Holy Grail of EM Radiation

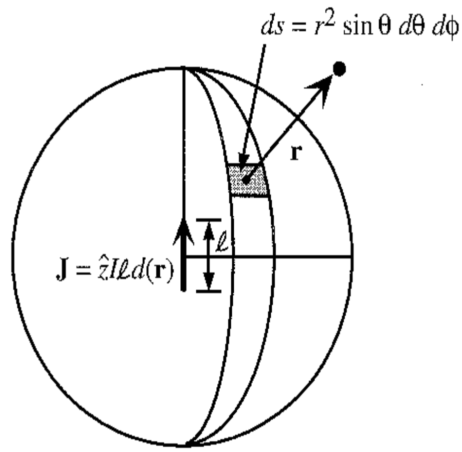
Exact:

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A}$$

$$\mathbf{E}(\mathbf{r}) = \iiint \left\{ -jkZ\,\mathbf{J}(\mathbf{r}')\,G(\mathbf{r},\mathbf{r}') - \underbrace{\frac{jZ}{k}\nabla'\cdot\mathbf{J}(\mathbf{r}')}_{\rho(\mathbf{r}')/\varepsilon}\nabla\,G(\mathbf{r},\mathbf{r}') \right\} dv'$$

$$\mathbf{H}(\mathbf{r}) = (\mathbf{J}\to\mathbf{M},\quad \mathbf{M}\to-\mathbf{J},\quad Z\to Y,\quad \varepsilon\to\mu)$$

Dipole radiation



$$E_\theta\big|_{\text{ff}} = jkZ_0(I\ell)\sin\theta\frac{e^{-jkr}}{4\pi r}$$

$$E_\phi\big|_{\text{ff}} = 0$$

$$S = \text{Rad. Power Density} = \frac{1}{2}\text{Re}(\mathbf{E}\times\mathbf{H}) = \frac{1}{2}\frac{1}{Z_0}|E_\theta|^2 = \frac{k^2(I\ell)^2}{2(4\pi)^2}\frac{1}{r^2}\underbrace{\sin^2\theta}_{\text{pattern}}$$

Total power radiated:

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi}\frac{1}{2Z_0}|E_\theta|^2\,ds = \frac{k^2(I\ell)^2}{2(4\pi)^2}\underbrace{\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,d\theta}_{8\pi/3}$$

in which the first integral applies to the surface of a sphere around the antenna. Thus $ds = \sin\theta\, d\theta\, d\phi$.

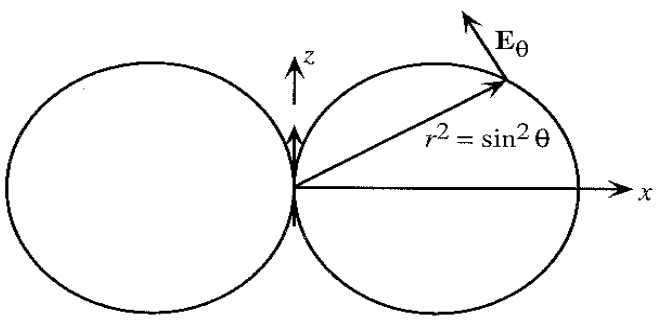
What is plotted:

$$U = r^2 S = \frac{r^2}{2Z_0}|E_\theta|^2 = \frac{k^2(I\ell)^2}{2(4\pi)^2}\sin^2\theta$$

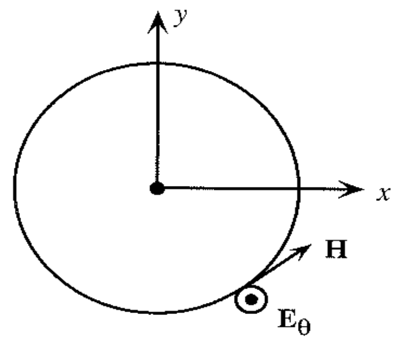
$$U_{\max} = \frac{k^2 I\ell}{2(4\pi)^2}$$

$$\text{Antenna Power Pattern} = \frac{U(\theta,\phi)}{U_{\max}} = \sin^2\theta$$

What is plotted is $U(\theta,\phi)/U_{\max}$.



