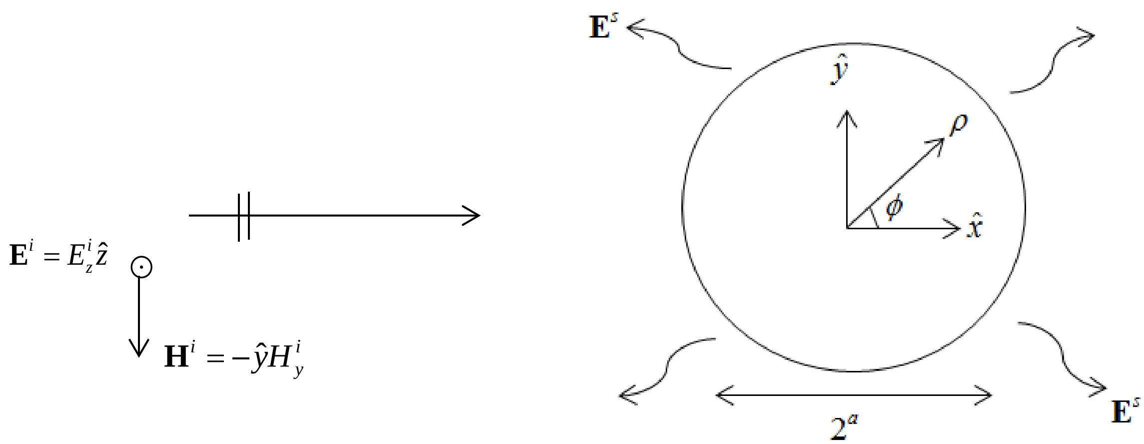
Scattering From an Infinite PEC Cylinder (Balanis 11.5 pp. 607-617)

Formulation for TM_z incidence

To characterize the interaction of a plane wave with a cylindrical object, we must first represent the plane wave as a sum of cylindrical waves.

Note that this is possible because both plane waves and cylindrical waves for a "complete set" of modes. Specifically, <u>any</u> fields may be written as either a series of plan waves or a series of cylindrical waves.

For TM_z incidence: ($\mathbf{E}$ field along axis of cylinder)



The incident plane wave takes the form

$$\mathbf{E}^i = \hat{z} E_z^i = \hat{z} E_0 e^{-j\beta x} = \hat{z} E_0 e^{-j\beta\rho\cos\phi}$$

$$\boxed{\mathbf{E}^i = \hat{z} E_z^i = \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho) e^{jn\phi} = \sum_{n=-\infty}^{\infty} A_n e^{jn\phi}}$$

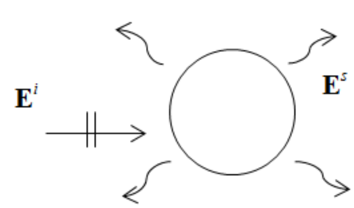
The above is similar to a Fourier series in cylindrical coordinates. Specifically, we have,

$A_n = \dfrac{1}{2\pi}\int_0^{2\pi} e^{-j\beta\rho\cos\phi} e^{-jn\phi} d\phi = j^{-n} J_n(\beta\rho) = a_n J_n(\beta\rho)$. Note that $J_n(\beta\rho)$ is small for $\beta\rho >> n$.

Thus, the # of required modes is proportional to the electrical size (radius) of the cylinder. For very small cylinders $(\beta_\rho \to o)$, and we only need the $n = 0$ term.

The <u>total</u> field in presence of the cylinder is the sum of the incident field plus the <u>scattered</u> field.

$$\mathbf{E}^t(\rho,\phi) = \mathbf{E}^i(\rho,\phi) + \mathbf{E}^s(\rho,\phi)$$



The scattered field can also be written as the sum of cylindrical waves:

$$\mathbf{E} = \hat{z}E_0 \sum_{n=-\infty}^{\infty} c_n \underbrace{H_n^{(2)}(\beta\rho)e^{jn\phi}}_{\text{outgoing wave}}$$

In this, the scattered fields are outward traveling. Also, c_n are the coefficienst of each Hankel $H_n^{(\rho)}$.

The Hankel functions can be expressed in terms of the 1st and 2nd kind. Bessel functions, *viz*.

$H_n^{(1)}(\beta\rho) = J_n(\beta\rho) + jY_n(\beta\rho)$ (inward $e^{-j\beta\rho}$ traveling wave)

$H_n^{(2)}(\beta\rho) = J_n(\beta\rho) - jY_n(\beta\rho)$ (outward $e^{+j\beta\rho}$ traveling wave)

This is exactly analogous to:

$e^{+j\beta x} = \cos\beta x + j\sin\beta x$ (traveling wave along $-x$)

$e^{-j\beta x} = \cos\beta x - j\sin\beta x$ (traveling wave along $+x$)

Note that the Bessel and Handel functions are the only valid solutions of Maxwell's equations in source free regions. When multiplied by the appropriate angular dependence, i.e. $e^{jn\phi}$ for 2D problems, we have the solutions

