3D Waveguides

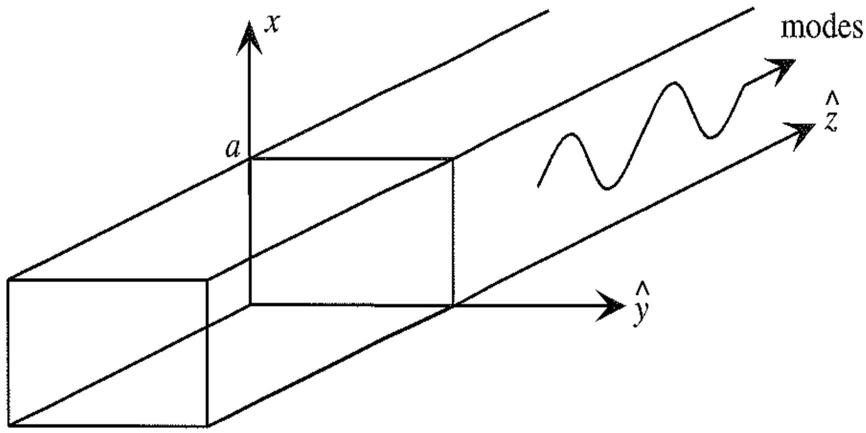

Figure 1: Waveguide geometry.

Solution for the fields within the guide can be written as a summation of modes. As done for 2D waveguides, we will decompose the modes down to TE and TM. Since propagation is along z (a choice), we will refer to the modes as TE^z and TM^z (i.e., transverse electric to z and transverse magnetic to z). As usual the TE^z modes will have $H_z \neq 0$, $E_z = 0$ and the TM^z modes will be associated with $H_z = 0$, $E_z \neq 0$. Further, since $H_z = \hat{z} \cdot \frac{1}{\mu} \nabla \times (\hat{z} A_z) = 0$, and $E_z = \hat{z} \cdot \frac{1}{\varepsilon} \nabla \times (\hat{z} F_z) = 0$, we can associate the TE^z modes with the electric vector potential $\mathbf{F} = \hat{z} F_z$ and the TM^z modes with the magnetic vector potential $\mathbf{A} = \hat{z} A_z$. All of these statements are summarized below.

$$\mathbf{E} = \mathbf{E}^{\mathrm{TE}} + \mathbf{E}^{\mathrm{TM}} = \sum_n \sum_m a_{nm} \, \mathbf{e}_{nm}^{\mathrm{TE}}(x,y) \, e^{-jk_{zmn}^{\mathrm{TE}} z} + \sum_n \sum_m b_{nm} \, \mathbf{e}_{nm}^{\mathrm{TM}}(x,y) \, e^{-jk_{zmn}^{\mathrm{TM}} z}$$

$$\mathbf{H} = \frac{-1}{j\omega\mu} \nabla \times \mathbf{E}$$

Alternatively,

$$\mathbf{E} = -j\omega \mathbf{A} + \frac{1}{j\omega\mu\varepsilon} \nabla\nabla \cdot \mathbf{A} - \frac{1}{\varepsilon} \nabla \times \mathbf{F}$$

$$\mathbf{E}^{\mathrm{TE}} = -\frac{1}{\varepsilon} \nabla \times (\hat{z} F_z), \qquad \mathbf{H}^{\mathrm{TE}} = \hat{z}(-j\omega F_z)$$

$$\mathbf{E}^{\mathrm{TM}} = -j\omega(\hat{z} A_z), \qquad \mathbf{H}^{\mathrm{TM}} = +\frac{1}{\mu} \nabla \times (\hat{z} A_z)$$

TM_z modes	TE_z modes
$\mathbf{A} = \hat{z}A_z, \left(\Pi_{ez} = \frac{1}{j\omega\mu\varepsilon}A_z\right)$	$\mathbf{F} = \hat{z}F_z, \left(\Pi_{mz} = \frac{1}{j\omega\mu\varepsilon}F_z\right)$
$E_x = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x\,\partial z} = +\frac{\partial^2 \Pi_{ez}}{\partial x\,\partial z}$	$H_x = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\,\partial z} = \frac{\partial^2 \Pi_{mz}}{\partial x\,\partial z}$
$E_y = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y\,\partial z} = +\frac{\partial^2 \Pi_{ez}}{\partial y\,\partial z}$	$H_y = -\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\,\partial z} = \frac{\partial^2 \Pi_{mz}}{\partial y\,\partial z}$
$E_z = -\frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)A_z = \left(\frac{\partial^2}{\partial z^2} + k_0^2\right)\Pi_{ez}$	$H_z = -\frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + k^2\right)F_z = \left(\frac{\partial^2}{\partial z^2} + \beta^2\right)\Pi_{mz}$
$H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = \frac{-1}{j\omega\varepsilon}\frac{\partial \Pi_{ez}}{\partial y}$	$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} = +\frac{1}{j\omega\mu}\frac{\partial \Pi_{mz}}{\partial y}$
$H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} = \frac{+1}{j\omega\varepsilon}\frac{\partial \Pi_{ez}}{\partial x}$	$E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} = -\frac{1}{j\omega\mu}\frac{\partial \Pi_{mz}}{\partial x}$
	$E_z = 0$

Expressions assume an $e^{\pm jk_z^{\mathrm{TM}}z}$ dependence.

TM^z modes

Using the method of separation of variables, we let

$$A_z = \psi^{\mathrm{TM}}(x,y,z) = B^{\mathrm{TM}}\, f(x)\, g(y)\, e^{-jk_z^{\mathrm{TM}}z}$$

(see Harrington p. 148 or Balanis p. 362). Here, we chose $h(z) = e^{-jk_z^{\mathrm{TM}}z}$ since we expect propagation along the $+z$ direction. For propagation along the $-z$ direction, $h(z) = e^{+jk_z^{\mathrm{TM}}z}$, and if both $+z$ and $-z$ propagation is allowed then $h(z) = A_3 e^{-jk_z^{\mathrm{TM}}z} + B_3 e^{+jk_z^{\mathrm{TM}}z} = A_3' \cos k_z^{\mathrm{TM}}z + B_3' \sin k_z^{\mathrm{TM}}z$, where A_3, B_3 are constants to be determined on the basis of boundary conditions and the excitation.

