# Theorems: Poynting, Uniqueness, Superposition and Duality

## Poynting Theorem

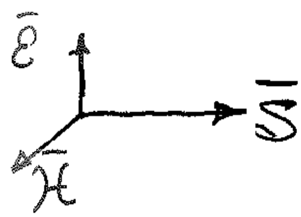
The quantity

$$\overline{\mathcal{S}} = \overline{\mathcal{E}} \times \overline{\mathcal{H}} \qquad \text{time dependent field (not harmonic)}$$

$$\langle \mathcal{S} \rangle = \frac{1}{T}\int_0^T \overline{\mathcal{E}} \times \overline{\mathcal{H}}\, dt$$

($\mathcal{S}$ is the average power over a cycle of the wave) has units of Power Density V/m $\times$ A/m $\rightarrow$ VA/m$^2$ = W/m$^2$ and represents the instantaneous power density carried by the EM field.

$\rightarrow$ Power is carried in the direction of $\overline{\mathcal{S}}$:




Direction of power
(in complex media, direction is ambiguous)

We would like to examine the constituents of this power density (what is it made of? what carries the power?). Also, we would like to specialize our analysis to time harmonic fields, viz., we consider

$$\overline{\mathcal{E}} = \mathrm{Re}\{\mathbf{E}e^{j\omega t}\} = \hat{e}E_\mathrm{m}\cos(\omega t + \alpha) = \hat{x}E_{x0}\cos(\omega t + \phi_x) + \hat{y}E_{y0}\cos(\omega t + \phi_y) + \hat{z}E_{z0}\cos(\omega t + \phi_z)$$

$$\overline{\mathcal{H}} = \mathrm{Re}\{\mathbf{H}e^{j\omega t}\} = \hat{h}H_\mathrm{m}\cos(\omega t + \beta)$$

in which $\mathbf{E} = \hat{e}E_\mathrm{m}e^{j\alpha}$ and $\mathbf{H} = \hat{h}H_\mathrm{m}e^{j\beta}$. Thus,

$$\overline{\mathcal{E}} \times \overline{\mathcal{H}} = \tfrac{1}{2}\,\hat{e} \times \hat{h}\, E_\mathrm{m}H_\mathrm{m}[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)]$$

and since we are talking about sinusoidal fields, it is appropriate to consider the average power density

$$\begin{aligned}\langle \mathcal{S} \rangle &= \tfrac{1}{2}\,\hat{e} \times \hat{h}\, E_\mathrm{m}H_\mathrm{m}\cos(\alpha - \beta) = \tfrac{1}{2}\mathrm{Re}(\mathbf{E} \times \mathbf{H}^*) \\ &= \frac{1}{T}\int_0^T \overline{\mathcal{E}} \times \overline{\mathcal{H}}\, ds \qquad = \tfrac{1}{2}\mathrm{Re}[E_\mathrm{m}e^{j\alpha}H_\mathrm{m}e^{-j\beta}]\end{aligned}$$

What is important in the above result is that the average power density of the wave can be expressed in terms of phasor field quantities. That is, the instantaneous field quantities are not needed to compute the average power.

The quantity ("$*$" indicates complex conjugation)

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{1}$$

is known as the complex Poynting vector and has units of Watts/m$^2$. It represents the complex power density of the wave and it is therefore important to understand the source and nature of this power. To do so, we refer to Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + j\omega\varepsilon\mathbf{E} \tag{2}$$

$$\nabla \times \mathbf{E} = -\mathbf{M}_\mathrm{i} - j\omega\mu\mathbf{H} \tag{3}$$

and by dotting each equation with $\mathbf{E}$ or $\mathbf{H}^*$, we have

$$\mathbf{E} \cdot \nabla \times \mathbf{H}^* = \mathbf{J}_\mathrm{i}^* \cdot \mathbf{E} - j\omega\varepsilon^*\mathbf{E}^* \cdot \mathbf{E} = \mathbf{J}_\mathrm{i}^* \cdot \mathbf{E} - j\omega\varepsilon^*|\mathbf{E}|^2 \tag{4}$$

$$\mathbf{H}^* \cdot \nabla \times \mathbf{E} = -\mathbf{M}_\mathrm{i} \cdot \mathbf{H}^* - j\omega\mu\mathbf{H} \cdot \mathbf{H}^* = -\mathbf{M}^i \cdot \mathbf{H}^* - j\omega\mu|\mathbf{H}|^2 \tag{5}$$

From the vector identity [Van Bladel, 1985]

