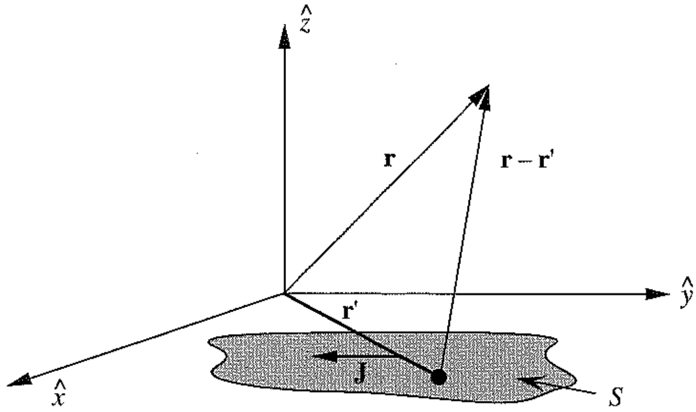
Solving complex problems using plane waves

Consider a current source in the (x,y) plane. This source may be due to some aperture antenna (reflector or horn) or due to some dipole array (Yagi-Uda, for example).



In either case, the source may be represented by the surface current density:

$$\mathbf{J}(x,y) = \hat{x}J_x(x,y) + \hat{y}J_y(x,y)$$

The usual approach for computing the field due to such a source is to first construct the vector potential

$$\mathbf{A} = \mu \iint_S \mathbf{J}(x',y') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \, dx' \, dy'$$

where $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ (field point) and $\mathbf{r}' = x'\hat{x} + y'\hat{y}$ (location of source), so that

$$\mathbf{E}_J = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A}$$

The above expression can be readily expanded given the current density $\mathbf{J}(x,y)$. However, the problem of computing the fields generated by $\mathbf{J}$ in the presence of a nearby object or even a simple dielectric interface becomes extremely complicated. An approach to simplify the analysis is to decompose $\mathbf{E}_J$ into plane waves. Then the problem becomes that of a plane wave analysis.

To rewrite $\mathbf{E}_J$ as a "sum" of plane waves we need to obtain its Fourier transform $\widetilde{E}_J$. Toward this step, we define the Fourier transform pair

$$\begin{aligned}
\mathcal{F}_{2D}\{\mathbf{F}(x,y)\} = \widetilde{\mathbf{F}}(k_x,k_y) &= \iint_{-\infty}^{\infty} \mathbf{F}(x,y)\, e^{-j(k_x x + k_y y)} \, dx \, dy \\
&= \hat{x}\widetilde{F}_x(k_x,k_y) + \hat{y}\widetilde{F}_y(k_x,k_y) \\
\mathbf{F}(x,y) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\mathbf{F}}_y(k_x,k_y)\, e^{+j(k_x x + k_y y)} \, dk_x \, dk_y
\end{aligned}$$

We also note the integral identity

$$\begin{aligned}
\frac{e^{-jkr}}{4\pi r} &= \frac{-j}{2(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk_z|z|}}{k_z} e^{j(k_x x + k_y y)} \, dk_x \, dk_y \\
r &= \sqrt{x^2+y^2+z^2}
\end{aligned}$$

That is,

$$\mathcal{F}_{2D}\left\{\frac{e^{-jkr}}{4\pi r}\right\} = \frac{-j}{2}\frac{e^{-jk_z|z|}}{k_z}$$

where

$$k_z = \begin{cases} \sqrt{k_0^2 - k_x^2 - k_y^2} & k_0^2 > k_x^2 + k_y^2 \\ -j\sqrt{k_x^2 + k_y^2 - k_0^2} & k_x^2 + k_y^2 > k_0^2 \end{cases}$$

