Integral form of Maxwell's equations

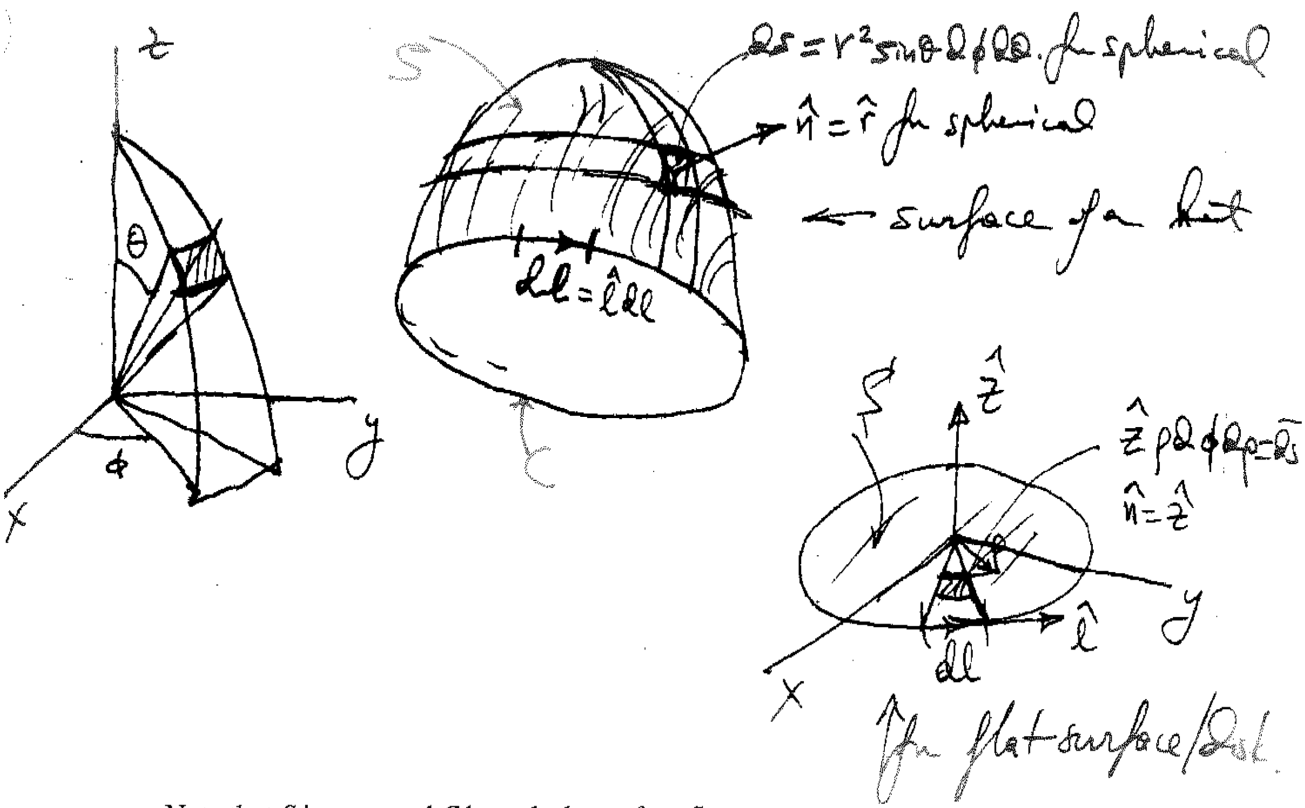
Faraday's Law

- Faraday's Law is the basic principle for electric machines (motors and generators).
- It provides the basic relation between electric and magnetic fields.
- Faraday's Law states that changing magnetic field causes a rise of potential across the conductor terminals (generator).
- emf around the closed path C = neg. rate of change of magnetic flux through the surface enclosed by C.

$$\oint \mathbf{E}\cdot \mathbf{dl} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot \overbrace{\hat{n}\, ds}^{\mathbf{ds}}$$

where $\mathbf{E}$ is the electric field in volts/meter and $\mathbf{B}$ is the magnetic flux density in web/m^2.

