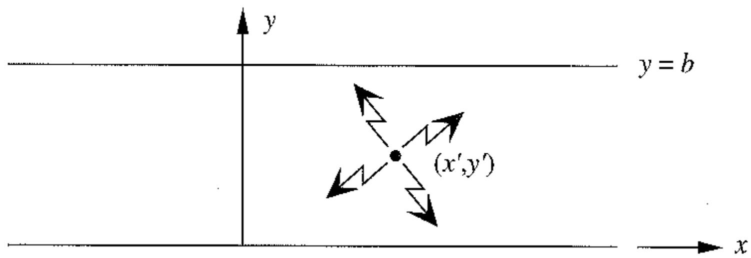
Two-Dimensional Green's Function

Consider the parallel plate waveguide excited by a line source at (x', y'):



We are interested in finding the Green's function of this configuration subject to the boundary conditions $G = 0$ at $y = 0, b$ and the radiation conditions at $x \to \pm\infty$.

We have

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \beta_0^2 G = -\delta(x - x')\,\delta(y - y'), \qquad \beta_0 = k_0 = \omega_0\sqrt{\mu\varepsilon}$$

and we propose the solution (by invoking separation of variables)

$$G = G_x G_y = G_x(x, x')\,G_y(y, y')$$

As done for the one-dimensional case we may choose (this sum is most appropriate for G_y since ψ_n will be sinusoidal and not decaying exponentials)

$$G_y = \sum_{n=-\infty}^{\infty} a_n \psi_n(y)$$

with

$$\psi_n''(y) + \beta_n^2 \psi_n(y) = 0, \qquad \psi_n = 0 \text{ at } y = 0, b$$

It follows that

$$\psi_n(y) = A \sin\frac{n\pi}{b}y = \sqrt{\frac{2}{b}}\,\sin\frac{n\pi}{b}y \qquad \text{with } \beta_n = \frac{n\pi}{b}$$

Note that $\psi_n(y)$ are the eigenfunctions and β_n are the associated eigenvalues of the problem.

We have chosen A so that

$$\int_0^b \psi(y)\,\psi_m(y)\,dy = \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

i.e., ψ_n are orthonormal eigenfunctions.

Then

$$G = \sqrt{\frac{2}{b}} \sum_n a_n\,G_x(x, x') \sin\frac{n\pi}{b}y = \sum_n a_n\,\psi_n(y)\,G_x(x, x')$$

Substituting this into the D.E. satisfied by G gives

$$\sum_n a_n[G_x''\psi_n + G_x\psi_n'' + \beta_0^2 G_x\psi_n] = -\delta(x - x')\,\delta(y - y')$$

Next we multiply both sides by ψ_m and integrate with respect to y to get (this step eliminates $\delta(y-y')$ and the y dependence)

$$a_m[G_x'' + (\beta_0^2 - \beta_m^2)G_x] = -\delta(x-x')\,\psi_m(y')$$

This leads to

$$\widetilde{G}_x'' + (\beta_0^2 - \beta_n^2)\widetilde{G}_x = -\delta(x-x')$$

where

$$\widetilde{G}_x = G_x \underbrace{\frac{a_n}{\psi_n(y')}}_{\text{constant}}$$

Clearly, we have now reduced the 2D problem down to a corresponding 1D problem.

The solution of this latter D.E. was already given earlier and we have (since only the radiation condition need be satisfied)

$$\widetilde{G}_x = \left\{ \begin{array}{ll} \dfrac{e^{-j\beta_{gn}x}e^{+j\beta_{gn}x'}}{2j\beta_{gn}} & x > x' \\ \dfrac{e^{-j\beta_{gn}x'}e^{+j\beta_{gn}x}}{2j\beta_{gn}} & x < x' \end{array} \right\} = \frac{e^{-j\beta_{gn}|x-x'|}}{2j\beta_{gn}}$$

with

$$\beta_{gn} = \begin{cases} \sqrt{\beta_0^2 - \beta_n^2} & \beta_0 > \beta_n, \quad \beta_0 = k_0 = \omega_0\sqrt{\mu_0\varepsilon_0} \\ -j\sqrt{\beta_n^2 - \beta_0^2} & \beta_0 < \beta_n \end{cases}$$

Returning back to G, we now have

$$G = G_x G_y = \sum_n a_n \psi_n(y)\, G_x = \sum_n a_n\, \psi_n(y) \left[\frac{\psi_n(y')}{a_n}\right] \widetilde{G}_x$$

$$\Longrightarrow G = \sum_n \psi_n(y)\,\psi_n(y')\,\frac{e^{-j\beta_{gn}|x-x'|}}{2j\beta_{gn}}$$

$$\boxed{G = \frac{2}{b}\sum_n \sin\frac{n\pi}{b}y \sin\frac{n\pi}{b}y'\,\frac{e^{-j\beta_{gn}|x-x'|}}{2j\beta_{gn}}}$$

