Applying the unit vector relations of (26) and (27), (28) reduces to

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x) \tag{29}$$

This may be expressed concisely in determinant form as follows:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{30}$$

1-9 INTRODUCTION TO COORDINATE SYSTEMS

As an introduction to the subject of coordinate systems, let us compare the rectangular and polar two-coordinate systems. Here the basic relations can be shown in a two-dimensional flat space (plane of the page). These ideas are then easily extended to three-dimensional systems which are discussed later. The coordinate systems we will use are all orthogonal systems, that is, their axes are mutually perpendicular (at 90° angles).

The position of a point P in a flat plane may be described by its perpendicular distance x from a y axis and its perpendicular distance y from an x axis as in Fig. 1-15*a*. The x and y axes of this *rectangular two-coordinate system* intersect at right angles. The point of intersection ($x = y = 0$) is called the *origin*.

The point P may also be located in *polar coordinates* as the radial distance r from the origin and the angle ϕ to the radial line as measured from a reference direction (x axis) (Fig. 1-15*b*). Thus, P is at the intersection of a circle of radius r and a straight line at the angle ϕ. The point for which $r = 0$ is the origin.

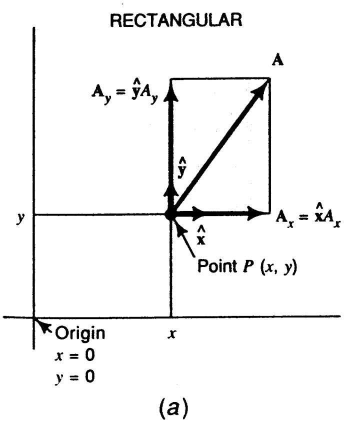


FIGURE 1-15
(*a*) Point P in rectangular coordinates (x, y) with vector $\mathbf{A}$ resolved into its rectangular components $(\mathbf{A}_x, \mathbf{A}_y)$. (*Continued on next page*)

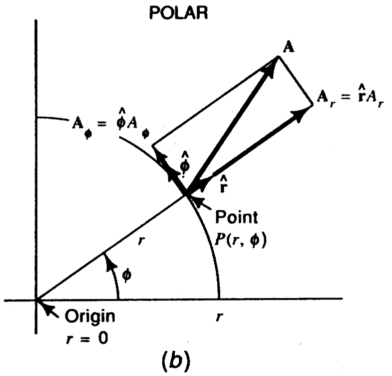


FIGURE 1-15 *(continued)*
(*b*) Point P in polar coordinates (r, ϕ) with vector $\mathbf{A}$ resolved into its polar components $(\mathbf{A}_r, \mathbf{A}_\phi)$.

Comparing the rectangular and polar systems in Fig. 1-15*a* and *b*, we note that

$$x = r\cos\phi \qquad y = r\sin\phi \tag{1}$$

and conversely that

$$r = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{2}$$

Thus, the coordinates of the point P are transformed from polar into rectangular coordinates by (1) and from rectangular into polar by (2).

Suppose next that we have a vector $\mathbf{A}$ at the point P with coordinates (x, y) or (r, ϕ).[†] Referring to Fig. 1-15*a*, the vector $\mathbf{A}$ may be expressed in terms of its rectangular components as

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y \tag{3}$$

or in terms of its polar components (Fig. 1-15*b*) as

$$\mathbf{A} = \hat{\mathbf{r}}A_r + \boldsymbol{\phi}A_\phi \tag{4}$$

It follows that

$$A_r = A_x\cos\phi + A_y\sin\phi \qquad A_\phi = -A_x\sin\phi + A_y\cos\phi \tag{5}$$

and that

$$A_x = A_r\cos\phi - A_\phi\sin\phi \qquad A_y = A_r\sin\phi + A_\phi\cos\phi \tag{6}$$

[†] Note that A is *not* the distance from the origin (r is the distance). Note also that

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{A_r^2 + A_\phi^2}$$

Thus, the vector **A** at the point P is transformed from rectangular into polar components by (5) and from polar into rectangular components by (6).

Turning now to three-coordinate systems, the most common are the rectangular (Fig. 1-16*a*), the cylindrical (Fig. 1-16*b*), and the spherical (Fig. 1-16*c*).