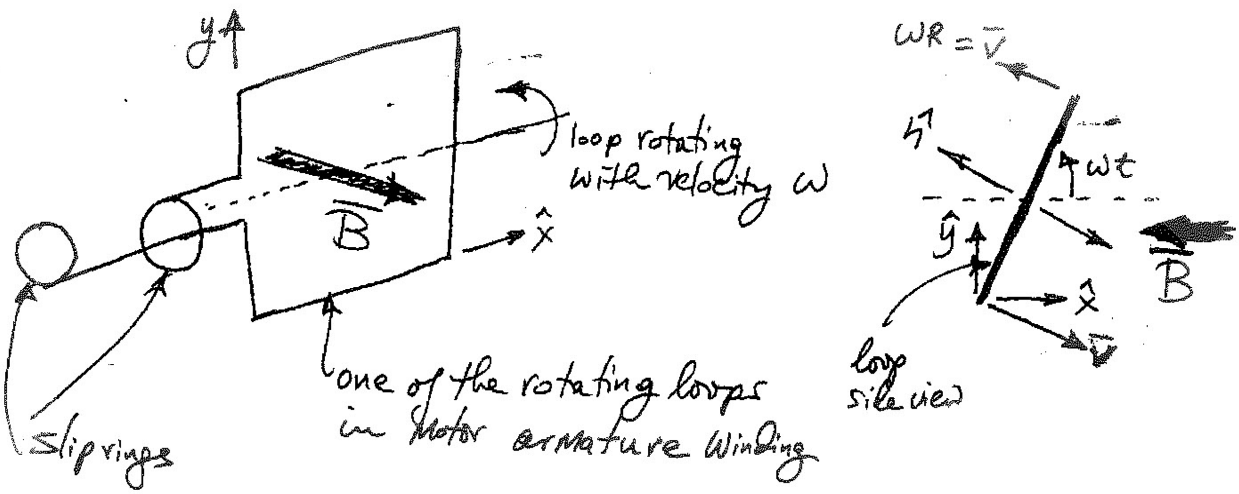
Applicable relationship between C and S



- Note that S is open and C bounds the surface S.

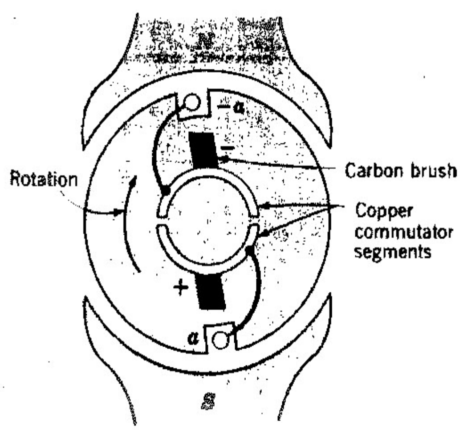
Faraday's Law example

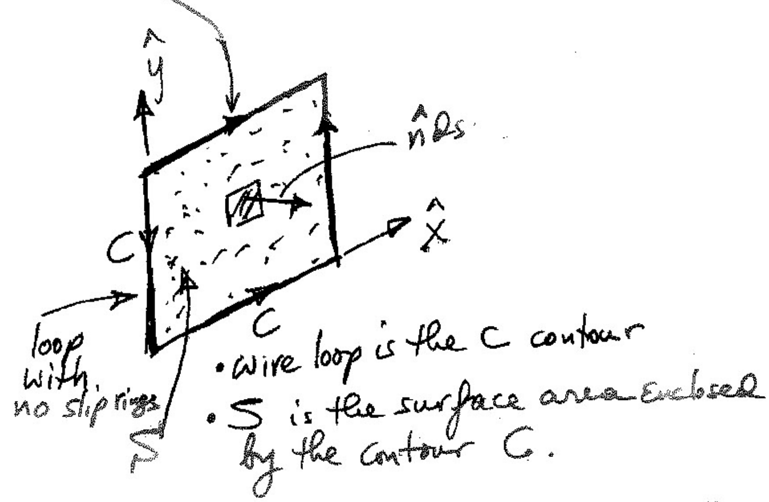
Basic principles of motors and generators



- A potential $\mathcal{V}$ is seen across these rings due to the rotating loop in a **B** (static or dynamic) field. This **B** is due to an externally applied field from the stator magnet, for example.