E-plane pattern



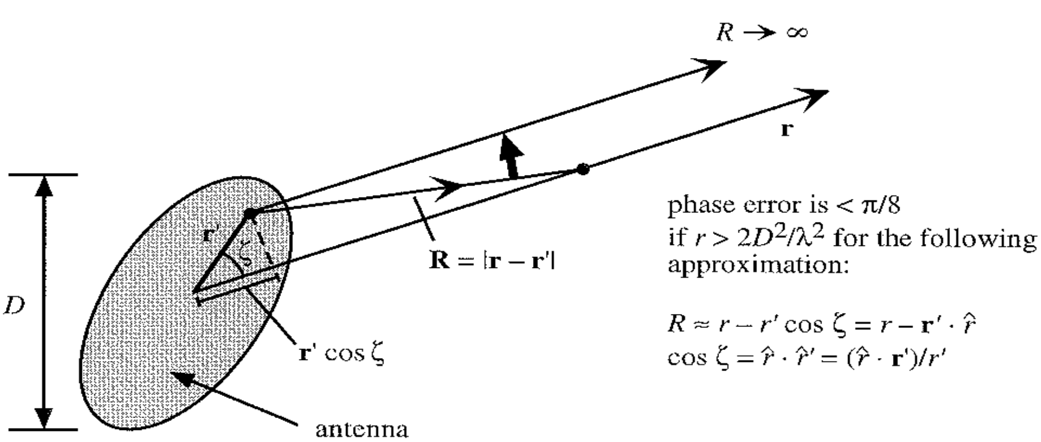
H-plane pattern

$$\text{Directivity (max directive gain)} = D_0 = \frac{4\pi\, U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{8\pi/3} = \frac{12}{8} = \frac{3}{2}$$

$$\text{or} \quad 20\log D = 3.5\ \text{dB}$$

Horns have directivity of 15–30 dB.
Reflectors have directivity of 30–80 dB.
Cellular phone patch antennas have a directivity of 4–7 dB.

Far zone fields



Using the approximation

$$R = r - r'\cos\zeta = r - \mathbf{r}'\cdot\hat{r}$$
$$\cos\zeta = \hat{r}\cdot\hat{r}' = \frac{\hat{r}\cdot\mathbf{r}'}{|\mathbf{r}'|}$$

we have

$$\frac{e^{-jkR}}{R} \approx \frac{e^{-jk(r-r'\cos\zeta)}}{r} = \frac{e^{-jkr}}{r}e^{jkr'\cos\zeta}$$

The above approximation is valid for $r > 2D^2/\lambda^2$ (i.e., phase error is less than $\pi/8$).

Next, by neglecting all terms with $1/R^2$, $1/R^3$ dependence, we have

$$\mathbf{E}_{\text{ff}} = jk\frac{e^{-jkr}}{4\pi r}\iiint\left[\hat{r}\times\mathbf{M}(\mathbf{r}') + Z_0(\hat{r}\times(\hat{r}\times\mathbf{J}(\mathbf{r}')))\right]e^{jk\mathbf{r}'\cdot\hat{r}}\,d\upsilon'$$
$$\mathbf{E}_{\text{ff}} = j\omega Z_0\,\hat{r}\times\mathbf{F} + j\omega\,\hat{r}\times(\hat{r}\times\mathbf{A}) = -Z_0\hat{r}\times\mathbf{H}_{\text{ff}}$$
$$E_{\text{ff}_\theta} = -j\omega Z_0 F_\phi - j\omega A_\theta = Z_0 H_{\text{ff}_\phi} \quad \text{(no } r \text{ component)}$$
$$E_{\text{ff}_\theta} = j\omega Z_0 F_\theta - j\omega A_\phi = -Z_0 H_{\text{ff}_\theta} \quad \text{(no } r \text{ component)}$$

$$A_{\substack{\theta\\\phi}} = \frac{\mu}{4\pi r}e^{-jkr}N_{\substack{\theta\\\phi}} \qquad N_{\substack{\theta\\\phi}} = \left\{\begin{matrix}\hat{\theta}\\\hat{\phi}\end{matrix}\right\}\cdot\iiint \mathbf{J}e^{jk\mathbf{r}'\cdot\hat{r}}\,d\upsilon'$$
$$F_{\substack{\theta\\\phi}} = \frac{\varepsilon}{4\pi r}e^{-jkr}L_{\substack{\theta\\\phi}} \qquad L_{\substack{\theta\\\phi}} = \left\{\begin{matrix}\hat{\theta}\\\hat{\phi}\end{matrix}\right\}\cdot\iiint \mathbf{M}e^{jk\mathbf{r}'\cdot\hat{r}}\,d\upsilon'$$