From the above expression for A_z, we have that

$$E_x = -B^{\mathrm{TM}}\frac{k_z^{\mathrm{TM}}}{\omega\mu\varepsilon}\left[\frac{\partial f(x)}{\partial x}\right] g(y)\, e^{-jk_z^{\mathrm{TM}}z}$$
$$E_y = -B^{\mathrm{TM}}\frac{k_z^{\mathrm{TM}}}{\omega\mu\varepsilon} f(x) \left[\frac{\partial g(y)}{\partial y}\right] e^{-jk_z^{\mathrm{TM}}z}$$
$$E_z = -B^{\mathrm{TM}}\frac{j}{\omega\mu\varepsilon}\left[k^2 - (k_z^{\mathrm{TM}})^2\right] f(x)\, g(y)\, e^{-jk_z^{\mathrm{TM}}z}$$

Given that $f(x)$ and $g(y)$ must satisfy the wave equation and that

1) $E_x(x, y=0) = E_x(x, y=b) = 0$

2) $E_y(x=0, y) = E_y(x=a, y) = 0$

we readily conclude that

1) $g(y=0)=0, \quad g(y=b)=0,$
implying $g(y)=\sin\left(\frac{n\pi y}{b}\right), \; n=1,2,3,\ldots$

2) $f(x=0)=0, \quad f(x=a)=0,$
implying $f(x)=\sin\left(\frac{m\pi}{a}x\right), \; m=1,2,3,\ldots$

Thus,

$$\psi_{mn}^{\mathrm{TM}}(x,y,z)=B_{mn}^{\mathrm{TM}}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_{z_{mn}}^{\mathrm{TM}}z}$$

with

$$k_{z_{mn}}^{\mathrm{TM}}=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$$

as dictated by the wave equation

$$\nabla^2 A_z+k^2A_z=\nabla^2\psi_{mn}^{\mathrm{TM}}+k^2\psi_{mn}^{\mathrm{TM}}=0$$

Here each of the wave functions ψ_{mn}^{TM} represents a possible solution or mode of the fields that may be supported inside the metallic waveguide. For this case, this mode will be referred to as the TM_{mn}^{z} mode. As stated earlier, the complete solution will be the sum of all modes. Specifically,

$$E_z^{\mathrm{TM}}=\sum_m\sum_n E_{0_{mn}}^{\mathrm{TM}}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_{z_{mn}}^{\mathrm{TM}}z}$$
$$E_x^{\mathrm{TM}}=\sum_m\sum_n\frac{jk_{z_{mn}}^{\mathrm{TM}}\left(\frac{m\pi}{a}\right)}{\left[k^2-(k_{z_{mn}}^{\mathrm{TM}})^2\right]}E_{0_{mn}}^{\mathrm{TM}}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-jk_{z_{mn}}^{\mathrm{TM}}z}$$
$$E_y^{\mathrm{TM}}=\sum_m\sum_n\frac{jk_{z_{mn}}^{\mathrm{TM}}\left(\frac{n\pi}{b}\right)}{\left[k^2-(k_{z_{mn}}^{\mathrm{TM}})^2\right]}E_{0_{mn}}^{\mathrm{TM}}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-jk_{z_{mn}}^{\mathrm{TM}}z}$$

where

$$E_{0_{mn}}^{\mathrm{TM}}=\frac{[k^2-(k_z^{\mathrm{TM}})^2]B_{mn}^{\mathrm{TM}}}{j\omega\mu\varepsilon}$$

is a convenient set of constants to be determined from the excitation.

The TM_{mn}^{z} modes will propagate when $k_{z_{mn}}$ are real, else they will attenuate and vanish within a small fraction of a wavelength from the source. That is, the TM_{mn}^{z} mode propagates/exists inside the guide when

$$k^2>\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2$$

The cutoff frequency of the (mn)th mode occurs when

$$(k_c)_{mn}^{\mathrm{TM}}=\frac{2\pi}{\lambda_{mn}}=\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}, \quad \text{or} \quad (f_c^{\mathrm{TM}})_{mn}=\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

or

$$\lambda_{c_{mn}}^{\mathrm{TM}}=\frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}, \qquad n\geq 1, \quad m\geq 1$$

The wave frequency $f = \omega/2\pi$ must be above $(f_c)_{mn}$ for the (mn)th mode to propagate. It is convenient to express $k_{z_{mn}} = -j\gamma$ in terms of the cutoff wavenumber as

$$\gamma = jk_z = jk\sqrt{1-(f_c/f)^2}, \qquad f > f_c$$

We note that the modes vanish (since E_z vanishes) when either $m = 0$ or $n = 0$. That is, the lowest TM^z mode to propagate is the TM^z_{11} mode. As an example, when $b = 6.5$ in $= 16.51$ cm, $a = 3.25$ in $= 8.25$ cm (WR650/L band guide) the TM_{11} mode will propagate when

$$f > (f_c^{\mathrm{TM}})_{11} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 2.03 \text{ GHz}$$

It will be seen next that this frequency is higher than the cutoff frequency of certain TE^z modes (pp. 361, 364, 365 of Balanis). Thus it is not the lowest supporting mode, and therefore it is not of much importance since typically only the lowest propagating mode is utilized for transmission in waveguides.

TE^z modes

Again, using the method of separation of variables, we express the electric vector potential as

$$F_z = \psi^{\mathrm{TE}}(x,y,z) = A^{\mathrm{TE}} f(x)\, g(y)\, e^{-jk_z^{\mathrm{TE}} z}$$

(see Harrington, p. 149, or Balanis, p. 356). The procedure for determining $f(x)$ and $g(y)$ follows that used for the TM modes. Again, note that the function $h(z) = e^{-jk_z^{\mathrm{TE}} z}$ could be replaced by $h(z) = A_3 e^{-jk_z^{\mathrm{TE}} z} + B_3 e^{-jk_z^{\mathrm{TE}} z} = A_3' \cos(k_z^{\mathrm{TE}} z) + B_3' \sin(k_z^{\mathrm{TE}} z)$ if we were to consider propagation in both $+z$ and $-z$ directions.