Using the above definitions, we can now proceed to evaluate the Fourier transform of $\mathbf{A}$ and subsequently $\mathbf{E}_J$. Specifically, we note that

$$\mathbf{A} = \mu \iint \mathbf{J}(\mathbf{r}') \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\, dx'\, dy' = \mu\mathbf{J} * G$$

where "$*$" refers to convolution and $G = e^{-jkr}/4\pi r$. Thus,

$$\widetilde{\mathbf{A}}(k_x, k_y) = \mu\widetilde{\mathbf{J}}(k_x, k_y)\left(\frac{-j}{2}\right)\frac{e^{-jk_z|z|}}{k_z}$$

Note that since the transform is only with respect to x and y, the z dependence/variable is simply carried along. Next, noting that

$$\mathcal{F}\left\{\frac{\partial\mathbf{A}}{\partial x}\right\} = jk_x\widetilde{\mathbf{A}}, \qquad \mathcal{F}_{2D}\{\nabla\mathbf{A}\} = j\mathbf{k}_{2d}\cdot\widetilde{\mathbf{A}}(k_x, k_y)$$

where $\mathbf{k}_{2d} = \hat{x}k_x + \hat{y}k_y$, it follows that

$$\widetilde{\mathbf{E}}_J = -j\omega\widetilde{\mathbf{A}} - \frac{1}{j\omega\mu\varepsilon}\mathbf{k}_{2d}(\mathbf{k}_{2d}\cdot\widetilde{\mathbf{A}})$$

More explicitly, the x component of this field is given by

$$\widetilde{E}_{J_x}(k_x, k_y, z) = -j\frac{Z_0}{k_0}(k_0^2 - k_x^2)\,\widetilde{J}_x(k_x, k_y)\left(\frac{-j}{2}\right)\frac{e^{-jk_z|z|}}{k_z}$$

and after inverse transformation we get

$$E_{J_x}(x, y, z) = -\frac{Z_0}{2k_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{(k_0^2 - k_x^2)}{k_z}\,\widetilde{J}_x(k_x, k_y)\right] e^{-jk_z|z|}e^{-jk_x x}e^{-jk_y y}\, dk_x\, dk_y$$

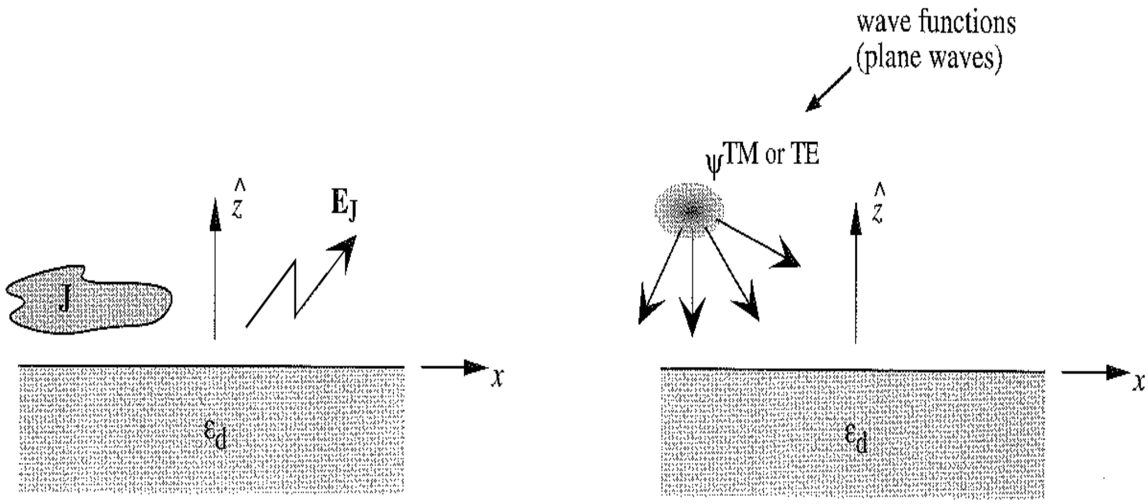
In this expression, we can identify the wave function

$$\psi(x, y, z) = e^{-j(k_x x + k_y y \pm k_z z)}, \qquad z \gtrless 0$$

and consequently E_{J_x} can be more compactly written as

$$E_{J_x}(x, y, z) = \iint_{-\infty}^{\infty} C(k_x, k_y)\,\psi(x, y, z)\, dk_x\, dk_y$$

where $C(k_x, k_y)$ is a spectral function identified from the previous integral. We next proceed to solve the problem due to the simple wave function excitation as illustrated below.