In *rectangular coordinates* a point P is specified by x, y, z, where these values are all measured from the origin (Fig. 1-16*a*). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. The unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ form a right-handed set, that is, turning $\hat{\mathbf{x}}$ into $\hat{\mathbf{y}}$ like a right-handed screw, we move in the $\hat{\mathbf{z}}$ direction (Fig. 1-7*a*).

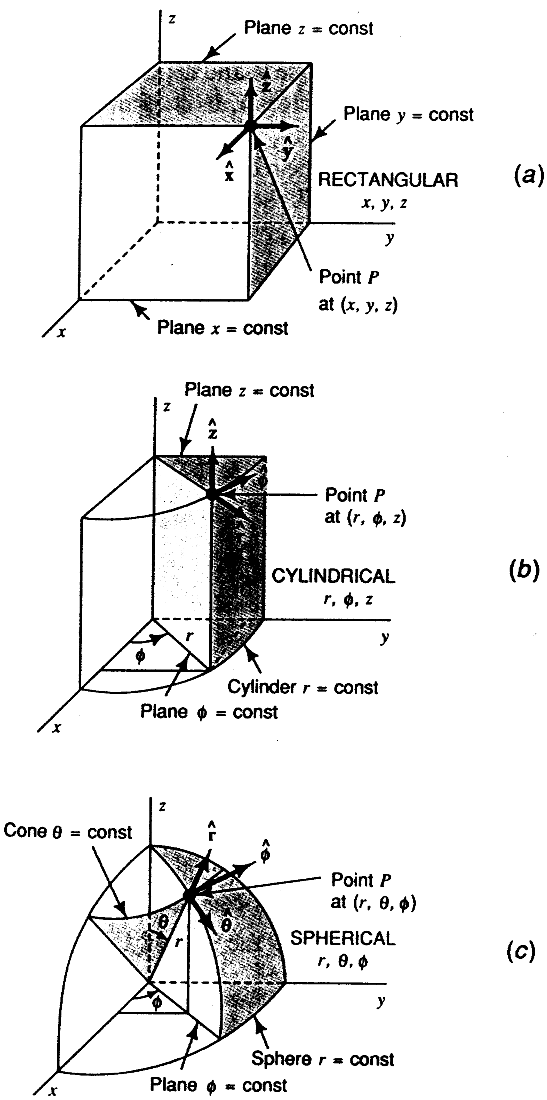


FIGURE 1-16
Coordinate systems. (*a*) Rectangular. (*b*) Cylindrical. (*c*) Spherical.

In *cylindrical coordinates* a point P is specified by r, ϕ, z, where ϕ is measured from the x axis (or x-z plane) (Fig. 1-16b). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{r}}$ perpendicular to the cylinder of radius r, $\hat{\boldsymbol{\phi}}$ perpendicular to the plane through the z axis at angle ϕ, and $\hat{\mathbf{z}}$ perpendicular to the x-y plane at distance z. The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$ form a right-handed set.

We note that the polar (two-coordinate) system of Fig. 1-15b is the same as the x-y plane coordinates of a cylindrical system (Fig. 1-16b).

In *spherical coordinates* a point P is specified by r, θ, ϕ, where r is measured from the origin, θ is measured from the z axis, and ϕ is measured from the x axis (or x-z plane) (Fig. 1-16c). With z axis up, as in Fig. 1-16c, θ is sometimes called the *zenith* angle and ϕ the *azimuth* angle. A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\mathbf{r}}$ perpendicular to the sphere of radius r, $\hat{\boldsymbol{\theta}}$ perpendicular to the cone of angle θ, and $\hat{\boldsymbol{\phi}}$ perpendicular to the plane through the z axis at angle ϕ. The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ form a right-handed set.

Length elements (dL) and volume elements (dv) for the rectangular, cylindrical, and spherical coordinate systems are shown in Fig. 1-17.

