Review of Differential Equations

1) First-order D.E.

E. Kreyszig, *Advanced Engineering Mathematics*, J. Wiley, 4th ed.

$$
\begin{aligned}
&y' + f(x)\,y = r(x) \qquad \text{Inhomogeneous} \\
&y' + f(x)\,y = 0 \qquad \text{homogeneous}
\end{aligned}
\tag{1}
$$

a) homogeneous solution

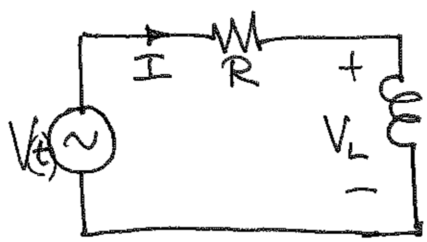
$$y = c_0 e^{-h}, \qquad h = \int f(x)\,dx$$

b) inhomogeneous solution

$$y(x) = c_0 e^{-h} + e^{-h} \int e^{h} r(x)\,dx$$

Example

Find the current of the circuit



$$Y_L = \frac{1}{sL} = \frac{1}{j\omega L}$$

or $Z_L = j\omega L = sL$

$$V_L = L\frac{dI}{dt}$$

Applying Kirchoff's voltage law we have

$$V(t) = RI(t) + L\frac{dI(t)}{dt} \Rightarrow I' + \frac{R}{L}I = \frac{V(t)}{L}$$

i.e., from (1)

$$f(t) = \frac{R}{L} \text{ and } r(t) = \frac{V(t)}{L} \Rightarrow h = \frac{R}{L}t$$

Choose $V(t) = \sin\omega t$. Then

$$I(t) = c_0 e^{-(R/L)t} + e^{-(R/L)t} \int e^{+(R/L)t} \frac{\sin\omega t}{L}\,dt$$

Using integration by parts

$$
\begin{aligned}
I(t) &= c_0 e^{-(R/L)t} + \frac{1}{R^2 + (\omega L)^2}(R\sin\omega t - \omega L\cos\omega t) \\
&= \underbrace{c_0 e^{-(R/L)t}}_{\text{transient response}} + \underbrace{\frac{1}{R^2 + (\omega L)^2}(R\sin\omega t - \omega L\cos\omega t)}_{\text{steady state response}}
\end{aligned}
$$

$$\delta = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Note: $c_0 = I(t = 0)$.

Second-order D.E. (see Chapters 2 and 4 of your text)

General form:

$$y'' + f(x)\,y' + g(x)\,y = r(x) \qquad \text{inhomogeneous}$$
$$y'' + f(x)\,y' + g(x)\,y = 0 \qquad \text{homogeneous}$$

1) Cauchy's equation or Euler's equation

$$\underline{x^2y'' + axy' + by = 0}$$

Solution is $y = x^r$, substituting in D.E. we have

$$r^2 + (a-1)r + b = 0 \Rightarrow r = r_{1,2} \Rightarrow$$

$$\text{Complete solution is } \boxed{y(x) = c_1x^{r_1} + c_2x^{r_2}}$$

$$\text{If } r = r_1 = r_2 \text{ then } \boxed{y(x) = c_1x^r + c_2 \ln rx^r}$$

2) Homogeneous solution with constant coefficients

$$\underline{y'' + ay' + by' = 0}$$

A solution is $y = e^{\lambda x}$. Substituting in D.E. we have

$$\lambda^2 + a\lambda + b = 0 \Rightarrow \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Case I $\lambda_{1,2}$ real
then $\underline{y = y_{\text{h}} = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}}$

Case II $\lambda_{1,2} = \pm j\beta$ $(a = 0)$
then $\underline{y = y_{\text{p}} = \beta_1 \cos \beta x + \beta_2 \sin \beta_2 x}$

Case III $\lambda_{1,2} - \alpha \pm j\beta$
then $\underline{y = y_{\text{p}} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)}$
The constants as you well know are computed with the appropriate initial conditions.