If the solution to $\psi(x,y,z)$ is

$$R(k_x,k_y)\,\psi(x,y,z)$$

then the resulting radiated field in the presence of the dielectric interface is (x component only)

$$E_{J_x}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(k_x,k_y)\,R(k_x,k_y)\,\psi(x,y,z)\,dk_x\,dk_y$$

In conclusion, to solve problems involving arbitrary sources, an approach to simplify the problem is as follows:

1) Decompose the radiated field into a summation/integral of plane waves or wave functions. This process is referred to as *plane wave or spectral representation* and amounts to performing a Fourier transformation of E as done above.

2) Obtain the solution for each plane wave and reinsert it into the spectral or Fourier integral to get the solution for the arbitrary source.

Summary of Equations for Potentials

See Balanis, pp. 256–266; Harrington, pp. 77, 127

1) Magnetic vector potential

$$\mathbf{H}_e = \frac{1}{\mu}\nabla\times\mathbf{A} \qquad (\text{Harrington uses } \mathbf{H}_e = \nabla\times\mathbf{A})$$

$$\mathbf{E}_e = -j\omega\mathbf{A} - \nabla\Phi_e = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{A} \qquad (\nabla\cdot\mathbf{A} = -j\omega\mu\varepsilon\Phi_e)$$

In terms of Hertz potentials

$$\mathbf{E}_e = k^2\mathbf{\Pi}_e + \nabla\nabla\cdot\mathbf{\Pi}_e, \qquad \mathbf{\Pi}_e = \frac{\mathbf{A}}{j\omega\mu\varepsilon}$$

Wave equations satisfied by A and Φ_e

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J} \qquad \nabla^2\mathbf{\Pi}_e + k^2\mathbf{\Pi}_e = -\frac{\mathbf{J}}{j\omega\varepsilon}$$

$$\nabla^2\Phi_e + k^2\Phi_e = -\frac{\rho}{\varepsilon} \qquad \rho = -\frac{1}{j\omega}\nabla\cdot\mathbf{J}$$

Solution (free space, $k \to k_0$)

$$\mathbf{A} = \mu \underset{\substack{\text{line, area}\\ \text{or volume}}}{\int} \mathbf{J}(\mathbf{r}')\,G_0(\mathbf{r},\mathbf{r}')\,d\mathbf{r}' \qquad G_0(\mathbf{r},\mathbf{r}') = \begin{cases} \dfrac{e^{-jk_0|z-z'|}}{2jk_0} & \text{1D} \\ -\dfrac{j}{4}H_0^{(2)}(k_0|\boldsymbol{\rho}-\boldsymbol{\rho}'|) & \text{2D} \\ \dfrac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{3D} \end{cases}$$

Radiation conditions satisfied by G_0

$$\text{1D:} \quad \frac{\partial G_0}{\partial z} + jk_0G_0 = 0, \qquad z\to\infty$$

$$\text{2D:} \quad \sqrt{\rho}\left(\frac{\partial G_0}{\partial\rho} + jk_0G_0\right) = 0, \qquad \rho\to\infty$$

$$\text{3D:} \quad r\left(\frac{\partial G_0}{\partial r} + jk_0G_0\right) = 0, \qquad r\to\infty$$

$$\Phi_e(\mathbf{r}) = \underset{\substack{\text{line, area}\\ \text{or volume}}}{\int} \frac{\rho(\mathbf{r}')}{\varepsilon}G_0(\mathbf{r},\mathbf{r}')\,d\mathbf{r}', \qquad \mathbf{\Pi}_e(\mathbf{r}) = \int\frac{\mathbf{J}(\mathbf{r}')}{j\omega\varepsilon}G_0(\mathbf{r},\mathbf{r}')\,d\mathbf{r}'$$