An infinitesimal length in the *rectangular* system is given by

$$dL = \sqrt{dx^2 + dy^2 + dz^2} \tag{7}$$

and an infinitesimal volume by

$$dv = dx\,dy\,dz \tag{8}$$

In the *cylindrical* system the corresponding quantities are

$$dL = \sqrt{dr^2 + r^2\,d\phi^2 + dz^2} \tag{9}$$

and

$$dv = dr\,r\,d\phi\,dz \tag{10}$$

In the *spherical* system we have

$$dL = \sqrt{dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2} \tag{11}$$

and

$$dv = dr\,r\,d\theta\,r\sin\theta\,d\phi \tag{12}$$

Referring to Fig. 1-18, the projection x of the scalar distance r on the x axis is given by $r\cos\alpha$ where α is the angle between r and the x axis. The projection of r on the y axis is given by $r\cos\beta$, and the projection on the z axis by $r\cos\gamma$. Note that $\gamma = \theta$ so $\cos\gamma = \cos\theta$.

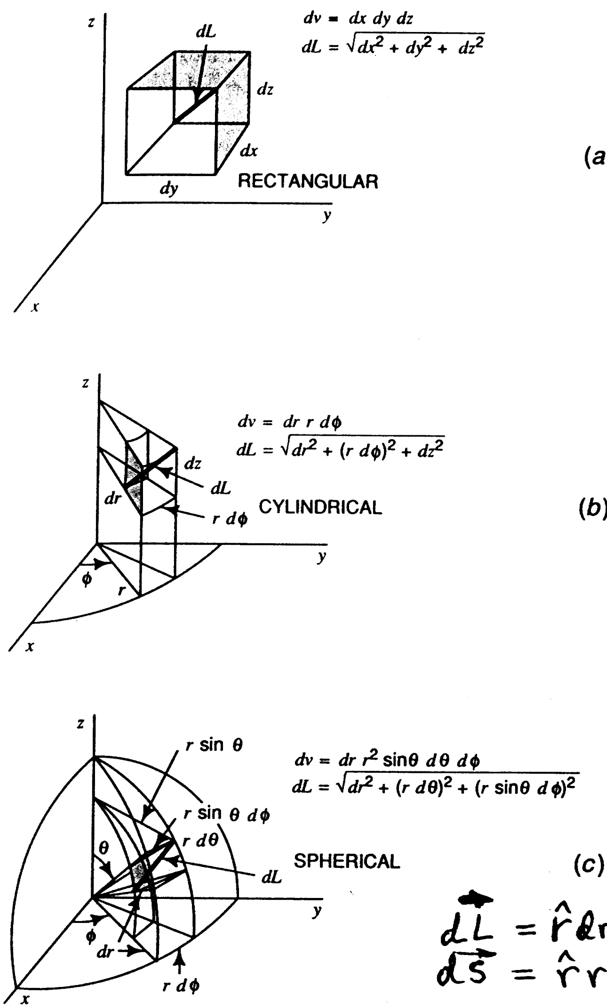
The quantities $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines*. From the theorem of Pythagoras,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{13}$$

The direction cosines $\cos\alpha$ and $\cos\beta$ are related to the spherical angles θ and ϕ as follows:

$$\cos\alpha = \sin\theta\cos\phi$$

$$\cos\beta = \sin\theta\sin\phi$$



$\vec{dL} = \hat{x}dx + \hat{y}dy + \hat{z}dz$
$\vec{ds} = \hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy$

$\vec{dL} = \hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$
$\vec{ds} = \hat{r}rd\phi dz + \hat{\phi}drdz + \hat{z}rdrd\phi$

$\vec{dL} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$
$\vec{ds} = \hat{r}r^2\sin\theta d\phi d\theta + \hat{\theta}r\sin\theta d\phi dr + \hat{\phi}rdrd\theta$

FIGURE 1-17
Elemental lengths and volumes in rectangular, cylindrical, and spherical coordinates.

This may be shown in two steps. First, we obtain the projection of r on the x-y (horizontal plane). From Fig. 1-18 this is seen to be $r \sin\theta$. Then, we obtain the projection of $r \sin\theta$ on the x axis which is $r \sin\theta \cos\phi = r \cos\alpha$.

In a similar way, we find that the projection of r on the y axis is $r \sin\theta \sin\phi = r \cos\beta$.

