SIMPLIFIED EXPRESSIONS FOR Z-PROPAGATING WAVES

Assume

$$\mathbf{E}_{\mathrm{g}}(x,y,z) = \mathbf{E}(x,y)\, e^{-\gamma z}$$
$$\mathbf{H}_{\mathrm{g}}(x,y,z) = \mathbf{H}(x,y)\, e^{-\gamma z}$$

where from Maxwell's equations ("g" refers to guided wave)

$$\nabla \times \mathbf{E}_{\mathrm{g}} = -j\omega\mu \mathbf{H}_{\mathrm{g}}, \qquad \nabla \times \mathbf{H}_{\mathrm{g}} = +j\omega\epsilon \mathbf{E}_{\mathrm{g}}$$

Expanding these we have

$$\begin{aligned}
\frac{\partial E_z}{\partial y} + \gamma E_y &= -j\omega\mu H_x & \qquad \frac{\partial H_z}{\partial y} + \gamma H_y &= j\omega\epsilon E_x \\
\gamma E_x + \frac{\partial E_z}{\partial x} &= +j\omega\mu H_y & \qquad -\gamma H_x - \frac{\partial H_z}{\partial x} &= j\omega\epsilon E_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -j\omega\mu H_z & \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= j\omega\epsilon E_z
\end{aligned}$$

which can be further manipulated to express the x and y components in terms of the z components. We obtain

$$\begin{aligned}
H_x &= -\frac{1}{h^2}\left(\gamma\frac{\partial H_z}{\partial x} - j\omega\epsilon\frac{\partial E_z}{\partial y}\right) \\
H_y &= -\frac{1}{h^2}\left(\gamma\frac{\partial H_z}{\partial y} + j\omega\epsilon\frac{\partial E_z}{\partial x}\right) & \qquad h^2 = k^2 + \gamma^2 \\
E_x &= -\frac{1}{h^2}\left(\gamma\frac{\partial E_z}{\partial x} + j\omega\mu\frac{\partial H_z}{\partial y}\right) & \qquad k = \omega\sqrt{\mu\epsilon} \\
E_y &= -\frac{1}{h^2}\left(\gamma\frac{\partial E_z}{\partial y} - j\omega\mu\frac{\partial H_z}{\partial x}\right)
\end{aligned}$$

TM Waves ($H_z = 0$)

$$\begin{aligned}
H_x &= \frac{j\omega\epsilon}{h^2}\,\frac{\partial E_z}{\partial y} = -\frac{j\omega\epsilon}{\gamma}\,E_y \\
H_y &= -\frac{j\omega\epsilon}{h^2}\,\frac{\partial E_z}{\partial x} \\
E_x &= -\frac{\gamma}{h^2}\,\frac{\partial E_z}{\partial x} \\
E_y &= -\frac{\gamma}{h^2}\,\frac{\partial E_z}{\partial y}
\end{aligned}$$

NOTE: $Z_{\mathrm{TM}} = E_x/H_y = -E_y/H_x = \gamma/j\omega\epsilon =$ wave impedance $\neq \eta = Z = \sqrt{\mu/\epsilon}$. Also, $\mathbf{H} = (1/Z_{\mathrm{TM}})\hat{z} \times \mathbf{E}$ (for the transverse components only).

TE Waves $(E_z = 0)$

$$
\begin{aligned}
H_x &= -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} = -\frac{\gamma}{h^2}\left(\frac{h^2}{j\omega\mu}\right)E_y = -\frac{\gamma}{j\omega\mu}E_y \\
H_y &= -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y} \\
E_x &= -\frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \\
E_y &= \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}
\end{aligned}
$$

NOTE: $Z_{\mathrm{TE}} = -E_y/H_x = E_x/H_y = j\omega\mu/\gamma$ = wave impedance $\neq \eta = Z = \sqrt{\mu/\epsilon}$. Also for the transverse components $\mathbf{E} = -Z_{\mathrm{TE}}(\hat{z} \times \hat{\mathbf{H}})$.

