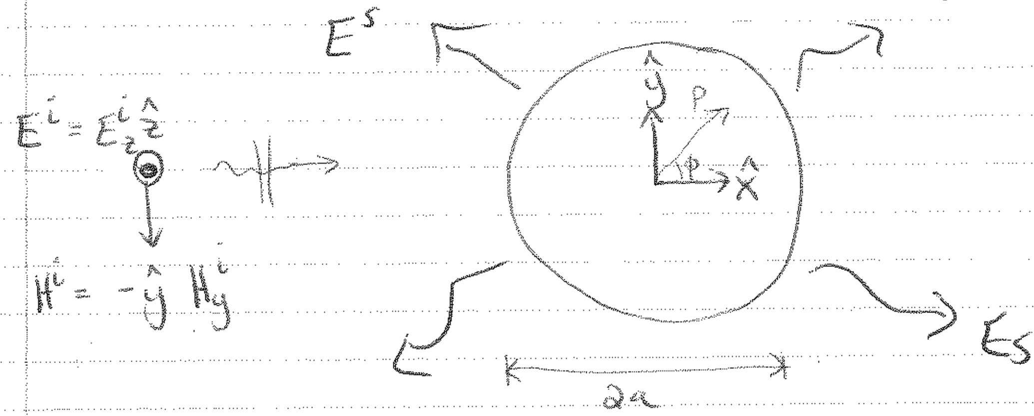
Scattering From an infinite PEC Cylinder (Balanis 11.5 pp. 607-617)

In order to characterize the interaction of a plane wave with a cylindrical object, we can represent the plane wave as a sum of cylindrical waves.

Note that this is possible because both plane waves & cylindrical waves form a "complete set" of modes. Thus <u>any</u> fields may be written as either a series of plane waves or as a series of cylindrical waves.

For TM_z incidence: ($\vec{E}$ field along axis of cylinder)



$$\vec{E}^i = \hat{z} E_z^i = \hat{z} E_0 e^{-j\beta x} = \hat{z} E_0 e^{-j\beta\rho\cos\phi}$$

$$\boxed{\vec{E}^i = \hat{z} E_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho) e^{jn\phi}} \quad \text{(Balanis 11-81a)}$$

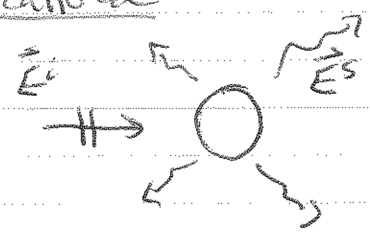
Note that $J_n(\beta\rho)$ is small for $\beta\rho \gg n$

Thus # of required modes is proportional to electrical size (radius) of cylinder

For very small cylinders ($\beta a \to 0$), we only need $n=0$ term

We can also represent the <u>total</u> field as the sum of the <u>incident</u> field plus the <u>scattered</u> field.

$$\vec{E}^t(\rho,\phi) = \vec{E}^i(\rho,\phi) + \vec{E}^s(\rho,\phi)$$



The scattered field can also be written as the sum of cylindrical waves:

$$\vec{E}_s = \hat{z}\, E_0 \sum_{n=-\infty}^{\infty} c_n \underbrace{H_n^{(2)}(\beta\rho)}_{\rightarrow\ \boxed{\text{scattered fields are outward traveling}}} e^{jn\phi}$$

where c_n is the coefficent of each Hankel fn
(note: our c_n differs from that in Balanis)

Hankel functions are defined by the Bessel fns of 1st & 2nd kind & are <u>traveling</u> waves

$$H_n^{(1)}(\beta\rho) = J_n(\beta\rho) + j\,Y_n(\beta\rho) \qquad \left(\begin{array}{l}\underline{\text{inward}}\ (-\beta\rho)\\ \text{traveling wave}\end{array}\right)$$

$$H_n^{(2)}(\beta\rho) = J_n(\beta\rho) - j\,Y_n(\beta\rho) \qquad \left(\begin{array}{l}\underline{\text{outward}}\ (+\beta\rho)\\ \text{traveling wave}\end{array}\right)$$

