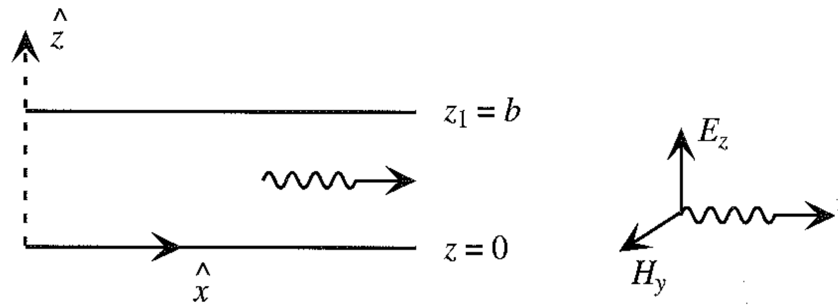
TM and TE mode derivation for a parallel plate waveguide



Based on the solution of the wave equation and the boundary conditions, we obtained the expressions

$$E_x = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x}, \qquad \gamma = jk_x$$

$$E_y = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x}$$

with

$$k_x^2 + k_z^2 = k_0^2 \quad \text{or} \quad -\gamma^2 + \left(\frac{n\pi}{z_1}\right)^2 = k_0^2 \quad \Rightarrow \quad \gamma = \sqrt{\left(\frac{n\pi}{z_1}\right) - k_0^2}$$

These two components are sufficient to define the rest of the field components in the guide. Note also that these expressions are "complete" as is the case with the Fourier expansions.

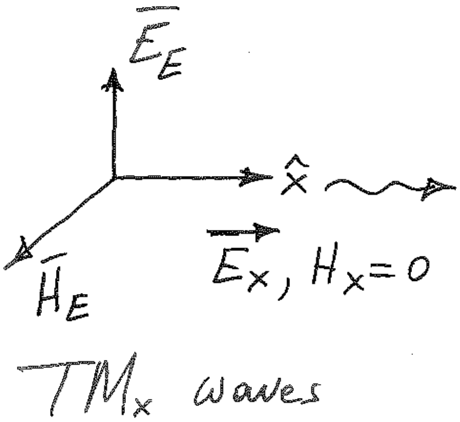
Although, we could begin now using (E_x, E_y) and get the rest of the field components, it is customary to use the pairs

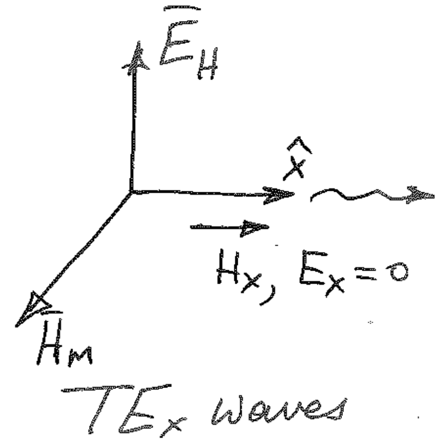
$$\begin{array}{ll} (E_x, H_x), & \text{for } x \text{ propagating waves} \\ (E_y, H_y), & \text{for } y \text{ propagating waves} \\ (E_z, H_z), & \text{for } z \text{ propagating waves} \end{array}$$

as the components for defining the others. For our case, we will select the (E_x, H_x) pair. In fact, we proceed with one more subdivision of these components. Specifically, using superposition, we can express the field components as follows (E and H waves or TM and TE modes)

$$\mathbf{E} = \mathbf{E}_E + \mathbf{E}_H$$

in which $\mathbf{E}_E$ is due to the E_x component (with $H_x = 0$), and $\mathbf{E}_H$ is due to the H_x components (with $E_x = 0$). The contribution from $\mathbf{E}_E$ is often referred to as the TM wave since the corresponding magnetic field is transverse to the direction of propagation.





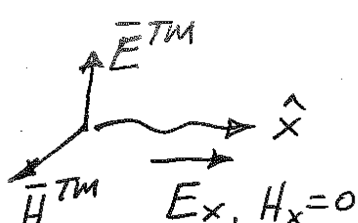
The contribution from $\mathbf{E}_H$ is referred to as the TE wave since the corresponding electric field is transverse to the direction of propagation as shown.