To determine the form of $f(x)$ and $g(y)$, we first look at the corresponding expressions for E_x and E_y, viz.

$$\begin{aligned}
E_x &= -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} = -\frac{A^{\mathrm{TE}}}{\varepsilon} f(x) \left[\frac{dg(y)}{dy}\right] e^{-jk_z^{\mathrm{TE}} z} \\
E_y &= +\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} = +\frac{A^{\mathrm{TE}}}{\varepsilon} \left[\frac{df(x)}{dx}\right] g(y)\, e^{-jk_z^{\mathrm{TE}} z} \\
E_z &= 0 \qquad \text{(for TE modes)}
\end{aligned}$$

Next, we note the boundary conditions to be satisfied by E_x and E_y. Specifically (as before for TM modes),

1) $E_x(x, y=0) = E_x(x, y=b) = 0$

2) $E_y(x=0, y) = E_y(x=a, y) = 0$

Each of these implies the following forms for $g(y)$ and $f(x)$, respectively:

1) $g(y) = \cos\left(\frac{n\pi}{b}y\right),\ n = 0, 1, 2, \ldots$

2) $f(x) = \cos\left(\frac{m\pi}{a}x\right)$, $m = 0, 1, 2, \ldots$

Thus,

$$\psi_{mn}^{\mathrm{TE}^z}(x,y,z) = A_{mn}^{\mathrm{TE}_x} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_{z_{mn}}^{\mathrm{TE}} z}$$

with

$$k_z^{\mathrm{TE}_{mn}} = \sqrt{k^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{m\pi}{a}\right)^2}$$

as dictated by the wave equation satisfied by $F_z/\psi_{mn}^{\mathrm{TE}^z}$. The associated field expressions are

$$H_z^{\mathrm{TE}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \underbrace{H_{0_{mn}}^{\mathrm{TE}} \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_{z_{mn}}^{\mathrm{TE}} z}}_{\psi_{mn}^{\mathrm{TE}}}$$

$$E_x^{\mathrm{TE}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{\frac{+j\omega\mu}{[k^2 - (k_{z_{mn}}^{\mathrm{TE}})^2]}\right\} \left(\frac{n\pi}{b}\right) H_{0_{mn}}^{\mathrm{TE}} \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-jk_{z_{mn}}^{\mathrm{TE}} z}$$

$$E_y^{\mathrm{TE}} = -\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left\{\frac{+j\omega\mu}{[k^2 - (k_{z_{mn}}^{\mathrm{TE}})^2]}\right\} \left(\frac{m\pi}{b}\right) H_{0_{mn}}^{\mathrm{TE}} \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) e^{-jk_{z_{mn}}^{\mathrm{TE}} z}$$

where $H_{0_{mn}}^{\mathrm{TE}}$ is a new constant proportional to the original constant A_{mn}^{TE}, viz.,

$$H_{0_{mn}}^{\mathrm{TE}} = \left[\frac{k^2 - (k_z^{\mathrm{TE}})^2}{j\omega\mu\varepsilon}\right] A_{mn}^{\mathrm{TE}}$$

The TE_{mn}^z modes will propagate when $k_{z_{mn}}^{\mathrm{TE}}$ is real. Else, they will attenuate. That is, the (mn)th mode propagates when

$$k^2 > \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Thus, the corresponding cutoff frequency for the (mn)th mode is

$$(k_c)_{mn}^{\mathrm{TE}} = \frac{2\pi}{\lambda_{mn}} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \qquad \text{cutoff wavenumber}$$

$$(f_c^{\mathrm{TE}})_{mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \qquad \text{cutoff frequency}$$

For the TE modes, if either $m = 0$ or $n = 0$, then H_z is associated with non-zero E_x or E_y, respectively. Thus, setting $m = 0$ or $n = 0$ yields a viable mode that can propagate in the waveguide. The corresponding cutoff frequency is

$$(f_c^{\mathrm{TE}})_{m=0,\ n=1} = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = 0.9085 \text{ GHz}, \qquad (\lambda_c^{\mathrm{TE}})_{01} = 2b = 33.02 \text{ cm}$$

or

$$(f_c^{\mathrm{TE}})_{m=1,\ n=0} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}}, \qquad (\lambda_c^{\mathrm{TE}})_{10} = 2a$$

That is, for $f > (f_c^{\mathrm{TE}})_{01}$ the TE_{01} mode propagates. Alternatively, the TE_{01} mode begins to propagate when $\lambda < 2b$ or $b > \lambda/2$. The TE_{10} mode will begin to propagate when $(\lambda < 2a)$ $a > \lambda/2$ and the TE_{20} mode will begin to propagate when $a > \lambda$, and so on. Whether TE_{01} or TE_{10} is the lowest order mode to propagate, this will depend on the values of a and b. Specifically, if $b > a$ then TE_{01} will be the lowest order mode. For $b = 6.5$ in, $a = 3.25$ in, the TE_{01} propagates for frequencies

$$f > (f_c^{\mathrm{TE}})_{01} = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = \frac{3\times 10^8}{2b} = \frac{0.3}{2b}\ \mathrm{GHz}$$

Typically, the chosen frequency is slightly above $(f_c)_{01}$, but is low enough so that no other mode is allowed to propagate.