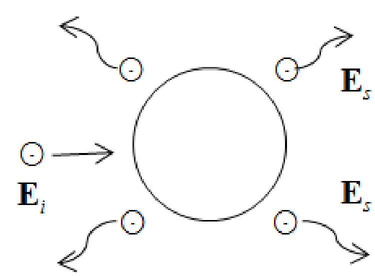
$$\Rightarrow J_n(\beta\rho)e^{jn\phi},\ Y_n(\beta\rho)e^{jn\phi}$$

$$H_n^{(1)}(\beta\rho)e^{jn\phi},\ H_n^{(2)}(\beta\rho)e^{jn\phi}$$

We can solve for scattered fields by applying the boundary conditions for the PEC cylinder, *viz.*

$$\Rightarrow \boxed{E_{\tan}^t = 0\big|_{\rho=0} = E_z^i(\rho=a) + E_z^s(\rho=\alpha) = 0}$$

$$\Rightarrow E_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta a)e^{jn\phi} + E_o \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(\beta a)e^{jn\phi} = 0$$



Since $e^{jn\phi}$ are orthogonal in $\phi \in [0, 2\pi]$, each "*n*" term must vanish independently. Enforcing this condition, we get

$$\Rightarrow j^{-n} J_n(\beta a) + c_n H_n^{(2)}(\beta a) = 0$$

$$\Rightarrow c_n = -j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}$$

Therefore, we can write the scattered field, $\mathbf{E}^s$, as

$$\boxed{\mathbf{E}^s = -\hat{z} E_0 \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta \rho) e^{jn\phi} \qquad (TM_z \text{ incidence})}$$

Scattered fields look like far away from the cylinder (Far Fields)

The Hankel functions have an <u>asymptotic</u> approximation for $\beta\rho \to \infty$,

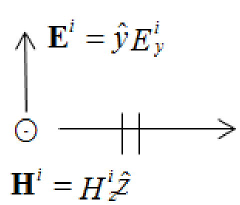
$$H_n^{(2)}(\beta\rho \to \text{large}) \sim j^n e^{-j\beta\rho} \sqrt{\frac{2j}{\pi\beta\rho}}$$

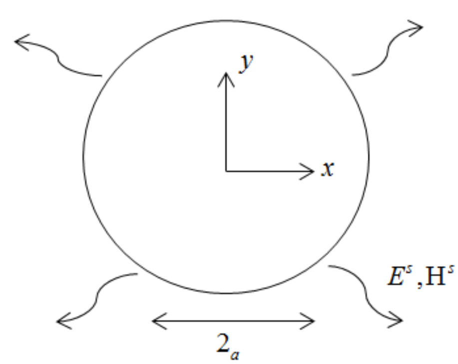
This is a wave that travels outwards $(+\rho)$ direction and decaying as $1/\sqrt{\rho}$, *vi.z* cylindrical wave. Using this asymptotic approximation for the Hankel fuction, $\mathbf{E}^s$ becomes

$$\Rightarrow \boxed{\mathbf{E}^s(\beta p \to \infty) \sim -\hat{z} E_0 \sqrt{\frac{2j}{\pi\beta}} \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \sum_{n=-\infty}^{\infty} \frac{J_n(\beta a)}{H_n^2(\beta a)} e^{jn\phi}}$$

Scattering from a PEC cylinder, TE_z incidence

$\mathbf{H}$ field parallel to cylinder's axis





As before, the incident plane wave in the series:

$$\mathbf{H}^i = \hat{z}H_0 e^{-j\beta x} = \hat{z}H_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n\left(\beta\rho\right) e^{jn\phi}$$

To enforce the boundary condition $\mathbf{E}_{\tan} = 0$ on the cylinder's surface, we proceed to evaluate $\mathbf{E}$.

$$\mathbf{E}^i = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H}^i = \frac{1}{j\omega\varepsilon}\left(\hat{\rho}\frac{1}{\rho}\frac{\partial H_z^i}{\partial\phi} - \hat{\phi}\frac{\partial H_z^i}{\partial\rho}\right)$$

giving

$$E_\rho^i = \frac{H_0}{j\omega\varepsilon}\left(\frac{1}{\rho}\sum_{n=-\infty}^{\infty}\left(jn\right) j^{-n} J_n\left(\beta\rho\right) e^{jn\phi}\right)$$

$$E_\phi^i = \frac{-H_0}{j\omega\varepsilon}\left(\beta\sum_{n=-\infty}^{\infty} j^{-n}\underbrace{\frac{\partial J_n\left(\beta\rho\right)}{\partial\left(\beta\rho\right)}}_{J_n'\left(\beta\rho\right)} e^{jn\phi}\right)$$

Similarly, the scattered field can be expressed as:

$$\boxed{\begin{aligned}
&\mathbf{H}^s = \hat{z}H_0 \sum_{n=-\infty}^{\infty} d_n H_n^{(2)}\left(\beta\rho\right) e^{jn\phi} \\
&E_\rho^s = \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z^s}{\partial\phi} = \frac{H_0}{\omega\varepsilon}\frac{1}{\rho}\sum_{n=-\infty}^{\infty} d_n n H_n^{(2)}\left(\beta\rho\right) e^{jn\phi} \\
&E_\phi^s = -\frac{\beta H_0}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} d_n H_n^{(2)\prime}\left(\beta\rho\right) e^{jn\phi}
\end{aligned}}$$