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* \tag{6}$$

we then obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = j\omega\varepsilon^*|\mathbf{E}|^2 - j\omega\mu|\mathbf{H}|^2 - \mathbf{J}_\mathrm{i}^* \cdot \mathbf{E} - \mathbf{M}_\mathrm{i} \cdot \mathbf{H}^* \tag{7}$$

which is an identity valid everywhere in space. Integrating both sides of this over a volume $V$ containing all sources, and invoking the divergence theorem yields

$$\frac{1}{2}\oiint_{S_\mathrm{c}} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = \frac{1}{2}\iiint_V \left[j\omega\varepsilon^*|\mathbf{E}|^2 - j\omega\mu|\mathbf{H}|^2 - \mathbf{J}_\mathrm{i}^* \cdot \mathbf{E} - \mathbf{M}_\mathrm{i} \cdot \mathbf{H}^*\right] dv \tag{8}$$

which is commonly referred to as *Poynting's theorem*. Since $S_\mathrm{c}$ is closed, based on energy conservation one deduces that the right hand side of (8) must represent the sum of the power stored or radiated, i.e., escaping, out of the volume $V$. Each term of the volume integral of (8) is associated with a specific type of power but before proceeding with their identification, it is instructive that $\varepsilon^*$ be first replaced by $\varepsilon_0\varepsilon_\mathrm{r} + j\frac{\sigma}{\omega}$. Equation (8) can then be rewritten as

$$\frac{1}{2}\mathrm{Re}\oiint_{S_\mathrm{c}} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = P_\mathrm{ei} + P_\mathrm{mi} - P_\mathrm{d} \tag{9}$$

$$\frac{1}{2}\mathrm{Im}\oiint_{S_\mathrm{c}} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = 2\omega[W_\mathrm{e} - W_\mathrm{m}] - \frac{1}{2}\mathrm{Im}\iiint_V [\mathbf{J}_\mathrm{i}^* \cdot \mathbf{E} + \mathbf{M}_\mathrm{i} \cdot \mathbf{H}^*]\, dv \tag{10}$$

where

$$P_{\mathrm{ei}} = -\frac{1}{2}\iiint_V \mathrm{Re}(\mathbf{J}_{\mathrm{i}}^* \cdot \mathbf{E})\, dv = \text{ average outgoing power due to the impressed current } \mathbf{J}_{\mathrm{i}} \tag{11}$$

$$P_{\mathrm{mi}} = -\frac{1}{2}\iiint_V \mathrm{Re}(\mathbf{M}_{\mathrm{i}} \cdot \mathbf{H}^*)\, dv = \text{ average outgoing power due to the impressed current } \mathbf{M}_{\mathrm{i}} \tag{12}$$

$$P_{\mathrm{d}} = \frac{1}{2}\iiint_V \sigma|\mathbf{E}|^2\, dv = \text{average power dissipated in } V \tag{13}$$

$$W_{\mathrm{e}} = \frac{1}{4}\iiint_V \varepsilon_0\varepsilon_{\mathrm{r}}|\mathbf{E}|^2\, dv = \text{average electric energy in } V \tag{14}$$

$$W_{\mathrm{m}} = \frac{1}{4}\iiint_V \mu_0\mu_{\mathrm{r}}|\mathbf{H}|^2\, dv = \text{average magnetic energy in } V \tag{15}$$

The time-averaged power delivered to the electromagnetic field outside $V$ is clearly the sum of $P_{\mathrm{ei}}$ and $P_{\mathrm{mi}}$, whereas $P_{\mathrm{d}}$ is that dissipated in $V$ due to conductor losses. Thus, we may consider

$$P_{\mathrm{av}} = \frac{1}{2}\mathrm{Re} \oiint_{S_{\mathrm{c}}} (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} \tag{16}$$

to be the average or radiated power outside $V$ if $\sigma$ is zero in $V$. Expression (10) gives the reactive power, i.e., that which is stored within $V$ and is not allowed to escape outside the boundary of $S_{\mathrm{c}}$.

## Uniqueness Theorem

Whenever one pursues a solution to a set of equations it is important to know *a priori* whether this solution is unique and if not, what are the required conditions for a unique solution. This is important because depending on the application, different analytical or numerical methods will likely be used for the solution of Maxwell's equations. Given that Maxwell's equations (subject to the appropriate boundary conditions) yield a unique solution, one is then comforted to know that any convenient method of analysis will yield the correct solution to the problem.