Case IV $\lambda = \lambda_1 = \lambda_2$ (real, of course complex roots appear always in conjugate pairs)
$\underline{y = y_{\text{p}} = (D_1 + D_2x)e^{\lambda x}}$

3) Non-homogeneous solution of D.E. with constant coefficients
(also inhomogeneous)

$$y'' + ay' + by = r(x)$$

general solution is

$$y(x) = y_{\text{h}} + y_{\text{p}}$$

where "h" indicates the homogeneous solution from 2) and "p" indicates the particular solution corresponding to $r(x)$.

The choices of $y_{\text{p}}(x)$ according to $r(x)$ are as follows:

$r(x)$	$y_p(x)$
e^{px}	Be^{px}
$AX^N;\ N = 0, 1, 2, \ldots$	$\sum_{n=0}^{N} B_n X^n$
$A \cos qx$	$B_1 \cos qx + B_2 \sin qx$
$A \sin qx$	$A_1 \cos qx + A_2 \sin qx$

Note: The constants B_i for $y_p(x)$ are found by substituting y_p in the D.E. and equating the coefficients of like basis functions.

A general method for finding y_p is also given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x)\, r(x)}{W}\, dx + y_2(x) \int \frac{y_1(x)\, r(x)}{W}\, dx$$

where y_1 and y_2 are the two homogeneous solutions to the D.E. and $W = y_1 y_2' - y_1' y_2$, referred to as the *Wronskian.*

4) Bessel's Differential Equation (sections 4.5, 4.6)

$$x^2 + y'' + xy' + (x^2 - v^2)y = 0 \qquad v = \text{real number}$$

The solution is found by using the power series substitution of $y = \sum_{m=0}^{\infty} c_m x^{m+k}$ and determining the coefficients c_m.

In particular,

$$c_m = 0 \qquad \text{for } m \text{ odd}$$
$$c_{2m} = -\frac{c_{2m-2}}{4m(v+m)}$$
$$c_0 = \frac{1}{2^v \Gamma(v+1)}$$

$$\Gamma(v) = \text{gamma function} = \int_0^{\infty} e^{-t} t^{v-1}\, dt; \qquad \Gamma(1/2) = \sqrt{\pi}$$
$$\text{for } v = n = \text{integer } \Gamma(n+1) = n!$$

The solution is written as

$$y(x) = A_1 J_v(x) + A_2 J_{-v}(x) \qquad \text{for } v \neq n \text{ since } J_{-n}(x) = (-1)^n J_n(x)$$

or in general as

$$y(x) = B_1 J_v(x) + B_2 Y_v(x) \qquad \text{for all } v$$

where $J_v(x)$ is the Bessel function of the first kind and of order v, given by

$$J_v(x) = X^v \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+v} m!\, \Gamma(v+m+1)}$$

For $v = n$, an integral form of $J_n(x)$ is

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(\sin\theta - n\theta)\, d\theta$$

$$\text{for small } x \Rightarrow J_v(x)\big|_{x\to 0} \approx \frac{1}{\Gamma(v+1)}\left(\frac{v}{2}\right)^v; \quad J_0(0) = 1 \;\; \& \;\; \underset{\substack{\uparrow \\ v\neq 0}}{J_v}(0) = 0$$

$$\text{for large } x \Rightarrow J_v(x)\big|_{x\to\infty}\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$$

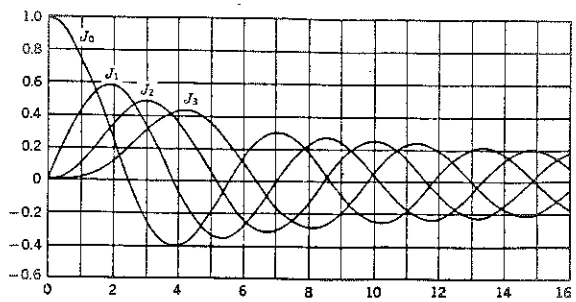


FIG. D-1. Bessel functions of the first kind.