E Waves and H Waves in Cartesian Coordinates

Consider a time-harmonic field in a source-free homogeneous region. Let

$$\frac{\partial^2 E_z}{\partial z^2} = \gamma^2 E_z \qquad \text{and} \qquad \frac{\partial^2 H_z}{\partial z^2} = \gamma^2 H_z. \tag{1}$$

Then it is convenient to use E_z and H_z as generating functions to determine the other field components as follows. Let

$$h^2 = \gamma^2 + \omega^2 \mu \varepsilon = \gamma^2 + k^2 \tag{2}$$

$$\begin{array}{ll}
\underline{E \text{ Waves}} & \underline{H \text{ Waves}}
\end{array}$$

$$E_x = \frac{1}{h^2}\frac{\partial^2 E_z}{\partial x \partial z} \qquad E_x = -j\omega\mu \frac{1}{h^2}\frac{\partial H_z}{\partial y} \tag{3}$$

$$E_y = \frac{1}{h^2}\frac{\partial^2 E_z}{\partial y \partial z} \qquad E_y = -j\omega\mu \frac{1}{h^2}\frac{\partial H_z}{\partial x} \tag{4}$$

$$H_x = j\omega\varepsilon \frac{1}{h^2}\frac{\partial E_z}{\partial y} \qquad E_z = 0 \tag{5}$$

$$H_y = -j\omega\varepsilon \frac{1}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{1}{h^2}\frac{\partial^2 H_z}{\partial x \partial z} \tag{6}$$

$$H_z = 0 \qquad H_y = \frac{1}{h^2}\frac{\partial^2 H_z}{\partial y \partial z} \tag{7}$$

A more general field can be expressed as the sum of an E wave and an H wave. Still more generality is obtained by considering the field to be the sum or integral of E and H waves having different values of γ.

Some of the solutions of (1) are:

$$\cosh(\gamma z), \quad \sinh(\gamma z), \quad e^{\gamma z}, \quad e^{-\gamma z} \tag{8}$$

$$\cos(\beta_z z), \quad \sin(\beta_z z), \quad e^{j\beta_z z}, \quad e^{-j\beta_z z} \tag{9}$$

where $\gamma = j\beta_z$.

If the field does not satisfy (1), the Fourier transform may be applied to express the field as a spectrum of waves that do satisfy (1). Thus, the field in a homogeneous source-free region is determined by E_z and H_z. However, uniqueness is spoiled by the possible existence of TEM waves having $E_z = 0$ *and* $H_z = 0$.

SIMPLIFIED EXPRESSIONS FOR *X*-PROPAGATING WAVES

Assume

$$\mathbf{E}_{\mathrm{g}} = \mathbf{E}(x,\cancel{y}z)\, e^{-\gamma x}$$
$$\mathbf{H}_{\mathrm{g}} = \mathbf{H}(x,\cancel{y}z)\, e^{-\gamma x}$$

("g" indicates a guided wave) where from Maxwell's equations

$$\nabla \times \mathbf{E}_{\mathrm{g}} = -j\omega\mu\mathbf{H}_{\mathrm{g}}, \qquad \nabla \times \mathbf{H}_{\mathrm{g}} = j\omega\varepsilon\mathbf{E}_{\mathrm{g}}$$

Expanding these we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$
$$\frac{\partial E_x}{\partial z} + \gamma E_z = -j\omega\mu H_y \qquad \frac{\partial H_x}{\partial z} + \gamma H_z = j\omega\varepsilon E_y$$
$$\gamma E_y + \frac{\partial E_x}{\partial y} = j\omega\mu H_z \qquad \gamma H_y + \frac{\partial H_x}{\partial y} = -j\omega\varepsilon E_z$$

