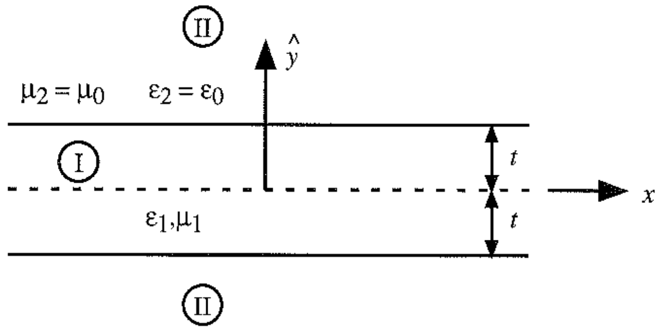
Dielectric Slab

(Harrington p. 168; Balanis p. 414)



We first select a set of modes/fields in each region that satisfy the wave equation. You could, of course, use vector potential as before, but we are skipping this step.

Even fields (H_z fields: TM case)

$$H_z^{\mathrm{I}} = Be^{-fx}\cosh(g_1 y)$$
$$H_z^{\mathrm{II}} = Ae^{-fx}e^{-g_2 y}$$

f is the same for both fields because $H_z^{\mathrm{I}} = H_z^{\mathrm{II}}$ for all x (like Snell's law). *Note:*

$$\sin j\alpha = j\sinh\alpha$$
$$\cos(\alpha) = \cosh j\alpha$$

E-fields:

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H} = \frac{\nabla\times(\hat{z}H_z)}{j\omega\varepsilon}$$
$$= -\frac{\hat{z}\times\nabla H_z}{j\omega\varepsilon} = -\frac{1}{j\omega\varepsilon}\left(\hat{y}\frac{\partial H_z}{\partial x} - \hat{x}\frac{\partial H_z}{\partial y}\right)$$
$$E_x^{\mathrm{I}} = Bg_1 e^{-fx}\sinh(g_1 y)\,\frac{1}{j\omega\varepsilon_1}$$
$$E_x^{\mathrm{II}} = -Ag_2 e^{-fx}e^{-g_2 y}\frac{1}{j\omega\varepsilon_0}$$

To determine the amplitude constants (A, B) and the propagation constants g_1, g_2 and f, we resort to the application of boundary condition. We have

$$H_z^{\mathrm{I}} = H_z^{\mathrm{II}} \quad \Rightarrow \quad 1.\ \boxed{Ae^{-g_2 t} = B\cosh g_1 t}$$
$$E_{\cancel{x}}^{\mathrm{I}} = E_{\cancel{x}}^{\mathrm{II}} \quad \Rightarrow \quad 2.\ \boxed{A\varepsilon_r g_2 e^{-g_2 t} = -g_1 B\sinh g_1 t}$$

Also, from the wave equation

$$\nabla^2 H_Z^{\mathrm{I,II}} + \beta^2 H_z^{\mathrm{I,II}} = 0 \quad \Rightarrow$$

$$3.\ \boxed{g_1^2 = -\underbrace{\beta_0^2 \mu_r \varepsilon_r}_{\beta_2^2} - f^2} \quad \Rightarrow$$

$$g_1^2 = -\beta_0^2 - \beta_0^2 \varepsilon_r \mu_r + g_2^2$$

$$\boxed{g_2^2 = +\beta_0^2 \varepsilon_r \mu_r - \beta_0^2 + g_1^2}$$

$$4.\ \boxed{g_2^2 = -\beta_0^2 - f^2}$$

For convenience we will take the ratio to eliminate A/B:

$$\frac{E_x^{\mathrm{I}}}{H_z^{\mathrm{I}}} = \frac{E_x^{\mathrm{II}}}{H_z^{\mathrm{II}}} = Z_{-y}$$

$$\hat{x} \times \hat{z} = -\hat{y}$$

where Z_{-y} represents the impedance of a wave traveling along the $-\hat{y}$ direction at $y = \pm t$. This gives the equation

$$\boxed{g_1 \tanh(g_1 t) + \varepsilon_r g_2 = 0}$$

or alternatively ($\widetilde{g}_1 = j g_1$)

$$\boxed{\widetilde{g}_1 t \tan(\widetilde{g}_1 t) = \varepsilon_r t g_2}$$

Note:

$$\sin ju = j \sinh u$$

$$\cos ju = \cosh u$$

$$\tan ju = j \tanh u$$

$$\tanh ju = j \tan u$$

Odd modes (H_z fields: TM case)

$$H_z^{\mathrm{I}} = B e^{-fx} \sinh(g_1 y)$$

$$H_z^{\mathrm{II}} = A e^{-fx} e^{-g_2 y}$$

and in this case the characteristic equation is

$$g_1 \coth(g_1 t) - \varepsilon_r g_2 = 0$$

$$\Rightarrow \quad \boxed{\widetilde{g}_1 \cot(\widetilde{g}_1 t) = -\varepsilon_r g_2, \text{ with } \widetilde{g}_1 = j g_1}$$