Thus (see Fig. 1-18), the scalar distance r of a spherical coordinate system transforms into rectangular coordinate distances

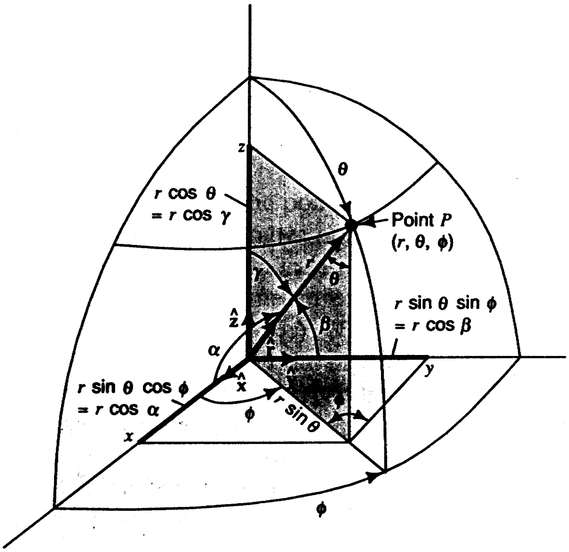


FIGURE 1-18
Scalar distance r to point P at (r, θ, ϕ) resolved into rectangular (x, y, z) components. Rectangular $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ unit vectors are also shown.

$$x = r\cos\alpha = r\sin\theta\cos\phi \tag{14}$$

$$y = r\cos\beta = r\sin\theta\sin\phi \tag{15}$$

$$z = r\cos\gamma = r\cos\theta \tag{16}$$

from which

$$\cos\alpha = \sin\theta\cos\phi \tag{17}$$

$$\cos\beta = \sin\theta\sin\phi \tag{18}$$

$$\cos\gamma = \cos\theta \tag{19}$$

As the converse of (14), (15), and (16), the spherical coordinate values (r, θ, ϕ) may be expressed in terms of rectangular coordinate distances as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0) \tag{20}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0 \leq \theta \leq \pi) \tag{21}$$

$$\phi = \tan^{-1}\frac{y}{x} \tag{22}$$

From these and similar *coordinate transformations* of spherical to rectangular and rectangular to spherical coordinates, we may express a vector **A** at some point P with spherical components A_r, A_θ, and A_ϕ as the rectangular components A_x, A_y, and A_z where

$$A_x = A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi \tag{23}$$

$$A_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi \tag{24}$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta \tag{25}$$

We note also from Fig. 1-18 that the dot products of the unit vector $\hat{\mathbf{r}}$ with the rectangular unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are equal to the direction cosines as given by

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{r}} = \cos \alpha = \sin \theta \cos \phi \tag{26}$$

$$\hat{\mathbf{y}} \cdot \hat{\mathbf{r}} = \cos \beta = \sin \theta \sin \phi \tag{27}$$

$$\hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \cos \gamma = \cos \theta \tag{28}$$

These and other dot product combinations are listed in Table 1-1.

In addition to rectangular, cylindrical, and spherical coordinate systems, there are many other systems such as the elliptical, spheroidal (both prolate and oblate), and paraboloidal systems. Although the number of possible systems is infinite, all of them can be treated in terms of a *generalized curvilinear coordinate system.* However, we will not need to deal with these systems in this book, the rectangular, cylindrical, and spherical systems being sufficient for our requirements.

The fundamental parameters of the rectangular, cylindrical, and spherical coordinate systems are summarized in Table 1-2.

Table 1-3 gives the unit vector dot products in rectangular coordinates for both rectangular-cylindrical and rectangular-spherical coordinates.

Table 1-4 summarizes the transformations between the three systems.

TABLE 1-1
Dot products of unit vectors in three coordinate systems

		Rectangular			Cylindrical			Spherical		
	$\cdot$	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{z}}$	$\hat{\mathbf{r}}$	$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\phi}}$
Rectangular	$\hat{\mathbf{x}}$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
	$\hat{\mathbf{y}}$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\hat{\mathbf{r}}$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\hat{\boldsymbol{\phi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
	$\hat{\mathbf{z}}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\hat{\mathbf{r}}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\hat{\boldsymbol{\theta}}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\hat{\boldsymbol{\phi}}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

Note that the unit vectors $\hat{\mathbf{r}}$ in the cylindrical and spherical systems are *not* the same. Example:

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \sin \theta \cos \phi$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin \theta \sin \phi$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos \theta$$

