathbf{E}$ $\nabla \cdot \mathbf{J}_\mathrm{i} = -j\omega\rho$ $\nabla \cdot \mathbf{M}_\mathrm{i} = -j\omega\rho_\mathrm{m}$	$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = -\mathbf{M}_s$ (impressed $\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = +\mathbf{J}_s$ surface charges)
$\nabla \cdot \varepsilon\mathbf{E} = \rho$ $\nabla \cdot \mu\mathbf{H} = \rho_\mathrm{m}$	$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$ $\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = \rho_{ms}$
$\mathbf{B} = \mu\mathbf{H}$ $\mathbf{D} = \varepsilon\mathbf{E}$	
$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon'(1 - j\tan\delta_e)$ $= \varepsilon_\mathrm{real} + \sigma/j\omega$ $\mu = \mu' - j\mu'' = \mu'(1 - j\tan\delta_\mathrm{m})$ $= \mu_\mathrm{real} + \sigma_\mathrm{m}/j\omega$ δ_e = electric loss tangent δ_m = magnetic loss tangent	**Special Cases** 1. Medium #2 is perfect (electric) conductor $\hat{n} \times \mathbf{H}_1 = \mathbf{J}_s$ $\hat{n} \times \mathbf{E}_1 = 0$ $\hat{n} \cdot \mathbf{B} = 0$ $\hat{n} \cdot \mathbf{D} = \rho_s y$ 2. Medium #2 is perfect magnetic conductor (PMC) $\hat{n} \times \mathbf{H}_1 = 0$ $\hat{n} \times \mathbf{E}_1 = -\mathbf{M}_s$ $\hat{n} \cdot \mathbf{B}_1 = \rho_{ms}$ $\hat{n} \cdot \mathbf{D} = 0$

Wave Equations

$$\boxed{Z_0 = \sqrt{\mu_0/\varepsilon}, \quad k_0 = \omega\sqrt{\mu_0\varepsilon_0}}$$

$$\nabla \times \left(\frac{\nabla \times \mathbf{E}}{\mu_\mathrm{r}}\right) - k_0^2\varepsilon_\mathrm{r}\mathbf{E} = -jk_0Z_0\mathbf{J}_\mathrm{i} - \nabla \times \left(\frac{\mathbf{M}_\mathrm{i}}{\mu_\mathrm{r}}\right)$$

or

$$\frac{1}{\mu_\mathrm{r}}\nabla \times \nabla \times \mathbf{E} - k_0^2\varepsilon_\mathrm{r}\mathbf{E} + \nabla\left(\frac{1}{\mu_\mathrm{r}}\right) \times \nabla \times \mathbf{E} = -jk_0Z_0\mathbf{J}_\mathrm{i} - \nabla \times \left(\frac{\mathbf{M}_\mathrm{i}}{\mu_\mathrm{r}}\right)$$

For homogeneous media: $\nabla^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) + k_0^2\varepsilon_\mathrm{r}\mu_\mathrm{r}\mathbf{E} = jk_0Z_0\mu_\mathrm{r}\mathbf{J}_\mathrm{i} + \nabla \times \mathbf{M}_\mathrm{i}$

Homogeneous and source-free: $\nabla^2\mathbf{E} + k_0^2\varepsilon_\mathrm{r}\mu_\mathrm{r}\mathbf{E} = jk_0Z_0\mu_\mathrm{r}\mathbf{J}_\mathrm{i} + \nabla \times \mathbf{M}_\mathrm{i}$