Impedance of a mode

The impedance of a given TE^z_{mn} or TM^z_{mn} mode is obtained by taking the ratio of the transverse components to z. That is,

$$(Z_0)^{\mathrm{TM}}_{mn} = \underbrace{\frac{E_x^{\mathrm{TM}_{mn}}}{H_y^{\mathrm{TM}_{mn}}}}_{\text{since } \hat{x}\times\hat{y}=\hat{z}} = \underbrace{-\frac{E_y^{\mathrm{TM}}}{H_x^{\mathrm{TM}}}}_{\text{since } \hat{y}\times\hat{x}=-\hat{z}} = \frac{k_z^{\mathrm{TM}_{mn}}}{\omega\varepsilon}$$

and if $jk = \gamma = \alpha + j\beta$, then

$$(Z_0)^{\mathrm{TM}}_{mn} = \begin{cases} \dfrac{\beta}{\omega\varepsilon} = \dfrac{k\sqrt{1-\left(\frac{f_c}{f}\right)^2}}{\omega\varepsilon} = \sqrt{\dfrac{\mu}{\varepsilon}}\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} & f > (f_c)_{mn} \text{ propagating} \\ -j\dfrac{\alpha}{\omega\varepsilon} & f < (f_c)_{mn} \text{ attenuating} \end{cases}$$

implying that the attenuating modes have imaginary reactive impedance.

Similarly, for the TE_{mn} modes

$$(Z_0)^{\mathrm{TE}}_{mn} = \frac{E_x^{\mathrm{TE}_{mn}}}{H_y^{\mathrm{TE}_{mn}}} = -\frac{E_y^{\mathrm{TE}_{mn}}}{H_x^{\mathrm{TE}_{mn}}} = \frac{\omega\mu}{k_z^{\mathrm{TE}_{mn}}}$$

$$(Z_0)^{\mathrm{TE}}_{mn} = \begin{cases} \sqrt{\dfrac{\mu}{\varepsilon}}\Big/\sqrt{1-\left(\dfrac{f_c}{f}\right)^2} & f > (f_c)_{mn} \\ +j\sqrt{\dfrac{\mu}{\varepsilon}}\Big/\sqrt{\left(\dfrac{f_c}{f}\right)^2-1} & f < (f_c)_{mn} \end{cases}$$

Note that $(Z_0)^{\mathrm{TE}}_{mn} \to \infty$ when $f = f_c$, and $(Z_0)^{\mathrm{TM}}_{mn} \to 0$ when $f = f_c$.

Power carried by waveguide fields

The power carried by each mode along z is found by integrating the power density over the cross section of the waveguide. Specifically,

$$P_{mn} = \int_0^a \int_0^b (\mathbf{S}_{mn}\cdot\hat{z})\, dx\, dy$$

(See scanned figure)

Figure 2: Plot of the TE and TM mode impedances as a function of frequency. Note that for $f < f_c$, $Z^{\mathrm{TE}}_{mn} = jX$ and $Z^{\mathrm{TM}}_{mn} = -jX$. For $f > f_c$, $Z^{\mathrm{TE,TM}} = R$.

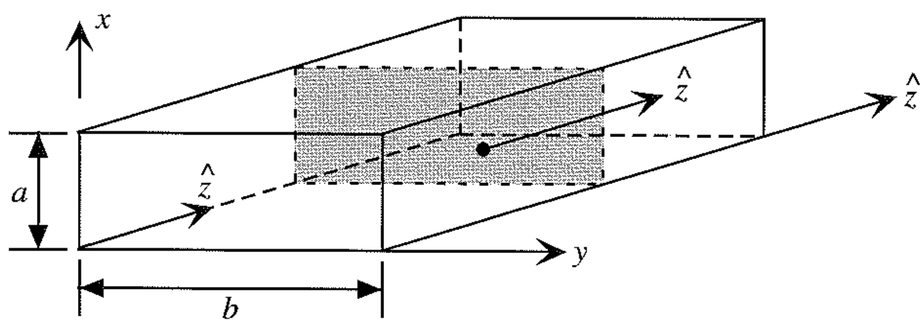

Figure 3: Cavity cross section for power computation.

where

$$\mathbf{S}_{mn} = \frac{1}{2}\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) = \hat{z}\frac{1}{2}\mathrm{Re}(E_x H_y^* - E_y H_x^*)$$

(see p. 172 of Harrington, p. 374 of Balanis)

For the lowest order TE_{10} mode, we have

$$\begin{aligned}
E_y^{\mathrm{TE}_{10}} &= \frac{-j\omega\mu}{k^2-(k_{z_{10}})^2}\left(\frac{\pi}{a}\right) H^{\mathrm{TE}}_{0_{mn}} \sin\left(\frac{\pi}{a}x\right) e^{-jk_{z_{mn}}z} \\
&= -\frac{1}{\varepsilon}\left(\frac{\pi}{a}\right) A^{\mathrm{TE}}_{10} \sin\left(\frac{\pi}{a}x\right) e^{-jk_{z_{mn}}z} \\
E_x &= 0 = E_z \\
H_x^{\mathrm{TE}_{10}} &= A^{\mathrm{TE}}_{10}\frac{k_{z_{10}}}{\omega\mu\varepsilon}\left(\frac{\pi}{a}\right) \sin\left(\frac{\pi}{a}x\right) e^{-jk_{z_{mn}}z} \\
H_z^{\mathrm{TE}_{10}} &= -j\frac{A^{\mathrm{TE}}_{10}}{\omega\mu\varepsilon}\left(\frac{\pi}{a}\right)^2 \cos\left(\frac{\pi}{a}x\right) e^{-jk_{z_{mn}}z}
\end{aligned}$$

In general, the (mn)th mode power is given by

$$P^{\mathrm{TE}}_{mn} = |A^{\mathrm{TE}}_{mn}|^2 \frac{k_{z_{mn}}}{2\omega\mu\varepsilon^2}\left(\frac{a}{\varepsilon_{0m}}\right)\left(\frac{b}{\varepsilon_{0n}}\right)\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

and

$$P_{\mathrm{total}} = \sum_{m,n} P^{\mathrm{TE}}_{mn} + \sum_{m,n} P^{\mathrm{TM}}_{mn}$$

where

$$\varepsilon_{0m} = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

since

$$\int_0^a \cos^2\left(\frac{m\pi}{a}x\right)\, dx = \int_0^a \sin^2\left(\frac{m\pi}{a}x\right)\, dx = \frac{a}{2}, \quad m \neq 0$$

and

$$= \int_0^a \cos^2\left(\frac{m\pi}{a}x\right)\, dx = a, \quad m = 0$$

Thus, for the TE_{10} mode

$$P_{10}^{\mathrm{TE}} = |A_{10}^{\mathrm{TE}}|^2 \frac{k_{z10}}{2\omega\mu\varepsilon^2}\left(\frac{a}{2}\right)(b)\left(\frac{\pi}{a}\right)^2$$

$$= |A_{10}^{\mathrm{TE}}|^2 \frac{\sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}}{2\omega\mu\varepsilon^2}\left(\frac{\pi}{a}\right)^2\left(\frac{ab}{2}\right)$$

