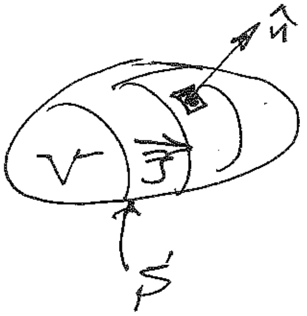
Green's 2nd Identity & Field Representations

Green's 2nd Identity is a mathematical expression stating the following



$$\iiint_V \left[\bar{Q}\cdot\nabla\times(\nabla\times\bar{P}) - \bar{P}\cdot(\nabla\times\nabla\times\bar{Q})\right]dv =$$

$$\oiint_S \left[\bar{P}\times(\nabla\times\bar{Q}) - Q\times(\nabla\times\bar{P})\right]\cdot\hat{n}\,ds \quad (1)$$

Where $\bar{Q}$ and $\bar{P}$ are some vectors defined within V.

We like to prove and use this identity to derive the known field expression (2)

$$\bar{E} = -j\omega\bar{A} + \frac{1}{j\omega\mu\varepsilon}\nabla(\nabla\cdot\bar{A}) - \nabla\times\bar{F}$$

$$= \oiint_S \left\{ -j\omega\mu\,\bar{J}_{eq}(\bar{r}')G(\bar{r},\bar{r}') + \frac{1}{j\omega\varepsilon}\nabla\left(\nabla\cdot\bar{J}_{eq}(\bar{r}')G(\bar{r},\bar{r}')\right) - \bar{M}_{eq}(\bar{r}')\times\nabla G(\bar{r},\bar{r}')\right\}dv'$$

$$\boxed{\nabla\times(\bar{M}(\bar{r}')G) = G\underbrace{\nabla\times\bar{M}}_{0\text{ because }\bar{M}\text{ is a function of }\bar{r}'} - \bar{M}\times\nabla G}$$

Where $\bar{J}_{eq} = \hat{n}\times\bar{H}$ and $\bar{M}_{eq} = \bar{E}\times\hat{n}$. In Essence, this result validates the equivalence principle/theorem.

Identity Proof

We begin by setting

$\bar{P} = \bar{E}(\bar{r})$, $\bar{Q} = \hat{a}\,G(\bar{r},\bar{r}')$, and noting the vector wave equation

$$\nabla\times\nabla\times\bar{E} - \beta^2\bar{E} = -j\omega\bar{J} \quad \text{source in V}$$

Also, $\nabla \times \bar{E} = -j\omega\mu\bar{H}$. Further since $\bar{Q} = \hat{a}\, G(\bar{r}, \bar{r}')$, (2)
it follows that

$$\nabla \times \bar{Q} = \nabla G \times \hat{a}$$

$$\nabla \times \nabla \times \bar{Q} = \hat{a}\beta^2 G + \nabla(\hat{a} \cdot \nabla G).$$

To use the above expression in Green's 2nd ~~Identity~~, we will rewrite it using the divergence theorem. Specifically, we note that

$$\nabla \cdot [\bar{P} \times \nabla \times \bar{Q}] = (\nabla \times \bar{P}) \cdot (\nabla \times \bar{Q}) - \bar{P} \cdot [\nabla \times (\nabla \times \bar{Q})]$$

$$\nabla \cdot [\bar{Q} \times (\nabla \times \bar{P})] = (\nabla \times \bar{Q}) \cdot (\nabla \times \bar{P}) - \bar{Q} \cdot [\nabla \times (\nabla \times \bar{P})]$$

and by subtracting these, we get

$$\nabla \cdot [\bar{P} \times (\nabla \times \bar{Q})] - \nabla \cdot [\bar{Q} \times (\nabla \times \bar{P})] = \bar{Q} \cdot \nabla \times \nabla \times \bar{P} - \bar{P} \cdot \nabla \times \nabla \times \bar{Q}.$$

Therefore, the LHS of (1) becomes

$$\iiint_V \{\bar{Q} \cdot (\nabla \times (\nabla \times \bar{P})) - \bar{P} \cdot (\nabla \times (\nabla \times \bar{Q}))\}\, dv = \iiint \nabla \cdot [\bar{P} \times (\nabla \times \bar{Q}) - \bar{Q} \times (\nabla \times \bar{P})]\, dv.$$

$$= \oiint_S \hat{n} \cdot \{\bar{P} \times (\nabla \times \bar{Q}) - \bar{Q} \times (\nabla \times \bar{P})\}\, ds \qquad (3)$$

where we used the Divergence theorem in the last step. Since the latter is identical to the RHS of (1), the result (3) proves the identity (1).

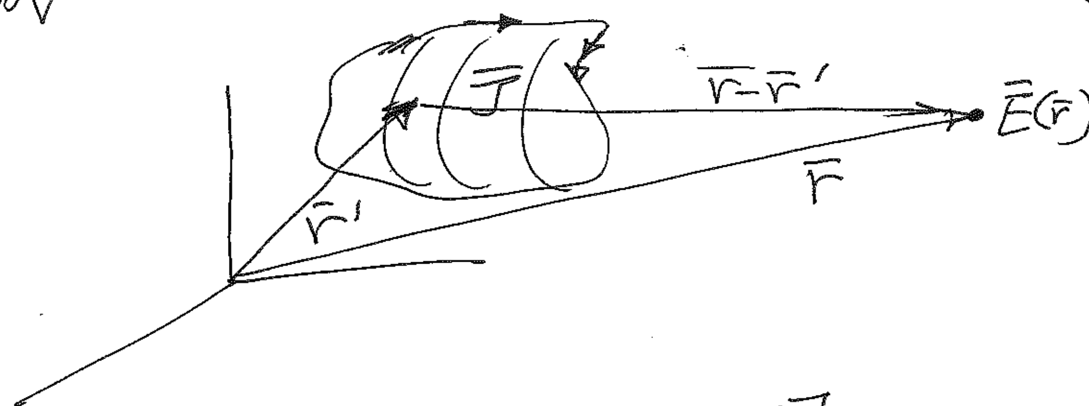
Green's 2nd Identity & Equivalence Principle