This is exactly analogous to:

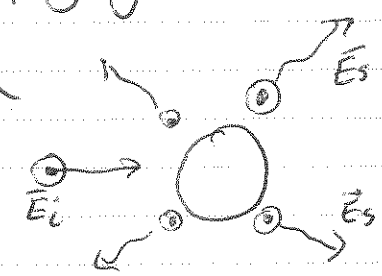
$$e^{+j\beta x} = \cos(\beta x) + j\sin(\beta x) \qquad \left(\begin{array}{l}\text{left } (-x) \text{ traveling}\\ \text{wave}\end{array}\right)$$

$$e^{-j\beta x} = \cos(\beta x) - j\sin(\beta x) \qquad \left(\begin{array}{l}\text{right } (+x) \text{ traveling}\\ \text{wave}\end{array}\right)$$

Also note that the Bessel & Hankel fns. are only valid solutions of M.E. in source free regions when multiplied by appropriate angular variation, ie. $e^{jn\phi}$ for 2d problems.

$$\Rightarrow \quad J_n(\beta\rho)e^{jn\phi}, \quad Y_n(\beta\rho)e^{jn\phi}$$
$$H^{(1)}_n(\beta\rho)e^{jn\phi}, \quad H^{(2)}_n(\beta\rho)e^{jn\phi}$$

We solve for scattered fields by applying B.C.s on PEC cylinder, $E^t_{tan}=0\big|_{\rho=a}$



$$\Rightarrow \boxed{E^i_z(\rho=a) + E^s_z(\rho=a) = 0}$$

$$\Rightarrow E_0\sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta a)e^{jn\phi} + E_0\sum_{n=-\infty}^{\infty} c_n H^{(2)}_n(\beta a)e^{jn\phi} = 0$$

Since $e^{jn\phi}$ are orthogonal in $\phi \in [0, 2\pi)$ each n term must vanish independantly

$$\Rightarrow j^{-n} J_n(\beta a) + c_n H^{(2)}_n(\beta a) = 0$$

$$\Rightarrow \boxed{c_n = -j^{-n}\frac{J_n(\beta a)}{H^{(2)}_n(\beta a)}}$$

We can therefore write the scattered E field:

$$\boxed{\vec{E}_s = -E_0 \hat{z} \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta \rho) e^{jn\phi}}$$

(TM$_z$ incidence)

What do the scattered fields look like far away from the cylinder? (Far Fields)

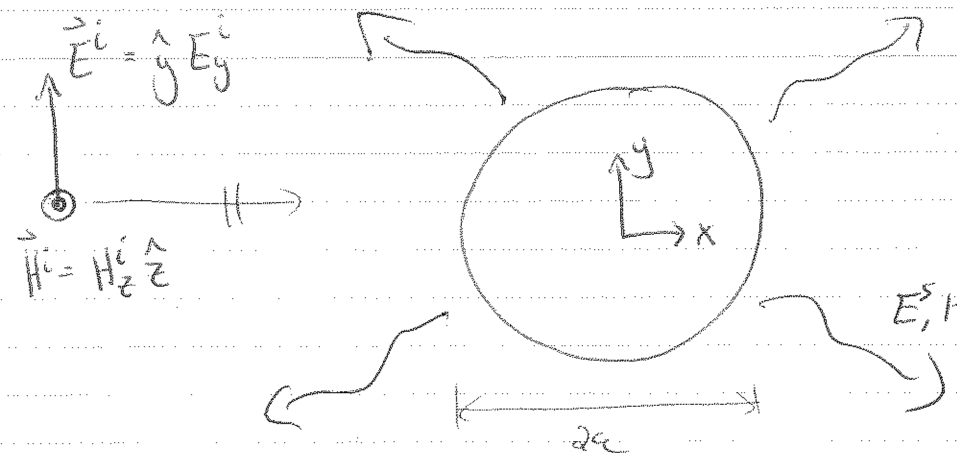
The Hankel functions have an asymtotic approximation for $\beta\rho \to \infty$

$$H_n^{(2)}(\beta\rho \to \text{large}) \approx \sqrt{\frac{2j}{\pi \beta \rho}}\, j^n e^{-j\beta\rho}$$