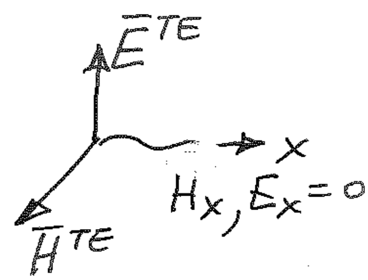
From the relations ($h^2 = \gamma^2 + k_0^2$)

$$
\begin{aligned}
H_y &= -\frac{1}{h^2}\left(\gamma\frac{\partial H_x}{\partial y} - j\omega\varepsilon\frac{\partial E_x}{\partial z}\right)\\
H_z &= -\frac{1}{h^2}\left(\gamma\frac{\partial H_x}{\partial z} - j\omega\varepsilon\frac{\partial E_x}{\partial y}\right)\\
E_y &= -\frac{1}{h^2}\left(j\omega\mu\frac{\partial H_x}{\partial z} + \gamma\frac{\partial E_x}{\partial y}\right)\\
E_z &= -\frac{1}{h^2}\left(-j\omega\mu\frac{\partial H_x}{\partial y} + \gamma\frac{\partial E_x}{\partial z}\right)
\end{aligned}
$$

it is not difficult to see that

E_x or TM fields	H_x or TE fields
H_y^{TM} or $H_{y_E} = +\dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_x}{\partial z}$	$H_y^{\mathrm{TE}} = H_{yH} = -\dfrac{\gamma}{h^2}\dfrac{\partial H_x}{\partial y} = 0$
H_z^{TM} or $H_{z_E} = +\dfrac{j\omega\varepsilon}{h^2}\dfrac{\partial E_x}{\partial y} = 0$	$H_z^{\mathrm{TE}} = H_{zH} = -\dfrac{\gamma}{h^2}\dfrac{\partial H_x}{\partial z}$
$E_y^{\mathrm{TM}} = E_{yE} = -\dfrac{\gamma}{h^2}\dfrac{\partial E_x}{\partial y} = 0$	$E_y^{\mathrm{TE}} = E_{yH} = -\dfrac{j\omega\mu}{h^2}\dfrac{\partial H_x}{\partial z}$
$E_z^{\mathrm{TM}} = E_{zE} = -\dfrac{\gamma}{h^2}\dfrac{\partial E_x}{\partial z}$	$E_z^{\mathrm{TE}} = E_{zH} = +\dfrac{j\omega\mu}{h^2}\dfrac{\partial H_x}{\partial y} = 0$





For our case, the derived fields are

$$E_x^{\mathrm{TM}} = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} = \sum_{n=0}^{\infty} E_{nx}^{\mathrm{TM}_x}, \qquad \gamma = jk_x = j\beta_g$$

$$E_y^{\mathrm{TE}} = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} = \sum_{n=0}^{\infty} E_{ny}^{\mathrm{TE}_x}$$

From the relation

$$E_y^{\mathrm{TE}} = -\frac{j\omega\mu}{h^2}\frac{\partial H_x}{\partial z}$$

it is clear that (note that $E_y^{\mathrm{TM}} = 0$ since there is no y dependence)

$$H_x^{\mathrm{TE}} = \sum B_n' \cos\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} = \sum_{n=0}^{\infty} H_{nx}^{\mathrm{TE}_x}$$

Here, $H_{nx}^{\mathrm{TE}_x}$ denotes the H_x components of the nth mode associated with the TE_x mode fields (only TE fields have a non-zero H_x component).

As an example for the TM modes we have

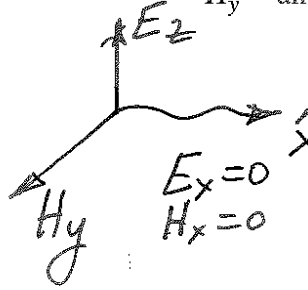
$$\begin{aligned}
H_{ny}^{\mathrm{TM}} &= A_n \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{z_1}\right) \cos\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} = A_{H_{ny}}^{\mathrm{TM}} \cos\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} \\
E_{nz}^{\mathrm{TM}} &= -A_n \frac{\gamma}{h^2}\left(\frac{n\pi}{z_1}\right) \cos\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} = A_{E_{nz}}^{\mathrm{TM}} \cos\left(\frac{n\pi}{z_1} z\right) e^{-\gamma x} \\
E_{nx}^{\mathrm{TM}} &= A_{E_{nx}}^{\mathrm{TM}} \sin\left(\frac{n\pi}{z_1} z\right)
\end{aligned}$$