To specialize Green's identity (1) to Electromagnetic fields, we set

$$\bar{P} = \bar{E} \quad \text{and} \quad \bar{Q} = \hat{a} G$$

Then the LHS of (1) becomes

$$LHS = \iiint_V \left\{ (\hat{a} G) \cdot (\beta^2 \bar{E} - j\omega\mu \bar{J}) - \bar{E} \cdot \left(\hat{a} G \beta^2 + \nabla(\hat{a}\cdot\nabla G) \right) \right\} dv$$



$$LHS = \iiint_V \left[-j\omega\mu G (\hat{a}\cdot\bar{J}) - \bar{E}\cdot\nabla(\hat{a}\cdot\nabla G) \right] dv$$

$$= \underbrace{\iiint_V \left[-j\omega\mu G (\hat{a}\cdot\bar{J}) + \psi \nabla\cdot\bar{E} \right.}_{LHS^+} \left. - \nabla\cdot\psi\bar{E} \right] dv$$

where $\psi = \hat{a}\cdot\nabla G$. Also, we note that $\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon} = \frac{\nabla\cdot\bar{J}}{j\omega\varepsilon}$.

$$LHS^+ = \iiint_V \left[-j\omega\mu (\hat{a}\cdot\bar{J}) G \right] dv + \iiint_V \frac{\nabla\cdot\bar{J}}{j\omega\varepsilon} (\hat{a}\cdot\nabla G)\, dv$$

(since $\hat{A}\cdot\bar{J}=0$, $\underbrace{\nabla\cdot\bar{J}\,\nabla G}$)

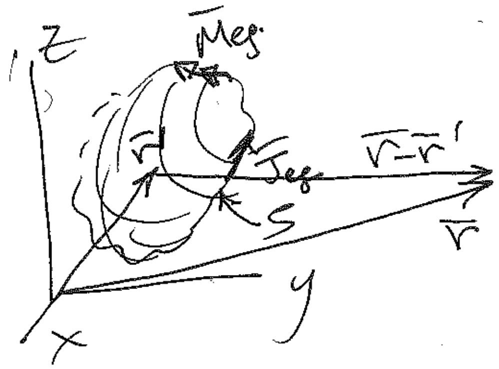
But, $\bar{E} = -j\omega\bar{A} + \frac{1}{j\omega\varepsilon\mu}\nabla\nabla\cdot\bar{A} = -j\omega\mu\iiint\bar{J}G + \frac{1}{j\omega\mu}\iiint\nabla\nabla\cdot\bar{J}G$

That is, the LHS of (1) is simply the radiated field by the sources $\bar{J}$ enclosed within V. With this understanding, we proceed to rewrite (1) as

$$\underbrace{\hat{a}\cdot\bar{E}(\;)}_{LHS^+} = \oiint_{S'} \left\{\bar{P}\times(\nabla\times\bar{Q}) - Q\times\nabla\times\bar{P}\right\}\cdot\hat{n}\,ds - \iiint \nabla\cdot\left[\bar{E}\,\hat{a}\cdot\nabla G\right]$$

$$\bar{P}=\bar{E},\quad Q=\hat{a}G$$

$$= \oiint_{S'} \left\{\bar{E}\times\underbrace{\nabla\times(\hat{a}G)}_{\nabla G\times\hat{a}} - (\hat{a}G)\times\underbrace{(\nabla\times\bar{E})}_{-j\omega\mu H}\right\}\cdot\hat{n}\,ds$$



$$\longrightarrow -\oiint_{S'} \underbrace{\bar{E}(\hat{a}\cdot\nabla G)\hat{n}\,ds}_{\hat{a}\cdot(\nabla G\,\bar{E}\cdot\hat{n})}$$

Therefore,

$$\hat{a}\cdot\bar{E} = \oiint_S \left\{\underbrace{\bar{E}\times(\nabla G\times\hat{a})\cdot\hat{n}}_{(\hat{n}\times\bar{E})\cdot(\nabla G\times\hat{a}) \,=\, \hat{a}\cdot[(\hat{n}\times\bar{E})\times\nabla G]} - \underbrace{\hat{a}G\times(-j\omega\mu H\cdot\hat{n})}_{-\hat{a}\cdot(j\omega\mu G\,\hat{n}\times H)} - \hat{a}\cdot\nabla G(\bar{E}\cdot\hat{n})\right\}ds$$

or

$$\hat{a}\cdot\bar{E}(\bar{r}) = \oiint_{S'} \hat{a}\cdot\left\{\underbrace{(\hat{n}\times\bar{E})}_{-M_{eq}}\times\nabla G - j\omega\mu\underbrace{(\hat{n}\times H)}_{J_{eq}}G + \underbrace{(\hat{n}\cdot\bar{E})}_{\frac{\rho_s}{\varepsilon}=\frac{\nabla_s\cdot\bar{J}}{j\omega\varepsilon}}\nabla G\right\}ds'$$

Upon cancelling out the $\hat{a}$ vector, we get

$$\bar{E}(\bar{r}) = \oiint_S\left\{-\bar{M}_{eq}\times\nabla G - j\omega\mu\,\bar{J}_{eq}\,G + \frac{\nabla_s\cdot\bar{J}}{j\omega\varepsilon}\nabla G\right\}ds'$$

$\bar{r}$ at any point in space

same Expression as obtained via equivalence theorem + vector potential