More explicitly,

$$P_{10}^{\mathrm{TE}} = -\int_0^a \int_0^b \frac{1}{2}\mathrm{Re}\left\{E_y H_x^*\right\}\, dx = +\int_0^a \int_0^b \frac{1}{2}\mathrm{Re}\left\{|E_y|^2/Z_{mn}^{\mathrm{TE}}\right\}$$

$$= \frac{1}{2}\frac{1}{\varepsilon^2}\left(\frac{\pi}{a}\right)^2 A_{10}^{\mathrm{TE}}\left(\frac{a}{2}\right)(b)\frac{k_{z10}}{\omega\mu}$$

$$= |A_{10}^{\mathrm{TE}}|^2 \frac{\sqrt{k^2 - (\pi/a)^2}}{2\omega\mu\varepsilon^2}\left(\frac{\pi}{a}\right)^2\left(\frac{ab}{2}\right)$$

Waveguide losses

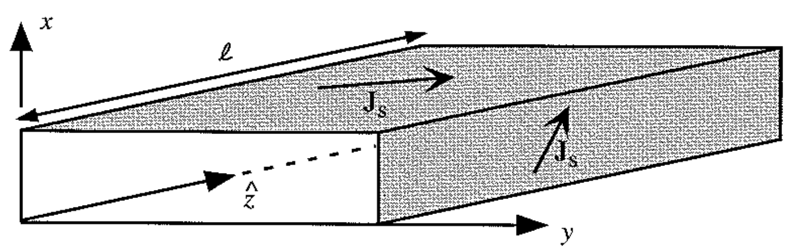

Figure 4: Surface currents for power loss computations.

Waveguide losses are caused by currents generated on the walls of the waveguide. Since the waveguide has finite conductivity, these currents generate heat due to I^2R losses.

The currents generated on the waveguide are given by

$$\mathbf{J}_s = \hat{n} \times (\mathbf{H}^+ - \mathbf{H}^-) = \hat{n} \times \mathbf{H}^+$$

since $\mathbf{H}^- = 0$ inside the conductor. Thus, each mode is responsible for a current density

$$\mathbf{J}_{s_{mn}} = \hat{n} \times \mathbf{H}_{mn}^{\mathrm{TE,TM}}$$

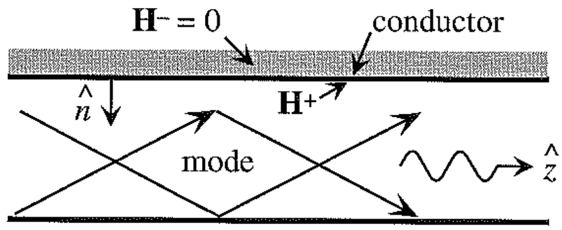

Figure 5: Definition of $\mathbf{H}^{\pm}$ fields used for surface current computations.

A good conductor is associated with a surface impedance due to its skin depth, viz.

$$Z = \sqrt{\frac{\mu_0}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0\left(1+\frac{\sigma}{j\omega\varepsilon_0}\right)}}$$

$$= \sqrt{\frac{j\omega\mu_0}{j\omega\varepsilon_0+\sigma}}\Bigg|_{\sigma\gg\omega\varepsilon_0} \approx \sqrt{\frac{j\omega\mu_0}{\sigma}} = \sqrt{\frac{\omega\mu_0}{2\sigma}}\underbrace{(1+j)}_{e^{j\pi/2}} = R_s + jX_s$$

Thus, dissipated power on walls is given by

$$P_c = \frac{1}{2}I^2R_s = \frac{R_s}{2}I\cdot I^*$$

$$= \frac{1}{2}R_s \iint_{\text{conductor surface}} \mathbf{J}_s\cdot\mathbf{J}_s^*\,ds$$

For the TE_{10} mode, we have

$$P_c\big|_{x=0\text{ wall (left vertical wall)}} = R_s|A_{10}^{TE}|^2\frac{b}{(\omega\mu\varepsilon)^2}\left(\frac{\pi}{a}\right)^4\ell$$

$$P_c\big|_{y=0\text{ wall (bottom horizontal wall)}} = R_s\frac{|A_{10}^{TE}|^2}{(\omega\mu\varepsilon)^2}\frac{a}{2}\left[\left(\frac{\pi}{a}\right)^2+(k_{z_{10}})^2\right]\left(\frac{\pi}{a}\right)^2\ell$$

where $\ell =$ length of the guide. Thus, total power dissipated per unit length is

$$\frac{(P_c)_{10}}{\ell} = R_s\frac{a}{2\left(\sqrt{\frac{\mu}{\varepsilon}}\right)^2}\frac{|A_{10}|^2}{\varepsilon^2}\left(\frac{\pi}{a}\right)^2\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

Since power decays as $e^{-2\alpha_c\ell}$, where α_c refers to the imaginary part of k_z due to conductor losses, the corresponding decay factor α_c is given by

$$\alpha_c = \left[\frac{(P_c)_{10}/\ell}{2P_{10}}\right] = \frac{R_s}{\left(\sqrt{\frac{\mu}{\varepsilon}}\right)b}\frac{\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\ \text{Nepers/meter}$$