Enforcing the boundary conditions: $\mathbf{E}_{\tan}^t\left(\rho = a\right) = 0$

gives

$$E_\phi^i\left(\rho = a\right) + E_\phi^s\left(\rho = a\right) = 0$$

$$-\frac{H_0\beta}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} j^{-n} J_n'\left(\beta a\right) e^{jn\phi} - \frac{H_0\beta}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} dn H_n^{(2)\prime}\left(\beta a\right) e^{jn\phi} = 0$$

Again, to enforce the boundary conditions, the individual summation terms must vanish. Enforcing this, gives

$$j^{-n} J_n{}'(\beta a) + d_n H_n^{(2)'}(\beta a) = 0$$

$$\boxed{d_n = -j^{-n} \frac{J_n{}'(\beta a)}{H_n^{(2)'}(\beta a)}}$$

Therefore,

$$\mathbf{H}_s = -\hat{z} H_o \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n{}'(\beta a)}{H_n^{(2)'}(\beta a)} H_n^{(2)}(\beta\rho) e^{jn\phi}$$

for TE incidence.

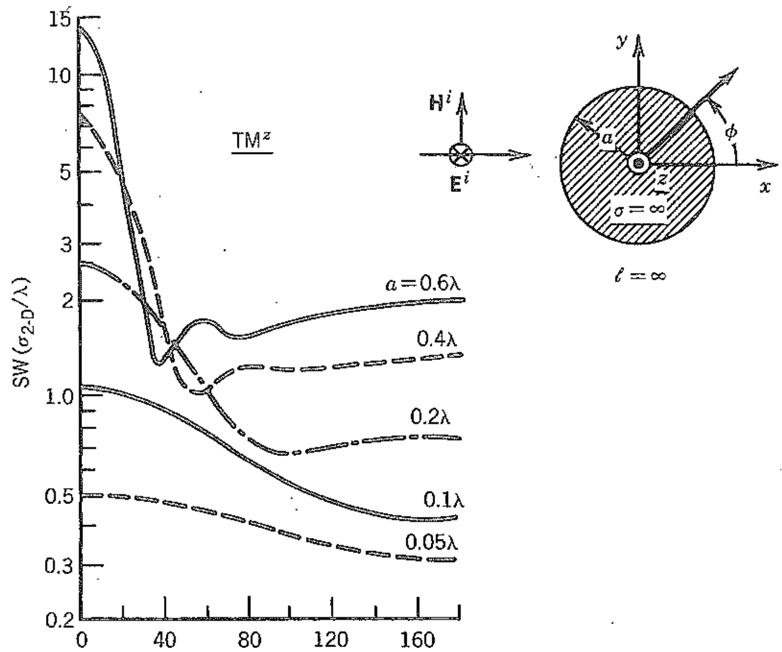


FIGURE 11-13 Two-dimensional TM^z bistatic scattering width (SW) of a circular conducting cylinder. (Courtesy J. H. Richmond, Ohio State University.)

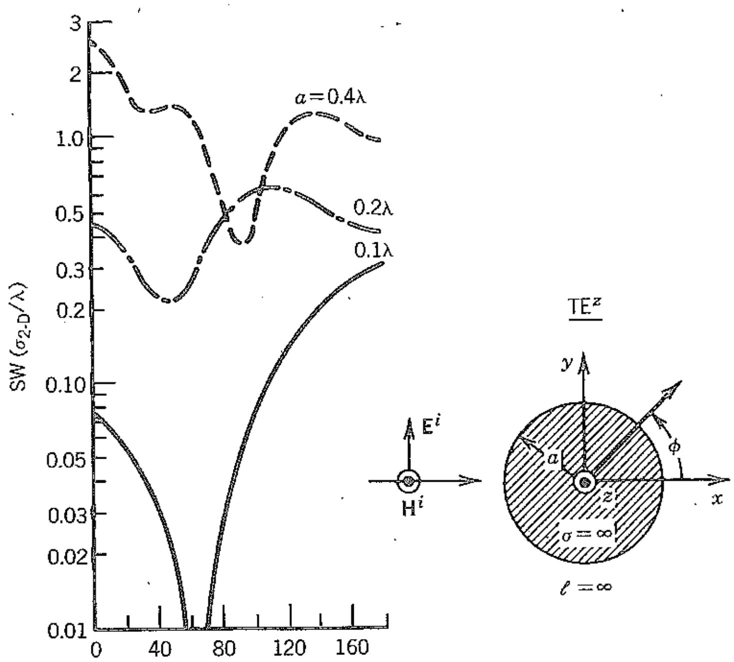


FIGURE 11-15 Two-dimensional TE^z bistatic scattering width (SW) of a circular conducting cylinder (Courtesy J. H. Richmond, Ohio State University.).