Note that for $n = 0$ (zeroth mode), then

$$\begin{aligned}
H_{0y}^{\mathrm{TM}} &= A_{H_{0y}}^{\mathrm{TM}} e^{-\gamma x} = A_{H_{0y}}^{\mathrm{TM}} e^{-j\beta_g x} \\
E_{0z}^{\mathrm{TM}} &= A_{E_{0z}}^{\mathrm{TM}} e^{-\gamma x} = A_{E_{0z}}^{\mathrm{TM}} e^{-j\beta_g x} \\
E_{0x}^{\mathrm{TM}} &= 0
\end{aligned}$$

which is the usual *plane wave* form since only transverse **E** and **H** fields exist. We also remark that for $n = 0$ the $\mathrm{TE}_{n=0}^{x}$ mode vanishes. That is, for $n = 0$ the non-zero field components are simply

$$H_y^{\mathrm{TM}} \text{ and } E_z^{\mathrm{TM}}$$



(TEM mode)
plane-wave-like mode.

and these are completely transverse to the propagation x. For this reason this field is referred to as *transverse electric and magnetic field* (TEM), and the components satisfy the usual plane wave relations. The TEM mode is the field existing in 2-wire and coax transmission lines, and will be seen later as not having a cutoff frequency. That is, it propagates unattenuated for all frequencies.

Summary of parallel plate waveguide modes

TM modes ($H_x = 0$)

$$E_{gx} = A_n \sin \frac{n\pi}{z_1} z e^{-\gamma x}$$

$$E_{gz} = -\frac{\gamma}{h^2}\frac{\partial E_x}{\partial z} = -A_n \frac{\gamma}{h^2}\left(\frac{n\pi}{z_1}\right)\cos\left(\frac{n\pi}{z_1}z\right) e^{-\gamma x}$$

$$\mathbf{H}_{\mathrm{TM}} = \frac{1}{Z_{\mathrm{TM}}}\hat{x} \times \mathbf{E}_{\mathrm{TM}}$$

$$Z_{\mathrm{TM}} = \frac{\gamma}{j\omega\varepsilon} = Z_0\sqrt{1-(\lambda_0/\lambda_{0c})^2} \qquad Z_0 = \sqrt{\mu/\varepsilon}$$

$$\gamma^2 + \beta_0^2 = \gamma^2 + \omega_0^2\mu\varepsilon = h^2$$

$$\gamma = j\beta_g$$

$$\beta_g = \sqrt{\omega_0^2\mu\varepsilon - (n\pi/z_1)^2} = \sqrt{\beta_0^2 - (n\pi/z_1)^2}, \qquad n = 0, 1, 2, 3, 4, \ldots$$

$$\upsilon_0 = \frac{1}{\sqrt{\mu\varepsilon}} = \text{velocity in unbounded medium}$$

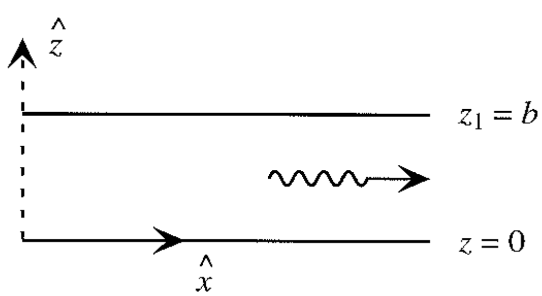
- For $n = 0$ we recover the TEM mode since $E_{gx} = 0$. In that case set $E_{gz} = A_0 e^{-\gamma x}$.

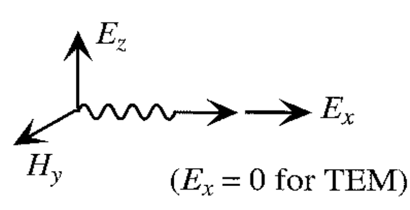
$$f_{0c} = \text{cutoff frequency of } n\text{th mode} = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2z_1\sqrt{\mu\varepsilon}}; \quad n = 0, 1, 2$$

$$\lambda_{0c} = \text{wavelength at cutoff frequency} = \frac{\upsilon_0}{f_{0c}} = \frac{1}{\sqrt{\mu\varepsilon}\, f_{0c}} = \frac{2z_1}{n}$$

$$\lambda_g = \text{wavelength in the guide} = \frac{2\pi}{\beta_g} = \frac{2\pi}{\beta_0\sqrt{1-(\omega_{0c}/\omega_0)^2}} = \frac{\lambda_0}{\sqrt{1-(\lambda_0/\lambda_{0c})^2}}$$

$$\lambda_0 = \text{wavelength of source} = \frac{\upsilon_0}{f_0} = \frac{2\pi}{\beta_0} = \frac{1}{f_0\sqrt{\mu\varepsilon}}$$