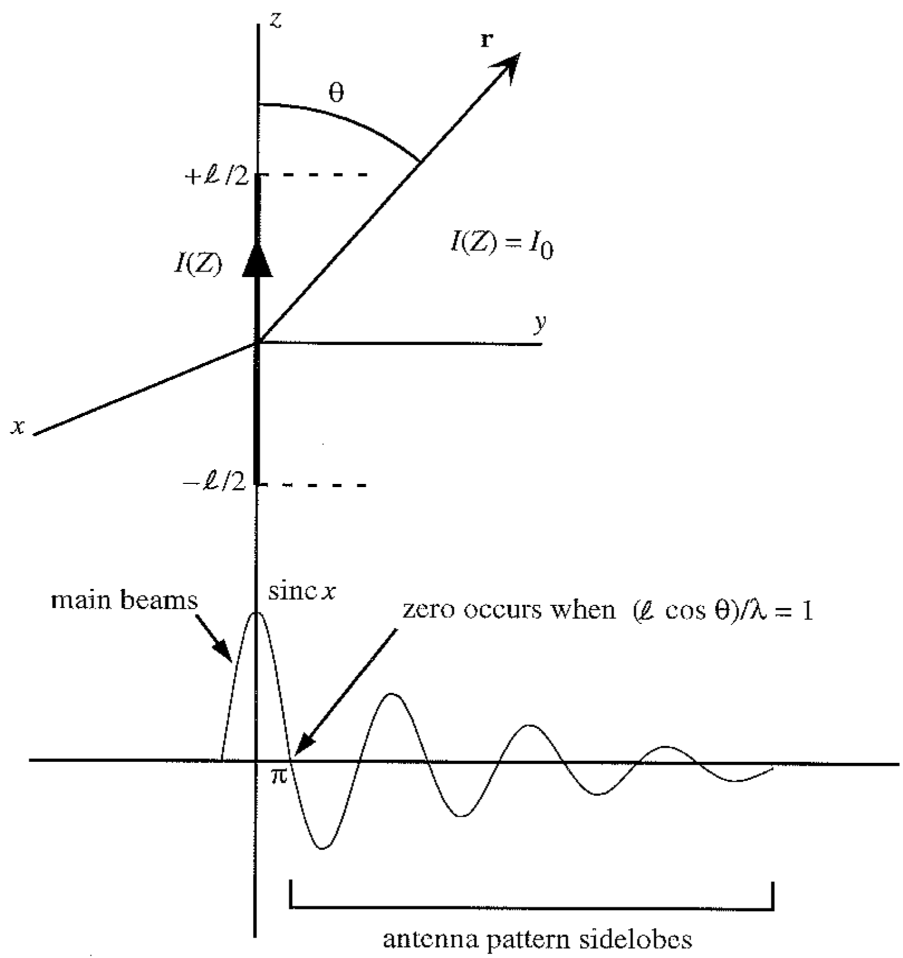
1) Far fields have no $\hat{r}$ component

2) Far field simplifications

$$\mathbf{E}^{\text{e}}_{\text{ff}} = -j\omega\mathbf{A}_{\text{transverse to }\hat{r}} \qquad H^{\text{m}}_{\text{ff}} = -j\omega F_{\text{trans.}}$$
$$H^{\text{e}} = \frac{\hat{r}\times\mathbf{E}^{\text{e}}_{\text{ff}}}{Z_0} \qquad E^{\text{m}} = -Z_0\hat{r}\times H^{\text{m}}_{\text{ff}}$$

In the above, superscripts "e" and "m" refer to electric and magnetic current densities, respectively.