These can be further manipulated to express the z and y components in terms of the x components. We obtain

$$H_y = -\frac{1}{h^2}\left(\gamma\frac{\partial H_x}{\partial y} - j\omega\varepsilon\frac{\partial E_x}{\partial z}\right)$$
$$H_z = -\frac{1}{h^2}\left(\gamma\frac{\partial H_x}{\partial z} + j\omega\varepsilon\frac{\partial E_x}{\partial y}\right)$$
$$E_y = -\frac{1}{h^2}\left(\gamma\frac{\partial E_x}{\partial y} + j\omega\mu\frac{\partial H_x}{\partial z}\right) \qquad h^2 = \beta_0^2 + \gamma^2$$
$$E_z = -\frac{1}{h^2}\left(\gamma\frac{\partial E_x}{\partial z} - j\omega\mu\frac{\partial H_x}{\partial y}\right) \qquad \beta_0 = \omega\sqrt{\mu\varepsilon}$$

TE and TM Fields Separable in the Cylindrical Coordinate System

The harmonic electromagnetic fields listed below satisfy Maxwell's equations in a homogeneous source-free region.

TE Fields	TM Fields
$E_\rho = -C\dfrac{j\omega\mu}{\rho}R\Phi'Z$	$E_\rho = CR'\Phi Z'$
$E_\phi = Cj\omega\mu R'\Phi Z$	$E_\phi = C\dfrac{1}{\rho}R\Phi' Z'$
$E_z = 0$	$E_z = C\beta^2 R\Phi Z$
$H_\rho = CR'\Phi Z'$	$H_\rho = C\dfrac{j\omega\varepsilon}{\rho}R\Phi' Z$
$H_\phi = C\dfrac{1}{\rho}R\Phi' Z'$	$H_\phi = -j\omega\varepsilon CR'\Phi Z$
$H_z = C\beta^2 R\Phi Z$	$H_z = 0$

C denotes an arbitrary constant. The time dependence $e^{j\omega t}$ is understood. R is a function of ρ only, Φ is a function of ϕ only, and Z is a function of z only. Primes indicate differentiation with respect to ρ, ϕ or z. The functions satisfy the following differential equations:

$$\rho\frac{d(\rho R')}{d\rho} + (k_\rho^2\rho^2 - m^2)R = 0$$

$$\Phi'' = -m^2\Phi$$

$$Z'' = -\beta^2 Z$$

where

$$k_\rho^2 + \beta_z^2 = \omega^2\mu\varepsilon$$

and k_ρ and h are constants. Note:

$$R' = k_\rho J_m'(k_\rho\rho) \quad \text{if } R = J_m(k_\rho\rho)$$
$$R' = k_\rho N_m'(k_\rho\rho) \quad \text{if } R = N_m(k_\rho\rho) = N_m(k_\rho\rho)$$
$$R' = k_\rho H_m'(k_\rho\rho) \quad \text{if } R = H_m(k_\rho\rho)$$

Some solutions of these differential equations are listed below.

$R(\rho) = J_m(k_\rho\rho)$	$\Phi(\phi) = \cos(m\phi)$	$Z(z) = \cos(\beta_z z)$
$N_m(k_\rho\rho)$	$\sin(m\phi)$	$\sin(\beta_z z)$
$H_m^{(1)}(k_\rho\rho)$	$e^{jm\phi}$	$e^{j\beta_z z}$
$H_m^{(2)}(k_\rho\rho)$	$e^{-jm\phi}$	e^{-jhz}

If $\beta = 0$, the radial function is $R(\rho) = \rho^{\pm m}$.

Note the definitions:

$$
\begin{aligned}
J_m(\cdot) &= \text{Bessel function of order } m \\
N_m(\cdot) &= \text{Neumann function of order } m \\
H_m^{(1)}(\cdot) &= m\text{th order Hankel function of the 1st kind} \\
H_m^{(2)}(\cdot) &= m\text{th order Hankel function of the 2nd kind}
\end{aligned}
$$