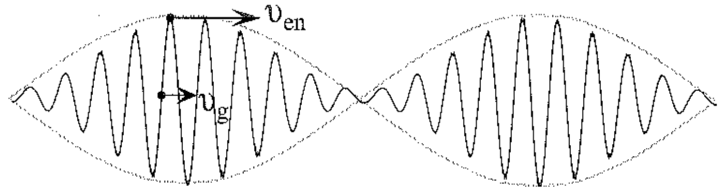




$$\upsilon_g = \text{phase velocity of the wave along } x = \frac{\omega_0}{\beta_g} = \frac{1}{\sqrt{\mu\varepsilon}\cos\theta} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\lambda_0/\lambda_{0c})^2}}$$

$$\upsilon_{en} = \text{velocity of envelope/energy or group velocity}$$

$$= \upsilon_0\cos\theta = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\left(\frac{\omega_{0c}}{\omega_0}\right)^2} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\left(\frac{\lambda_0}{\omega_{0c}}\right)^2}$$



- TM$_1$ ($n = 1$) exists for

$$f_0 > f_{0c} = \frac{1}{2z_1\sqrt{\mu\varepsilon}} \qquad \text{or} \qquad \lambda_{0c} < \frac{2z_1}{1} \quad \Rightarrow \quad z_1 > \frac{\lambda_{0c}}{2}$$

Note: υ_g can be greater than the speed of light, but υ_{en} is always less than the speed of light.

Note:

$$\upsilon_{en}\upsilon_g = \upsilon_0^2$$
$$\frac{1}{\lambda_g^2} + \frac{1}{\lambda_{0c}^2} = \frac{1}{\lambda_0^2}$$

TE$_x$ modes

$$H_{gx} = A_n \cos\left(\frac{n\pi z}{z_1}\right) e^{-\gamma x}$$
$$H_{gz} = A_n \left(\frac{\gamma}{h^2}\right)\left(\frac{n\pi}{z_1}\right) \sin\left(\frac{n\pi}{z_1}z\right) e^{-\gamma x}$$
$$E_{gy} = -\frac{j\omega\mu}{h^2}\frac{\partial H_x}{\partial z} \qquad (\text{Also, } E_{gz} = 0)$$
$$= -\frac{j\omega\mu}{h^2}\left(\frac{n\pi}{z_1}\right) \sin\left(\frac{n\pi z}{z_1}\right) e^{-\gamma x}$$
$$\mathbf{E}_{\mathrm{TE}} = -(\hat{x} \times \mathbf{H}_{\mathrm{TE}})Z_{\mathrm{TE}}$$
$$Z_{\mathrm{TE}} = \frac{j\omega\mu}{\gamma} = \frac{Z_0}{\sqrt{1-(\lambda_0/\lambda_{0c})^2}}$$

Note: TE$_1$ exists for $\lambda_{0c} > z_1/2$.

All parameters, such as γ, β_g, h, λ_{0c}, λ_g, v_g, v_{en} etc. are identical to those computed for the TM$_x$ modes. However, for $n = 0$, both E_{gy} and E_{gz} vanish because of the $\sin(n\pi z/z_1)$ term in their expression. Thus, $\mathbf{E} = 0$ for $n = 0$, implying that the lowest order TE mode to propagate occurs for $n \geq 1$. Lowest cutoff frequency is that of mode 1, viz.

$$f_{0c} = \left.\frac{n}{2z_1\sqrt{\mu\varepsilon}}\right|_{n=1} = \frac{1}{2z_1\sqrt{\mu\varepsilon}}$$