Example: Radiation from a constant wire current

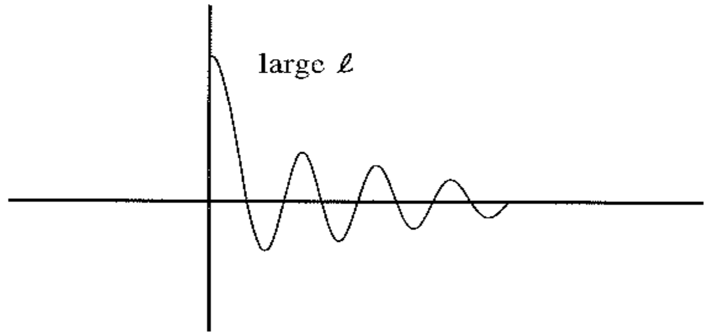


$$
\begin{aligned}
E^{\mathrm{e}}_{\mathrm{ff}} &= jkZ_0\,\frac{e^{-jkr}}{4\pi r}\,\hat{r}\times\hat{r}\times\mathbf{N}\\
\mathbf{N} &= I_0\hat{z}\int_{-\ell/2}^{\ell/2} e^{jk\hat{r}\cdot\mathbf{r}'}\,dz', \qquad \mathbf{r}' = z'\hat{z}, \quad \hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta\\
&= \hat{z}I_0\,\frac{e^{-jkz'\cos\theta}}{(-jk\cos\theta)}\Bigg]_{-\ell/2}^{\ell/2}\\
&= \hat{z}(I_0\ell)\,\frac{\sin\left(\frac{k\ell\cos\theta}{2}\right)}{\frac{k\ell}{2}\cos\theta} = \hat{z}I_0\ell\,\mathrm{sinc}\left(\frac{k\ell\cos\theta}{2}\right)
\end{aligned}
$$

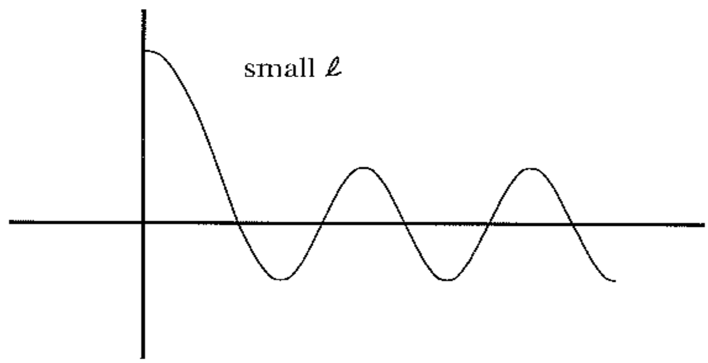
- As ℓ increases, so do the sidelobes.

$$\cos\theta_{\mathrm{null}} = \frac{1}{\ell/\lambda}$$

- As ℓ increases main beam narrows.

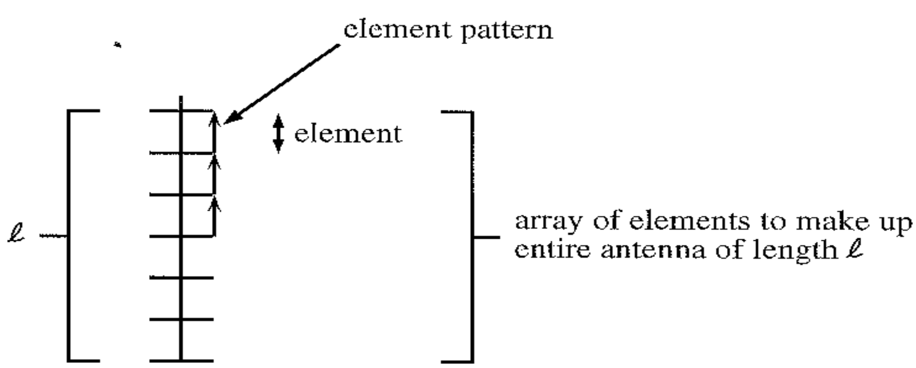


- As ℓ increases more sidelobes appear.