$$\mathcal{V} = \oint_C \mathbf{E}\cdot\mathbf{dl} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\underbrace{\hat{n}\,ds}_{\mathbf{ds}} = \underbrace{\oint_C \mathbf{v}\times\mathbf{B}\cdot\mathbf{dl}}_{\text{holds for constant } \mathbf{B} \text{ only}}$$





- Since the loop is rotating, $\hat{n}$ is dependent on time. At $t = 0$, $\hat{n} = \hat{z}$ and $\hat{n}\,ds = \hat{z}\,dx\,dy$. But at some time t, then

$$-\hat{n} = \hat{x}\cos\left(\frac{\pi}{2} - \omega t\right) - \hat{y}\sin\left(\frac{\pi}{2} - \omega t\right) \quad \Rightarrow$$

$$\hat{n} = -\hat{x}\sin\omega t + \hat{y}\cos\omega t$$

Thus,

$$\mathcal{V} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\hat{n}\,ds = +\frac{\partial}{\partial t}\iint_S [(\mathbf{B}\cdot\hat{x}\sin\omega t) - (\mathbf{B}\cdot\hat{y}\cos\omega t)]\,ds$$

If we assume that $\mathbf{B} = \hat{x}B_x$, then

$$\mathcal{V} = -B_x\frac{\partial}{\partial t}\left(\iint \hat{x}\cdot\hat{n}\,ds\right) = -B_x\frac{\partial}{\partial t}\left(-\sin\omega t\iint ds\right)$$
$$= +(B_x\omega\cos\omega t)\times\underbrace{(\text{Area of loop})}_{A}$$

If N loops form the winding, then

$$\mathcal{V} = N\omega B_x A\cos\omega t$$

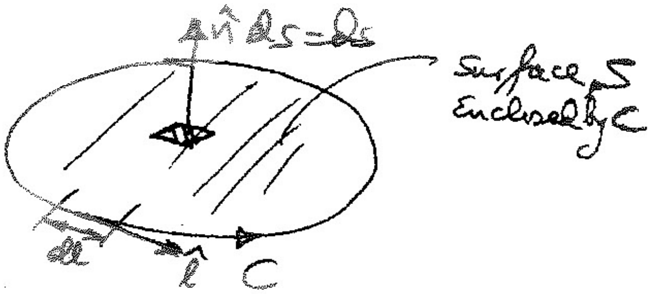
Note that the approach is valid even when $\mathbf{B}$ is time-varying, i.e., if $\mathbf{B} = \hat{x}\cos\omega_0 t$ as is usually the case. Note also that larger area, higher ω or ω_0, more turns and higher $\mathbf{B}$, all lead to higher voltages. Thus, we can use these parameters for design.

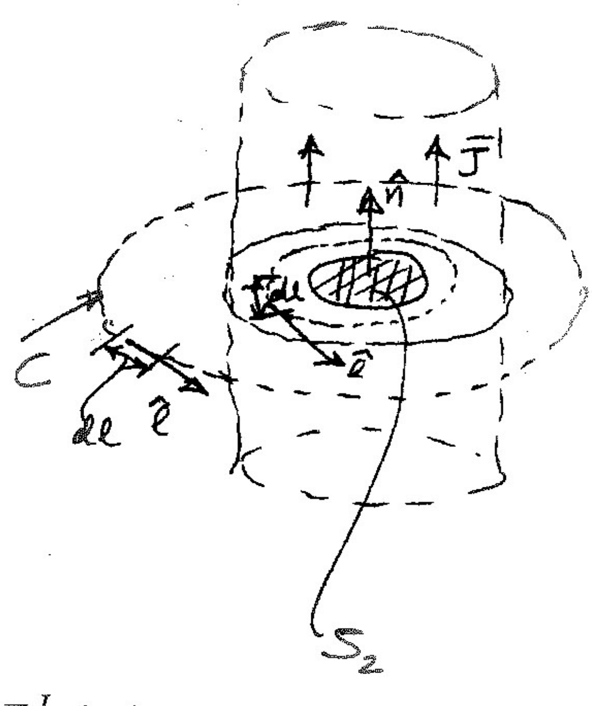
Ampère's Law example

$$\text{mmf} = \oint_C \mathbf{H}\cdot\mathbf{dl} = \iint_S \mathbf{J}\cdot\hat{n}\,ds$$

in which $\mathbf{H}$ is the magnetic field in A/m, $\mathbf{dl} = \hat{l}\,dl$, and $\mathbf{J}$ is the current density in A/m^2.