Power: (TM modes)

$$\begin{aligned}
\mathbf{S}_{\text{av}}\text{ (Power Density)} &= \frac{1}{2}\text{Re}\{\mathbf{E}\times\mathbf{H}^*\} \\
&= \frac{1}{2}\text{Re}\left\{\hat{z}E_{gz}\times(\hat{x}H_{gx}+\hat{x}H_{gx})\right\} \\
&= \frac{1}{2}\text{Re}\left\{\hat{y}E_{gz}H_{gz}^* - \hat{x}E_{gz}H_{gz}^*\right\} \\
\mathbf{S}_{\text{av}}\cdot\hat{x} = S_{x\text{av}} &= -\frac{1}{2}\text{Re}\left\{E_{gz}H_{gy}^*\right\} \\
&= \frac{1}{2}|A_n|^2\frac{\omega_0\varepsilon\beta_g}{h^2}\cos^2\left(\frac{n\pi z}{z_1}\right) \qquad \text{for } n\neq 0
\end{aligned}$$

For $n=0$,

$$S_{x\text{av}}\big|_{\text{TEM wave}} = \frac{1}{2}\frac{|A_0|^2}{Z_0}$$

$$\mathbf{P}_{\text{av}} = \hat{x}S_{x\text{av}}\text{ (Area)} = \hat{x}z_1y_1S_{x\text{av}}; \qquad y_1 = \text{width of parallel plates}$$

Example

Consider a parallel plate waveguide whose plates are separated by 3 cm (i.e., $z_1 = 3$ cm). The guide is excited by a signal whose carrier frequency is $f_0 = 10$ GHz.

(a) Find how many modes (TM and TE) will propagate.

(b) For each of the modes give the parameters β_g, λ_g, λ_{0c}, υ_g, υ_{en}, Z_{TE} and Z_{TM}.

Solution:

(a) Find cutoff mode frequencies.
Those modes with $f_{0c} < f_0$ will propagate

$$f_{0c} = \frac{n}{2z_1\sqrt{\mu\varepsilon}} = n\left(\frac{c}{2z_1}\right) = n\left(\frac{0.3}{2z_1}\right)\times 10^9 \text{ Hz}$$

$n=0 \quad f_{0c} = (f_{0c})_0 = 0 \quad$ (TEM mode, has no cutoff frequency)

$n=1 \quad f_{0c} = (f_{0c})_1 = 1\left(\frac{0.3}{2(0.03)}\right)\times 10^9 = 5\text{ GHz} < 10\text{ GHz} \quad (\lambda_{0c_1} = 0.06\text{ m})$

$\Rightarrow$ both TM_1 and TE_1 mode will propagate

$n=2 \quad f_{0c} = (f_{0c})_2 = 2\left(\frac{0.3}{2(0.03)}\right)\times 10^9 = 10\text{ GHz} = f_0 \quad (\lambda_{0c_2} = \lambda_0 = 0.03\text{ m})$

TM_2 and TE_2 can exist in the guide (near the source) but do not propagate

(b) Table of mode parameters

compute second ↓				compute first ↓				
$\cos\theta = \sqrt{1-\left(\frac{\lambda_0}{\lambda_{0c}}\right)^2}$	n	β_g (cm^{-1})	λ_g (cm)	λ_{0c} (cm)	υ_g	υ_g	Z_{TE}	Z_{TM}
1	0	2.094	3	∞	υ_0	υ_0	—	Z_0
0.866	1	1.813	3.465	6	$1.155\upsilon_0$	$0.866\upsilon_0$	$1.155Z_0$	$0.866Z_0$
0	2	0	∞	3 cm	∞	0	∞	0

$$\upsilon_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c$$

$$Z_0 = \sqrt{\mu_0/\varepsilon_0} \simeq 377\ \Omega$$

$$\beta_g = \beta_0\sqrt{1-(\lambda_0/\lambda_{0c})^2}, \quad \beta_0 = \frac{2\pi}{\lambda_0}$$

$$\lambda_g = \frac{2\pi}{\beta_g}$$

$$\lambda_{0c_n} = \frac{2z_1}{n}$$

$$\upsilon_g = \frac{\upsilon_0}{\sqrt{1-(\lambda_0/\lambda_{0c})^2}} = \upsilon_p \text{ in Balanis}$$

$$\upsilon_{en} = \upsilon_0\sqrt{1-(\lambda_0/\lambda_{0c})^2} = \upsilon_g \text{ in Balanis}$$

$$\cos\theta = \sqrt{1-(\lambda_0/\lambda_{0c})^2}$$