Note, this is an exponential traveling wave outwards ($+\rho$) decaying as $1/\sqrt{\rho}$, which is a cylindrical wave

$$\Rightarrow \boxed{\vec{E}_s^{FF}(\beta\rho \to \text{large}) \simeq -\hat{z} E_0 \sqrt{\frac{2j}{\pi\beta}} \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \sum_{n=-\infty}^{\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi}}$$

Scattering from a PEC Cylinder, TE_z incidence
($\vec{H}$ field along axis of cylinder)



Expand incident plane wave as cylindrical series:

$$\vec{H}^i = \hat{z} H_0 e^{-j\beta x} = \hat{z} H_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho) e^{jn\phi}$$

we must calculate $\vec{E}^i$ since the PEC B.C. require us to evaluate E Field

$$\vec{E}^i = \frac{1}{j\omega\varepsilon} \nabla \times \vec{H}^i = \frac{1}{j\omega\varepsilon}\left(\hat{\rho} \frac{1}{\rho} \frac{\partial H_z^i}{\partial \phi} - \hat{\phi} \frac{\partial H_z^i}{\partial \rho}\right) \quad \left(\begin{array}{c}\text{Balanis} \\ \text{p. 956}\end{array}\right)$$

$$E_\rho^i = \frac{H_0}{j\omega\varepsilon}\left(\frac{1}{\rho} \sum_{n=-\infty}^{\infty} (jn)\, j^{-n} J_n(\beta\rho) e^{jn\phi}\right)$$

$$E_\phi^i = \frac{-H_0}{j\omega\varepsilon}\left(\sum_{n=-\infty}^{\infty} j^{-n} \underbrace{\frac{\partial J_n(\beta\rho)}{\partial \beta\rho}}_{\substack{\text{Write this as } J_n'(\beta\rho)\text{, note} \\ \text{there are simple recursive def.} \\ \text{of Bessel Fn. derivatives}}} \cdot \beta\, e^{jn\phi}\right)$$

Similarly, we write the scattered field as:

$$\boxed{\begin{aligned}\vec{H}^s &= \hat{z} H_0 \sum_{n=-\infty}^{\infty} d_n H_n^{(2)}(\beta\rho) e^{jn\phi} \\ E_\rho^s &= \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z^s}{\partial\phi} = \frac{H_0}{\omega\varepsilon}\frac{1}{\rho}\sum_{n=-\infty}^{\infty} d_n\, n\, H_n^{(2)}(\beta\rho) e^{jn\phi} \\ E_\phi^s &= -\frac{\beta H_0}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} d_n H_n^{(2)\prime}(\beta\rho) e^{jn\phi}\end{aligned}}$$

Enforce B.C.: $\quad E^t_{tan}(\rho=a) = 0$

$$E_\phi^i(\rho=a) + E_\phi^s(\rho=a) = 0$$

$$-\frac{H_0\beta}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} j^{-n} J_n'(\beta a) e^{jn\phi} - \frac{H_0\beta}{j\omega\varepsilon}\sum_{n=-\infty}^{\infty} d_n H_n^{(2)\prime}(\beta a) e^{jn\phi} = 0$$

again, individual n terms must $\to 0$

$$j^{-n} J_n'(\beta a) + d_n H_n^{(2)\prime}(\beta a) = 0$$

$$\boxed{d_n = -j^{-n}\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)}}$$

$$\boxed{\vec{H}^s = -\hat{z} H_0 \sum_{n=-\infty}^{\infty} j^{-n}\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta\rho) e^{jn\phi}}$$

(TE_z incidence)