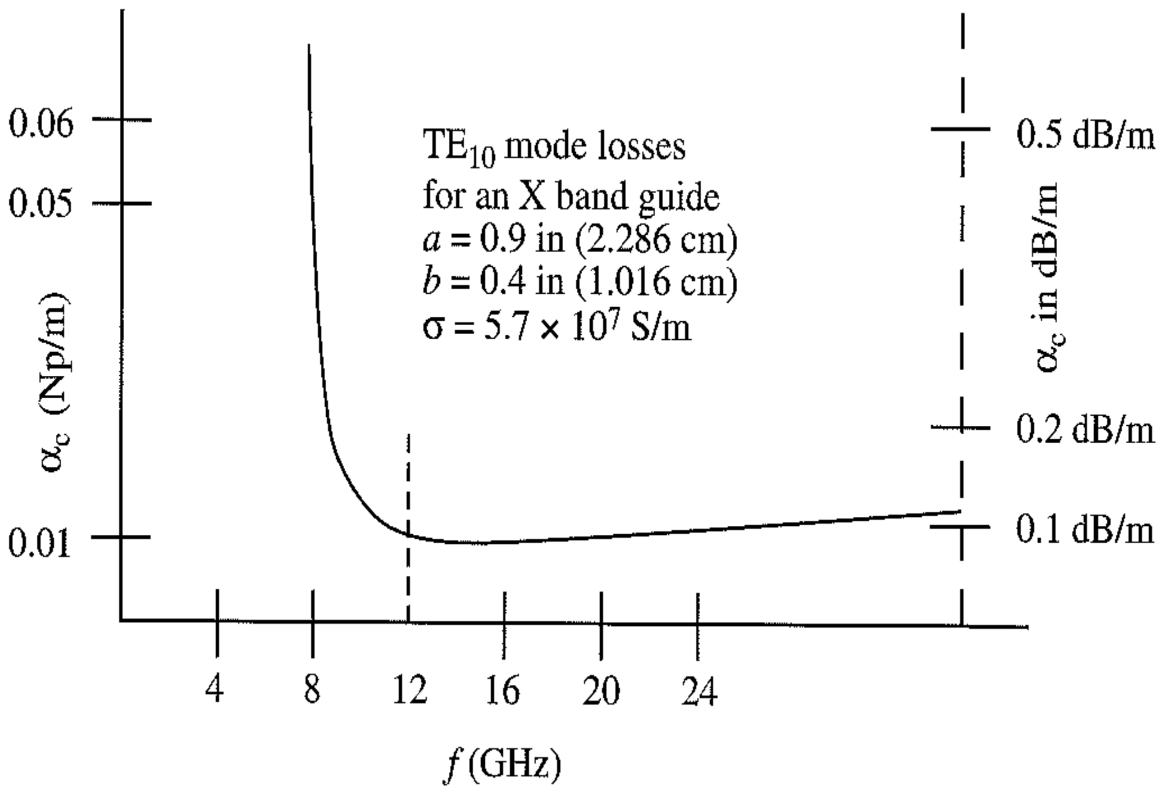

Figure 6: Attenuation coefficient for the TE$_{10}$ mode due to conductor losses.

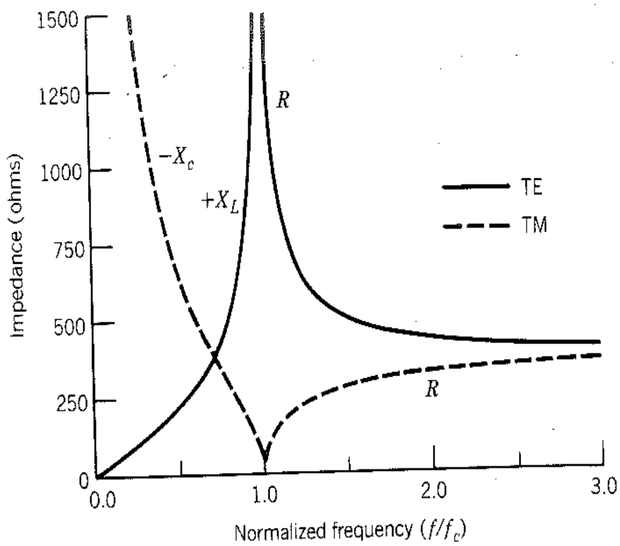

Figure 2: Plot of the TE and TM mode impedances as a function of frequency. Note that for $f < f_c$, $Z_{mn}^{\mathrm{TE}} = jX$ and $Z_{mn}^{\mathrm{TM}} = -jX$. For $f > f_c$, $Z^{\mathrm{TE,TM}} = R$.