Note:

$$\cot(ju) = -j \coth u$$

$$\coth(ju) = -j \cot u$$

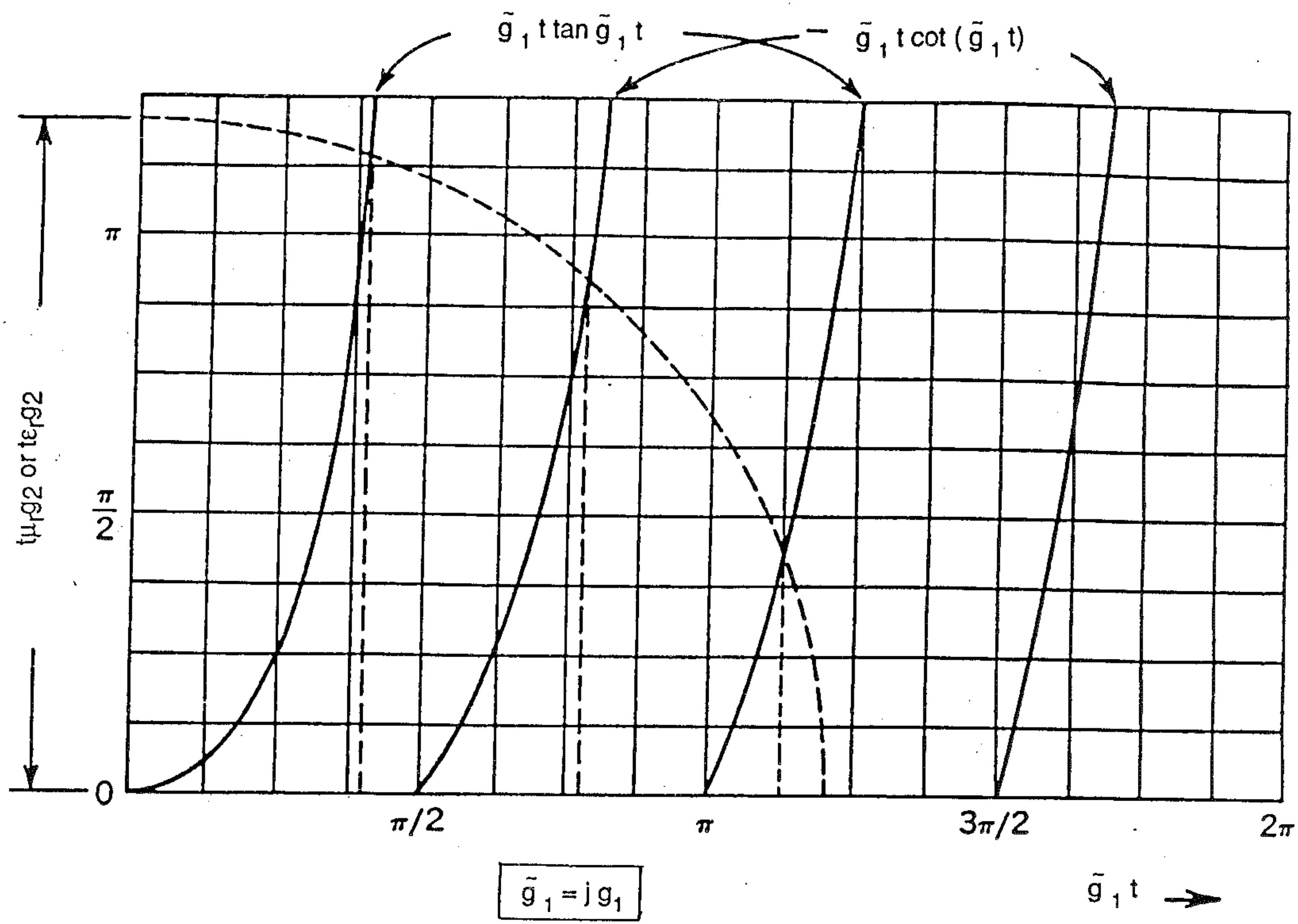


Figure 3. Graphical solution of the characteristic equation for the slab waveguide.