2) Electric vector potential

$$\mathbf{E}_\mathrm{m} = -\frac{1}{\varepsilon}\nabla\times\mathbf{F}$$

$$\mathbf{H}_\mathrm{m} = -j\omega\mathbf{F} - \nabla\Phi_\mathrm{m} = -j\omega\mathbf{F} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\mathbf{F} \qquad (\nabla\cdot\mathbf{F} = -j\omega\mu\varepsilon\Phi_\mathrm{m})$$

In terms of Hertz potentials

$$\mathbf{H}_\mathrm{m} = k^2\mathbf{\Pi}_\mathrm{m} + \nabla\nabla\cdot\mathbf{\Pi}_\mathrm{m}, \qquad \mathbf{\Pi}_\mathrm{m} = \frac{\mathbf{F}}{j\omega\mu\varepsilon}$$

Wave equations satisfied by F and Φ_m

$$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\varepsilon\mathbf{M} \qquad \nabla^2\mathbf{\Pi}_\mathrm{m} + k^2\mathbf{\Pi}_\mathrm{m} = -\frac{\mathbf{M}}{j\omega\mu}$$

$$\nabla^2\Phi_\mathrm{m} + k^2\Phi_\mathrm{m} = -\frac{\rho_\mathrm{m}}{\mu} \qquad \rho_\mathrm{m} = -\frac{1}{j\omega}\nabla\cdot\mathbf{M}$$

Solution (free space)

$$\mathbf{F}(\mathbf{r}) = \varepsilon \int_{\substack{\text{line, area}\\ \text{or volume}}} \mathbf{M}(\mathbf{r}')\, G_0(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'$$

$$\Phi_\mathrm{m}(\mathbf{r}) = \int_{\substack{\text{line, area}\\ \text{or volume}}} \frac{\rho_\mathrm{m}(\mathbf{r}')}{\mu}\, G_0(\mathbf{r},\mathbf{r}')\, d\mathbf{r}'$$

Duality

$$\begin{aligned}
\text{If } \mathbf{J}\to\mathbf{M} \quad &\Rightarrow \quad \mathbf{A}\to\mathbf{F} \text{ (provided } \varepsilon\leftrightarrow\mu)\\
& \quad \Phi_\mathrm{e}\to\Phi_\mathrm{m} \text{ (provided } \varepsilon\leftrightarrow\mu)\\
\text{If } \mathbf{M}\to -\mathbf{J} \quad &\Rightarrow \quad \mathbf{F}\to -\mathbf{A}\\
& \quad \Phi_\mathrm{m}\to -\Phi_\mathrm{e}
\end{aligned}$$

Superposition of solutions from J and M

$$\begin{aligned}
\mathbf{E} = \mathbf{E}_\mathrm{e} + \mathbf{E}_\mathrm{m} &= -j\frac{kZ}{\mu}\left(\mathbf{A} + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{A}\right) - \frac{1}{\varepsilon}\nabla\times\mathbf{F}\\
\updownarrow \text{Dual} \qquad &= k^2\mathbf{\Pi}_\mathrm{e} + \nabla\nabla\cdot\mathbf{\Pi}_\mathrm{e} - j\omega\mu\nabla\times\mathbf{\Pi}_\mathrm{m}\\
\mathbf{H} = \mathbf{H}_\mathrm{e} + \mathbf{H}_\mathrm{m} &= -j\frac{kY}{\varepsilon}\left(\mathbf{F} + \frac{1}{k^2}\nabla\nabla\cdot\mathbf{F}\right) + \frac{1}{\mu}\nabla\times\mathbf{A}\\
&= k^2\mathbf{\Pi}_\mathrm{m} + \nabla\nabla\cdot\mathbf{\Pi}_\mathrm{m} + j\omega\varepsilon\nabla\times\mathbf{\Pi}_\mathrm{e}
\end{aligned}$$