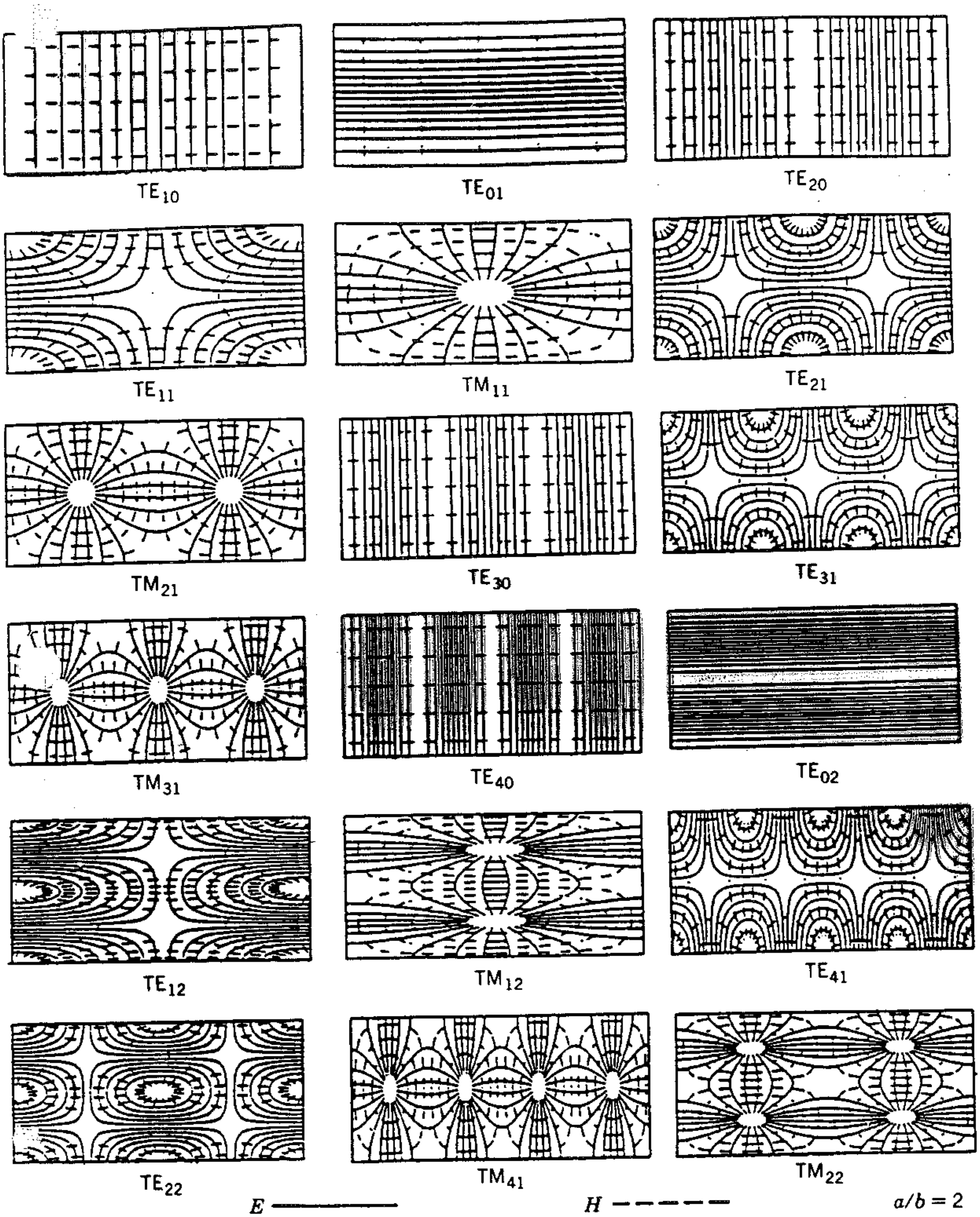

FIGURE 8-4 Field patterns for the first 18 TE^z and/or TM^z modes in a rectangular waveguide with $a/b = 2$ (*Source:* C. S. Lee, S. W. Lee, and S. L. Chuang, "Plot of modal field distribution in rectangular and circular waveguides," *IEEE Trans. Microwave Theory Tech.*, ©, 1985, IEEE.)

TABLE 8-1
Ratio of cutoff frequency of TE^z_{mn} mode to that of TE^z_{10}

$$R_{mn} = \frac{(f_c)^{TE^z}_{mn}}{(f_c)^{TE^z}_{10}} = \sqrt{m^2 + \left(\frac{na}{b}\right)^2} \qquad \left.\begin{array}{l} m = 0, 1, 2, \ldots \\ n = 0, 1, 2, \ldots \end{array}\right\} m = n \neq 0$$

$a/b \Rightarrow$	10	5	2.25	2	1
$m, n \Rightarrow$	1,0	1,0	1,0	1,0	1,0; 0,1
$R_{mn} \Rightarrow$	1	1	1	1	1
$m, n \Rightarrow$	2,0	2,0	2,0	2,0; 0,1	1,1
$R_{mn} \Rightarrow$	2	2	2	2	1.414
$m, n \Rightarrow$	3,0	3,0	0,1	1,1	2,0
$R_{mn} \Rightarrow$	3	3	2.25	2.236	2
$m, n \Rightarrow$	4,0	4,0	1,1	2,1	2,1; 1,2
$R_{mn} \Rightarrow$	4	4	2.462	2.828	2.236
$m, n \Rightarrow$	5,0	5,0; 0,1	3,0	3,0	2,2
$R_{mn} \Rightarrow$	5	5	3	3	2.828
$m, n \Rightarrow$	6,0	1,1	2,1	3,1	3,0; 0,3
$R_{mn} \Rightarrow$	6	5.099	3.010	3.606	3
$m, n \Rightarrow$	7,0	2,1	3,1	4,0; 0,2	3,1; 1,3
$R_{mn} \Rightarrow$	7	5.385	3.75	4	3.162
$m, n \Rightarrow$	8,0	3,1	4,0	1,2	3,2; 2,3
$R_{mn} \Rightarrow$	8	5.831	4	4.123	3.606
$m, n \Rightarrow$	9,0	6,0	0,2	4,1; 2,2	4,0; 0,4
$R_{mn} \Rightarrow$	9	6	4.5	4.472	4
$m, n \Rightarrow$	10,0; 0,1	4,1	4,1	5,0; 3,2	4,1; 1,4
$R_{mn} \Rightarrow$	10	6.403	4.589	5	4.123

TABLE 8-2
Ratio of cutoff frequency of TM^z_{mn} mode to that of TE^z_{10}

$$T_{mn} = \frac{(f_c)^{TM^z}_{mn}}{(f_c)^{TE^z}_{10}} = \sqrt{m^2 + \left(\frac{na}{b}\right)^2} \qquad \begin{array}{l} m = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \end{array}$$