In words: mmf around a closed path C equals the current through the surface enclosed by C.



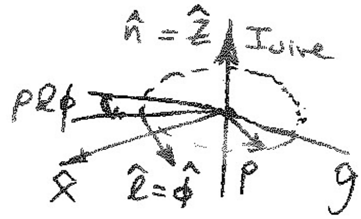


Note:

$$\iint_{S_2} \mathbf{J}\cdot\hat{n}\,ds = I_{\text{enclosed}}$$

where I_{enclosed} is the total current through S_2.

Thin wires

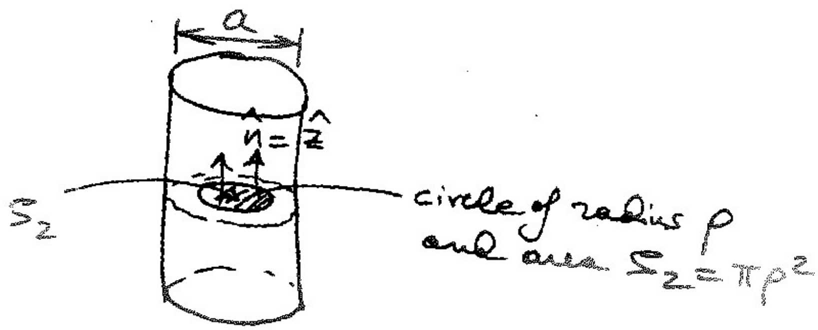


$$\oint_C \mathbf{H} \cdot \mathbf{dl} = I_{\text{wire}} = I_{\text{encl.}}$$

$$\int_0^{2\pi} \mathbf{H} \cdot \underbrace{\hat{\phi} \rho \, d\phi}_{dl} = \int_0^{2\pi} H_\phi \rho \, d\phi = 2\pi H_\phi \rho = I_{\text{wire}}$$

$$H_\phi = \frac{I_{\text{wire}}}{2\pi\rho}$$

Thick wire



$$\iint_{S_2} \mathbf{J} \cdot \hat{n} \, ds = I_{\text{encl.}}$$

If J is assumed constant through S_2, then

$$\underbrace{(\mathbf{J} \cdot \hat{n})}_{J_n = J_z} \pi\rho^2 = I_{\text{encl.}} \quad \Rightarrow \quad \boxed{J_n = J_z = \frac{I_{\text{encl.}}}{\pi\rho^2}}$$

When $\rho = a$, it follows that

$$\boxed{J_z = \frac{I_{\text{wire}}}{\pi a^2}}$$

For time-varying fields, Maxwell supplemented Ampère's equation (and made it to read similarly to Faraday's Law) as follows:

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = \iint_S \mathbf{J} \cdot \hat{n} \, ds + \frac{\partial}{\partial t} \iint_S \mathbf{D} \cdot \hat{n} \, ds \qquad \textbf{Ampère-Maxwell's Law}$$

where **D** is the electric flux density in C/m^2. Maxwell called the quantity

$$\frac{\partial \mathbf{D}}{\partial t} = \text{displacement current}$$

since it has the same mathematical effect as the current density $\mathbf{J}$. This becomes obvious when we rewrite Ampère-Maxwell's Law as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \hat{n}\, ds$$

Continuity Equation

A third independent equation to be added to the above two equations is the "charge conservation law":

$$\underbrace{\oiint_S \mathbf{J} \cdot \hat{n}\, ds}_{I} = -\frac{\partial}{\partial t} \underbrace{\iiint_V \rho\, dv}_{Q}$$

where ρ is the charge density in C/m^3.

- This is a generalization of the known relation

  $$I = -\frac{dQ}{dt}$$

  from circuit theory.
- Charge conservation provides the relation between moving charges and current.
- Its physical meaning is that: charge is neither created nor destroyed, *or*

  (neg) time rate of change of charge in V equals the current flowing out of V.