TABLE 1-2 Parameters of rectangular, cylindrical, and spherical coordinate systems

Coordinate system	Coordinates	Range	Unit vectors	Length elements	Coordinate surfaces	
Rectangular (Fig. 1-16*a*)	x	$-\infty$ to $+\infty$	$\hat{\mathbf{x}}$	dx	Plane	$x =$ constant
	y	$-\infty$ to $+\infty$	$\hat{\mathbf{y}}$	dy	Plane	$y =$ constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$	dz	Plane	$z =$ constant
Cylindrical (Fig. 1-16*b*)	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Cylinder	$r =$ constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\phi}}$	$r\,d\phi$	Plane	$\phi =$ constant
	z	$-\infty$ to $+\infty$	$\hat{\mathbf{z}}$	dz	Plane	$z =$ constant
Spherical (Fig. 1-16*c*)	r	0 to ∞	$\hat{\mathbf{r}}$	dr	Sphere	$r =$ constant
	θ	0 to π	$\hat{\boldsymbol{\theta}}$	$r\,d\theta$	Cone	$\theta =$ constant
	ϕ	0 to 2π	$\hat{\boldsymbol{\phi}}$	$r \sin\theta\, d\phi$	Plane	$\phi =$ constant

TABLE 1-3

Unit vector dot products for rectangular-cylindical and rectangular-spherical coordinates

Rectangular-cylindrical product in rectangular coordinates				Rectangular-spherical product in rectangular coordinates			
$\cdot$	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$	$\cdot$	$\hat{\mathbf{x}}$	$\hat{\mathbf{y}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	$\dfrac{y}{\sqrt{x^2+y^2}}$	0	$\hat{\mathbf{r}}$	$\dfrac{x}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{y}{\sqrt{x^2+y^2+z^2}}$	$\dfrac{z}{\sqrt{x^2+y^2+z^2}}$
$\hat{\boldsymbol{\phi}}$	$\dfrac{-y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0	$\hat{\boldsymbol{\theta}}$	$\dfrac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$\dfrac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}$	$-\dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$
$\hat{\mathbf{z}}$	0	0	1	$\hat{\boldsymbol{\phi}}$	$-\dfrac{y}{\sqrt{x^2+y^2}}$	$\dfrac{x}{\sqrt{x^2+y^2}}$	0
Example: $\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{y}} = \cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$				Example: $\hat{\mathbf{x}} \cdot \hat{\mathbf{r}} = \sin\theta \cos\phi = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$			

TABLE 1-4
Coordinate transformations

Rectangular to cylindrical

$$A_r = A_x \frac{x}{\sqrt{x^2 + y^2}} + A_y \frac{y}{\sqrt{x^2 + y^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2 + y^2}} + A_y \frac{x}{\sqrt{x^2 + y^2}}$$

$$A_z = A_z$$

Rectangular to spherical

$$A_r = A_x \frac{x}{\sqrt{x^2 + y^2 + z^2}} + A_y \frac{y}{\sqrt{x^2 + y^2 + z^2}} + A_z \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$A_\theta = A_x \frac{xz}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}} + A_y \frac{yz}{\sqrt{x^2 + y^2}\sqrt{x^2 + y^2 + z^2}} - A_z \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$A_\phi = -A_x \frac{y}{\sqrt{x^2 + y^2}} + A_y \frac{x}{\sqrt{x^2 + y^2}}$$

Cylindrical to rectangular

$$A_x = A_r \cos\phi - A_\phi \sin\phi$$

$$A_y = A_r \sin\phi + A_\phi \cos\phi$$

$$A_z = A_z$$

Spherical to rectangular

$$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$$

$$A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$$

$$A_z = A_r \cos\theta - A_\theta \sin\theta$$

1-10 SUMMARY

With the history of electromagnetism as background, dimensions, units, symbols, and equation numbering were explained. Finally, we discussed some of the introductory aspects of vector analysis and coordinate systems. We are now ready to begin the basics of electromagnetism with the Static Electric Fields of Chap. 2.

QUESTIONS

1-1. Thales knew about electricity and magnetism in the year 600 B.C. Why did it take 25 centuries before Maxwell formulated his theory unifying electricity and magnetism?

1-2. Who designed the Niagara Falls power station of 1895? What is the unit named for him?