The most common form of the uniqueness theorem is: *In a region $V$ completely occupied with dissipative media, a harmonic field* $(\mathbf{E},\mathbf{H})$ *is uniquely determined by the impressed currents in that region plus the tangential components of the electric or magnetic fields on the closed surface $S_{\mathrm{c}}$ bounding $V$.* This theorem may be proved by assuming for the moment that two solutions exist, denoted by $(\mathbf{E}_1,\mathbf{H}_1)$ and $(\mathbf{E}_2,\mathbf{H}_2)$. Both fields must, of course, satisfy Maxwell's equations (2) and (3) with the same impressed currents $(\mathbf{J}_{\mathrm{i}},\mathbf{M}_{\mathrm{i}})$. We have

$$\begin{aligned} \nabla \times \mathbf{H}_1 &= \mathbf{J}_{\mathrm{i}} + j\omega\varepsilon\mathbf{E}_1, & \nabla \times \mathbf{H}_2 &= \mathbf{J}_{\mathrm{i}} + j\omega\varepsilon\mathbf{E}_2 \\ \nabla \times \mathbf{E}_1 &= -\mathbf{M}_{\mathrm{i}} - j\omega\mu\mathbf{H}_1, & \nabla \times \mathbf{E}_2 &= -\mathbf{M}_{\mathrm{i}} - j\omega\mu\mathbf{H}_2 \end{aligned} \tag{17}$$

and when these are subtracted we obtain

$$\nabla \times \mathbf{H}' = j\omega\varepsilon\mathbf{E}' \tag{18}$$

$$\nabla \times \mathbf{E}' = -j\omega\mu\mathbf{H}' \tag{19}$$

where $\mathbf{E}' = \mathbf{E}_1 - \mathbf{E}_2$ and $\mathbf{H}' = \mathbf{H}_1 - \mathbf{H}_2$. To prove the theorem it is then necessary to show that $(\mathbf{E}', \mathbf{H}')$ are zero or equivalently, if no sources are enclosed by a volume $V$, the fields in that volume are zero for a given set of tangential electric and magnetic fields on $S_c$.

As a corollary to the uniqueness theorem, it can be shown that *if a harmonic field has a zero tangential electric or magnetic field on a surface enclosing a source-free region $V$ occupied by dissipative media, the field vanishes everywhere within $V$.*

The usual proof of the uniqueness theorem can be found in many electromagnetics texts (see for example Stratton).

## Superposition Theorem

The superposition theorem states that for a linear medium, the total field intensity due to two or more sources is equal to the sum of the field intensities attributed to each individual source radiating independent of the others. In particular, let us consider two electric sources $\mathbf{J}_1$ and $\mathbf{J}_2$. On the basis of the superposition theorem, to find the total field caused by the simultaneous presence of both sources, we can consider the field due to each individual source in isolation. The fields $(\mathbf{E}_1, \mathbf{H}_1)$ due to $\mathbf{J}_1$ satisfy the equations

$$\nabla \times \mathbf{H}_1 = \mathbf{J}_1 + j\omega\varepsilon\mathbf{E}_1 \tag{20}$$

$$\nabla \times \mathbf{E}_1 = -j\omega\mu\mathbf{H}_1 \tag{21}$$

and the fields corresponding to $\mathbf{J}_2$ satisfy

$$\nabla \times \mathbf{H}_2 = \mathbf{J}_2 + j\omega\varepsilon\mathbf{E}_2 \tag{22}$$

$$\nabla \times \mathbf{E}_2 = -j\omega\mu\mathbf{H}_2 \tag{23}$$

By adding these two sets of equations, it is clear that the total field due to both sources combined is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \qquad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \tag{24}$$

where $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$ are obtained by solving separately (20)–(21) and (22)–(23), respectively.

## Duality Theorem

The duality theorem relates to the interchangeability of the electric and magnetic fields, currents, charges or material properties. We observe from Maxwell's equations (2) and (3) that the first can be obtained from the second via the interchanges

$$\begin{array}{lcl} \mathbf{M} & \to & -\mathbf{J} \\ \mathbf{E} & \to & \mathbf{H} \\ \mathbf{H} & \to & -\mathbf{E} \\ \mu & \to & \varepsilon \end{array} \tag{25}$$

Similarly, (3) can be obtained from (2) via the interchanges

$$\begin{array}{rcl} \mathbf{J} & \rightarrow & \mathbf{M} \\ \mathbf{E} & \rightarrow & \mathbf{H} \\ \mathbf{H} & \rightarrow & -\mathbf{E} \\ \varepsilon & \rightarrow & \mu \end{array} \tag{26}$$

The duality theorem can reduce formulation and computational effort when one is able to invoke it for a particular application.