Basic recursive relationships for Bessel functions

$$\begin{aligned}
(x^v J_v)' &= x^v J_{v-1} \\
(x^{-v} J_v)' &= -x^{-v} J_{v+1} \\
J_v' &= J_{v-1} - \frac{v}{x} J_v \\
J_v' &= \frac{v}{x} J_v - J_{v+1} \\
J_v' &= \frac{1}{2}[J_{v-1} - J_{v+1}] \\
J_v &= \frac{x}{2v}[J_{v-1} + J_{v+1}] \\
J_{v+1} &= \frac{2v}{x} J_v - J_{v-1} \\
\int x^\mu J_v(x)\,dx &= x^\mu J_{v+1} + (v - \mu + 1)\int x^{\mu-1} J_{v+1}\,dx \\
\int x^\mu J_v(x)\,dx &= x^\mu J_{v-1} + (v + \mu - 1)\int x^{\mu-1} J_{v-1}\,dx \\
\int J_v\,dx &= -2J_{v-1} + \int J_{v-2}\,dx \\
\int J_v\,dx &= 2J_{v+1} + \int J_{v+2}\,dx
\end{aligned}$$

μ is a real number

$Y_v(x)$ are referred to as the Bessel functions of the second kind or Newman functions. They are given by

$$Y_v(x) = \frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}$$

$$\text{For small } x \Rightarrow Y_v(x)|_{x\to 0} = -\frac{(v-1)!}{\pi}\left(\frac{2}{x}\right)^v$$

$$\text{and for large } x \Rightarrow Y_v(x)|_{x\to 0} = \sqrt{\frac{2}{2\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$$

Note that we can also write the solution to Bessel's equation in terms of the Hankel functions given by

$$H_v^{(1)}(x) = J_v(x) + jY_v(x) \quad \text{and} \quad H_v^{(2)}(x) = J_v(x) - jY_v(x)$$

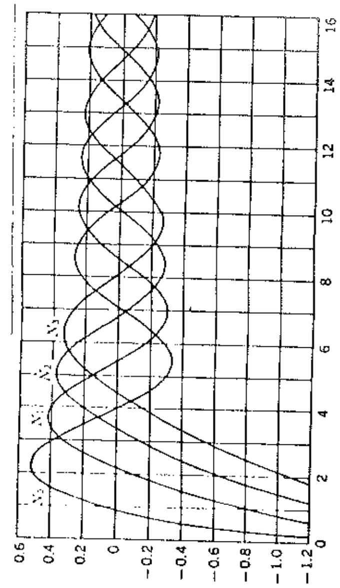


FIG. D-2. Bessel functions of the second kind.

5) Legendre's equation (section 4.3)

$$(1-x^2)y'' - 2xy' + n(n+1)y = 2 \qquad n\text{: integer}$$

Using $y = \sum_{n=0}^{\infty} c_m x^m$ a solution to Legendre's equation is

$$y(x) = c_1 P_n(x)$$

where $P_n(x)$ are Legendre's polynomials of the first kind, given by

$$P_n(x) = \sum_{m=1}^{n/2}(-1)^m \frac{(2n-2m)!\, x^{n-2m}}{2^n m!\,(n-m)!\,(n-2m)!} \qquad (n \text{ even})$$

$$P_n(x) = \sum_{m=1}^{(n-1)/2}(-1)^m \frac{(2n-2m)!\, x^{n-2m}}{2^n m!\,(n-m)!\,(n-2m)!} \qquad (n \text{ odd})$$

or using Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

Note

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

Also, note

$$P_n(-x) = (-1)^n P_n(x)$$

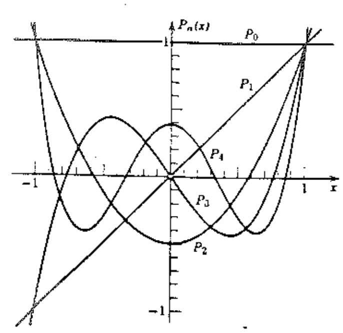


Fig. 74. Legendre polynomials

6) Sturm-Liouville Problem (section 4.8)

All of the homogeneous D.E. equations discussed above can be written in the form

$$[r(x)\, y']' + [q(x) + \lambda p(x)]y = 0$$

which is referred to as the Sturm-Liouville equation.

For example, if we choose

$$\text{a)} \quad r(x) = x, \quad q(x) = -\frac{v^2}{x}, \quad p(x) = x, \quad \& \quad \lambda = 1,$$

then we have

$$(xy')' + \left(-\frac{v^2}{x} + x\right)y = 0$$

which after multiplying by x reduces to Bessel's equation.

$$\text{b)} \quad r(x) = 1-x^2, \quad q(x) = 0, \quad \lambda = n(n+1), \quad \& \quad p(x) = 1,$$

then we have

$$[(1-x^2)y']' + n(n+1)y = 0 \Rightarrow -2xy' + (1-x)^2y'' + n(n+1)y = 0$$

which is Legendre's equation.