$a/b \Rightarrow$	10	5	2.25	2	1
$m, n \Rightarrow$	1, 1	1, 1	1, 1	1, 1	1, 1
$T_{mn} \Rightarrow$	10.05	5.10	2.46	2.23	1.414
$m, n \Rightarrow$	2, 1	2, 1	2, 1	2, 1	2, 1; 1, 2
$T_{mn} \Rightarrow$	10.19	5.38	3.01	2.83	2.236
$m, n \Rightarrow$	3, 1	3, 1	3, 1	3, 1	2, 2
$T_{mn} \Rightarrow$	10.44	6.00	3.75	3.61	2.828
$m, n \Rightarrow$	4, 1	4, 1	4, 1	1, 2	3, 1; 1, 3
$T_{mn} \Rightarrow$	10.77	6.40	4.59	4.12	3.162
$m, n \Rightarrow$	5, 1	5, 1	1, 2	4, 1; 2, 2	3, 2; 2, 3
$T_{mn} \Rightarrow$	11.18	7.07	5.09	4.47	3.606
$m, n \Rightarrow$	6, 1	6, 1	2, 2	3, 2	4, 1; 1, 4
$T_{mn} \Rightarrow$	11.66	7.81	5.38	5.00	4.123
$m, n \Rightarrow$	7, 1	7, 1	5, 1	5, 1	3, 3
$T_{mn} \Rightarrow$	12.21	8.60	5.48	5.39	4.243
$m, n \Rightarrow$	8, 1	8, 1	3, 2	4, 2	4, 2; 2, 4
$T_{mn} \Rightarrow$	12.81	9.43	5.83	5.66	4.472
$m, n \Rightarrow$	9, 1	1, 2	4, 2	1, 3	4, 3; 3, 4
$T_{mn} \Rightarrow$	13.82	10.04	6.40	6.08	5.00
$m, n \Rightarrow$	10, 1	2, 2	6, 1	2, 3	5, 1; 1, 5
$T_{mn} \Rightarrow$	14.14	10.20	6.41	6.32	5.09

TABLE 8-1
Ratio of cutoff frequency of TE^z_{mn} mode to that of TE^z_{10}

$$R_{mn} = \frac{(f_c)^{TE^z}_{mn}}{(f_c)^{TE^z}_{10}} = \sqrt{m^2 + \left(\frac{na}{b}\right)^2} \qquad \left.\begin{matrix} m = 0, 1, 2, \ldots \\ n = 0, 1, 2, \ldots \end{matrix}\right\} \; m = n \neq 0$$

$a/b \Rightarrow$	10	5	2.25	2	1
$m, n \Rightarrow$	1, 0	1, 0	1, 0	1, 0	1, 0; 0, 1
$R_{mn} \Rightarrow$	1	1	1	1	1
$m, n \Rightarrow$	2, 0	2, 0	2, 0	2, 0; 0, 1	1, 1
$R_{mn} \Rightarrow$	2	2	2	2	1.414
$m, n \Rightarrow$	3, 0	3, 0	0, 1	1, 1	2, 0
$R_{mn} \Rightarrow$	3	3	2.25	2.236	2
$m, n \Rightarrow$	4, 0	4, 0	1, 1	2, 1	2, 1; 1, 2
$R_{mn} \Rightarrow$	4	4	2.462	2.828	2.236
$m, n \Rightarrow$	5, 0	5, 0; 0, 1	3, 0	3, 0	2, 2
$R_{mn} \Rightarrow$	5	5	3	3	2.828
$m, n \Rightarrow$	6, 0	1, 1	2, 1	3, 1	3, 0; 0, 3
$R_{mn} \Rightarrow$	6	5.099	3.010	3.606	3
$m, n \Rightarrow$	7, 0	2, 1	3, 1	4, 0; 0, 2	3, 1; 1, 3
$R_{mn} \Rightarrow$	7	5.385	3.75	4	3.162
$m, n \Rightarrow$	8, 0	3, 1	4, 0	1, 2	3, 2; 2, 3
$R_{mn} \Rightarrow$	8	5.831	4	4.123	3.606
$m, n \Rightarrow$	9, 0	6, 0	0, 2	4, 1; 2, 2	4, 0; 0, 4
$R_{mn} \Rightarrow$	9	6	4.5	4.472	4
$m, n \Rightarrow$	10, 0; 0, 1	4, 1	4, 1	5, 0; 3, 2	4, 1; 1, 4
$R_{mn} \Rightarrow$	10	6.403	4.589	5	4.123

TABLE 8-2
Ratio of cutoff frequency of TM^z_{mn} mode to that of TE^z_{10}

$$T_{mn} = \frac{(f_c)^{TM^z}_{mn}}{(f_c)^{TE^z}_{10}} = \sqrt{m^2 + \left(\frac{na}{b}\right)^2} \qquad \begin{matrix} m = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \end{matrix}$$

$a/b \Rightarrow$	10	5	2.25	2	1
$m, n \Rightarrow$	1, 1	1, 1	1, 1	1, 1	1, 1
$T_{mn} \Rightarrow$	10.05	5.10	2.46	2.23	1.414
$m, n \Rightarrow$	2, 1	2, 1	2, 1	2, 1	2, 1; 1, 2
$T_{mn} \Rightarrow$	10.19	5.38	3.01	2.83	2.236
$m, n \Rightarrow$	3, 1	3, 1	3, 1	3, 1	2, 2
$T_{mn} \Rightarrow$	10.44	6.00	3.75	3.61	2.828
$m, n \Rightarrow$	4, 1	4, 1	4, 1	1, 2	3, 1; 1, 3
$T_{mn} \Rightarrow$	10.77	6.40	4.59	4.12	3.162
$m, n \Rightarrow$	5, 1	5, 1	1, 2	4, 1; 2, 2	3, 2; 2, 3
$T_{mn} \Rightarrow$	11.18	7.07	5.09	4.47	3.606
$m, n \Rightarrow$	6, 1	6, 1	2, 2	3, 2	4, 1; 1, 4
$T_{mn} \Rightarrow$	11.66	7.81	5.38	5.00	4.123
$m, n \Rightarrow$	7, 1	7, 1	5, 1	5, 1	3, 3
$T_{mn} \Rightarrow$	12.21	8.60	5.48	5.39	4.243
$m, n \Rightarrow$	8, 1	8, 1	3, 2	4, 2	4, 2; 2, 4
$T_{mn} \Rightarrow$	12.81	9.43	5.83	5.66	4.472
$m, n \Rightarrow$	9, 1	1, 2	4, 2	1, 3	4, 3; 3, 4
$T_{mn} \Rightarrow$	13.82	10.04	6.40	6.08	5.00
$m, n \Rightarrow$	10, 1	2, 2	6, 1	2, 3	5, 1; 1, 5
$T_{mn} \Rightarrow$	14.14	10.20	6.41	6.32	5.09