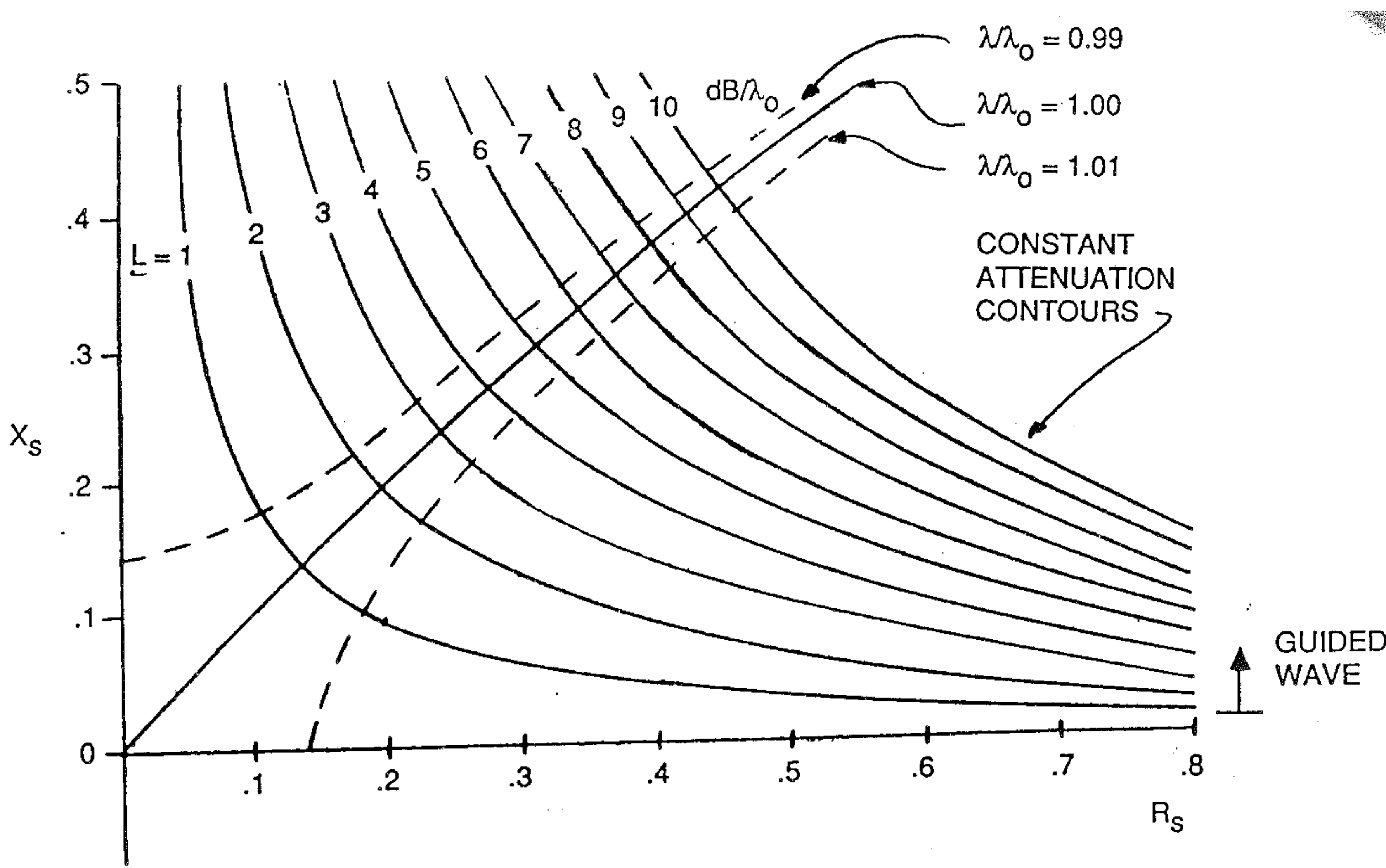


Figure 6. Contours of constant surface wave loss and velocity for an impedance plane.