7) Orthogonality of the basis functions which are solutions to the Sturm-Liouville D.E.

If the solutions to the Sturm-Liouville equation are chosen to also satisfy the conditions (Boundary Conditions)

$$A_1 y(a) + A_2 y'(a) = 0$$
$$B_1 y(b) + B_2 y'(b) = 0$$

where A_i, B_i, a, b are constants. Then it can be proven that (see section 4.8)

$$\langle e_m, e_n \rangle = \int_a^b p(x)\, e_m(x)\, e_n(x)\, dx = 0 \qquad \text{for } m \neq n$$

$e_m(x)$ and $e_n(x)$ are solutions to the D.E. and satisfy the Boundary Conditions. The above integral can be recognized as the interproduct of e_m and e_n with weighting function $p(x)$ as it appears in the Sturm-Liouville equation, i.e.,

$$\langle e_m, e_n \rangle = 0 \qquad \text{for } m \neq n$$

and, of course, for $m = n$ we have

$$\langle e_n, e_n \rangle = \|e_n\|^2$$

If $r(a) = r(b) = 0$ then the Boundary Conditions are always satisfied and therefore e_m and e_n are orthogonal for $n \neq m$ with such a choice of a and b.

Thus, choosing $e_n(x) = P_n(x)$, the Legendre polynomials, we then have that $p(x) = 1$. In addition, note that for $a = -1$ and $b = 1$, $r(-1) = r(1) = 0$. Therefore

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0 \qquad m \neq n$$

and it can be found that $\|P_n(x)\|^2 = \frac{2}{2n+1}$. The orthogonality of the Bessel functions will be discussed in Chapter 11 when we discuss the solution of $\nabla^2 V = 0$ in cylindrical coordinates. We will see in Chapter 11 that the Bessel functions are cylindrical functions, whereas the Legendre functions are spherical functions.

Further, note that if the set $e_m|_{n=1}^{\infty}$ is a complete orthogonal set then any appropriate function can be expressed as

$$f(x) = \sum_{n=0}^{\infty} c_n e_n(x)$$

where $a < x < b$ and $\langle e_n, e_m \rangle = \int_a^b e_n e_m = 0$, $m \neq n$. To find c_n we multiply by e_m and integrate both sides $\Rightarrow$

$$\int_a^b f(x)\, e_m(x)\, dx = \sum_{n=0}^{\infty} c_n \int_a^b e_n(x)\, e_m(x)\, dx \Rightarrow$$

$$c_m = \frac{\int_a^b f(x)\, e_m(x)}{\|e_m(x)\|^2}$$

Examples: e_n: $\sin nx$, $\cos nx$ and $\begin{matrix} a = -\pi \\ b = \pi \end{matrix}$ $\Rightarrow \|\sin nx\|^2 = \pi = \|\cos nx\|$ and $\|1\|^2 = 2\pi$.

Additional References

1) N.N. Lebedev, *Special Functions and Their Applications,* Dover Publications.

2) M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions,* etc.

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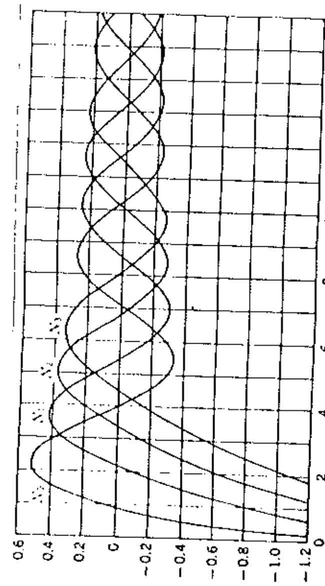

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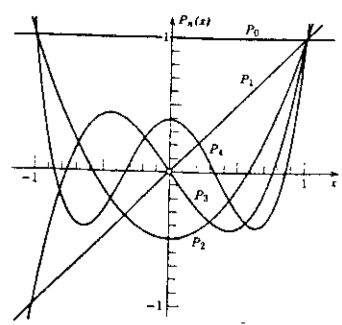

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