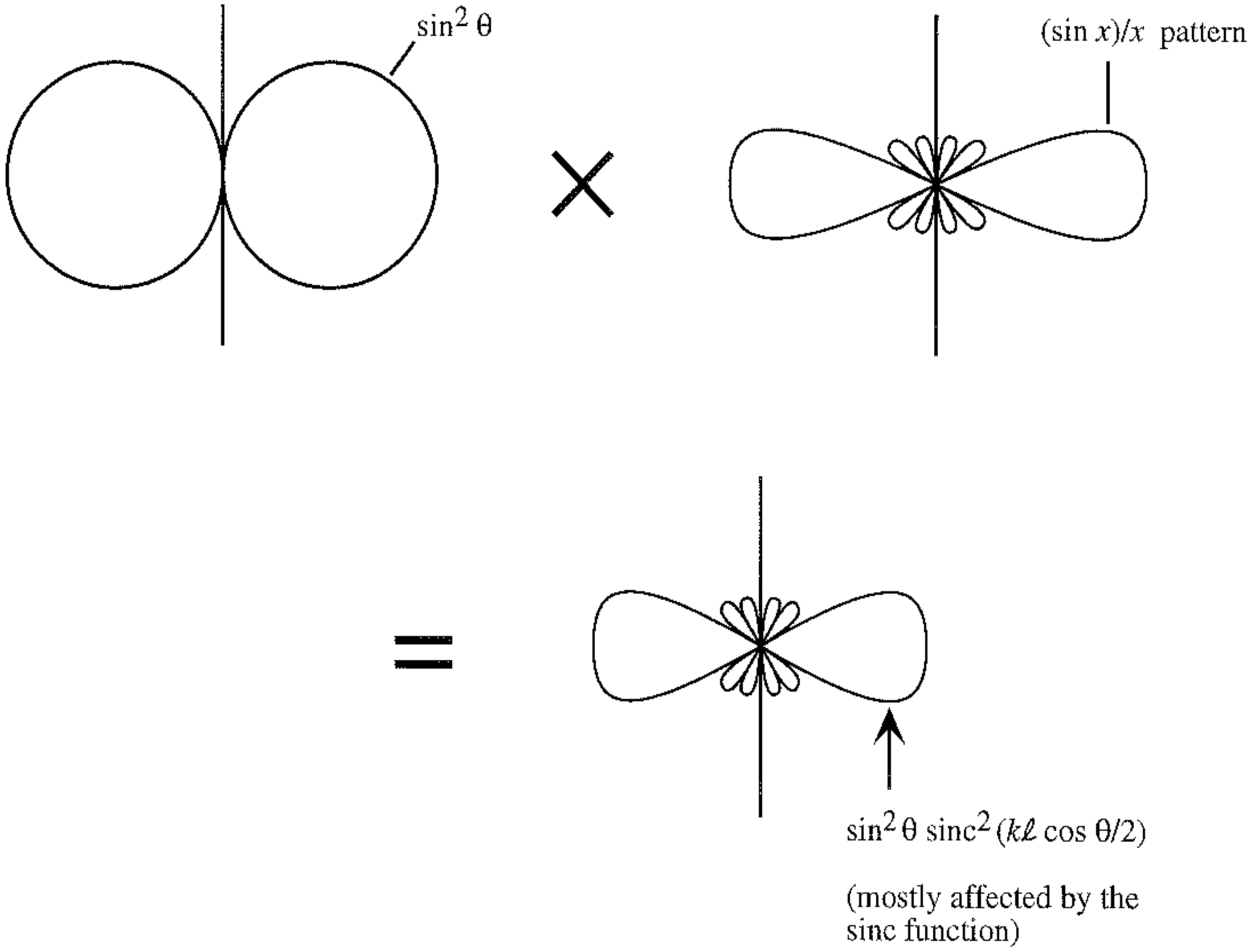


$$E_{\mathrm{ff}} = jkZ_0 \frac{e^{-jkr}}{4\pi r} \underbrace{\hat{r} \times \hat{r} \times \hat{z}}_{\hat{\theta}\sin\theta} N_z$$

$$E_{\mathrm{ff}_\theta} = jkZ_0(I_0\ell) \frac{e^{-jkr}}{4\pi r} \hat{\theta} \underbrace{\sin\theta \,\mathrm{sinc}\left(\frac{k\ell\cos\theta}{2}\right)}_{\text{field pattern}}$$



$$\text{Pattern} = \sin^2\theta \,\mathrm{sinc}^2\left[\frac{k\ell\cos\theta}{2}\right]$$


$\sin^2\theta$
$(\sin x)/x$ pattern
$\sin^2\theta\ \mathrm{sinc}^2(k\ell\cos\theta/2)$
(mostly affected by the sinc function)