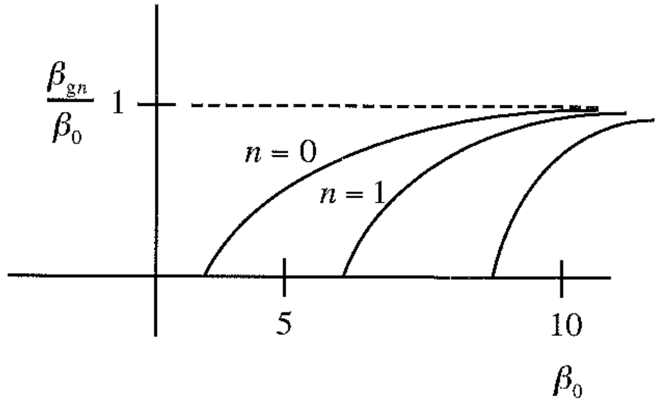
when $\beta_0 = 0$ (static case), it follows that

$$G = \sum_n \frac{1}{n\pi}\sin\frac{n\pi}{b}y \sin\frac{n\pi}{b}y'\, e^{-\beta_n|x-x'|}$$

and as expected G decays in the direction of $\pm x$, since there is no propagation at DC.

Each of the summands in the expression for G corresponds to a mode. The specific modal functions or eigenfunctions are given by

$$\phi_n(x,y) = C\sin\frac{n\pi}{b}y\, e^{\pm j\beta_{gn}x}, \qquad C \text{ is a constant}$$



These modes propagate in the $\pm x$ direction, and since $\phi_n = 0$ at $y = 0, b$ (on the PEC surfaces), they can represent the electric field component E_z (TE wave) or E_x (TM). The mode phase velocity is

$$\upsilon_{np} = \frac{\omega}{\beta_{gn}} = \frac{\upsilon_{0p}}{\sqrt{1-\left(\frac{n\pi}{\beta_0 b}\right)^2}} = \frac{c_0}{N\sqrt{1-\left(\frac{n\pi}{\beta_0 b}\right)^2}} > \upsilon_{0p}$$

in which c_0 is the speed of light. Note that υ_{np} is mode dependent and frequency dependent (i.e., the waveguide is dispersive). Also,

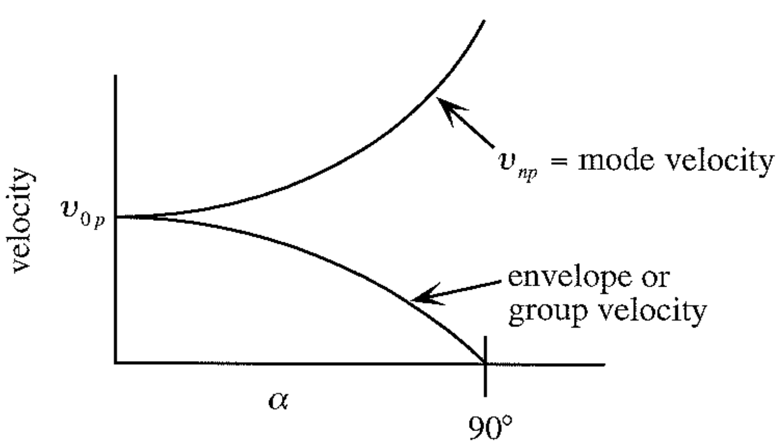
$$\upsilon_{0p}\big|_{n=0} = \frac{\omega}{\beta_{g0}} = \frac{\omega}{\beta_0} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{C_0}{N}$$

If we define

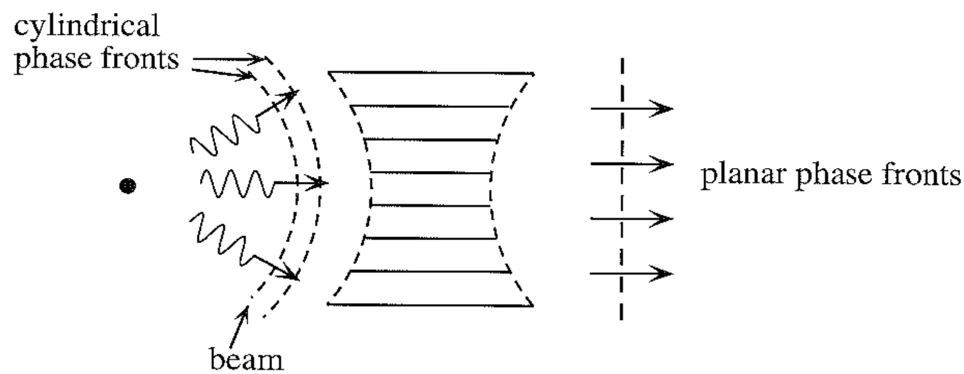
$$N_{\text{eq}} = N\sqrt{1-\left(\frac{n\pi}{\beta_0 b}\right)^2} \qquad \Longrightarrow \qquad \upsilon_{np} = \frac{c_0}{N_{\text{eq}}}$$

and we observe that

$$N_{\text{eq}} \leq N$$



This is the basis for designing microwave lenses.



Because the phase velocity in each guide formed by a pair of plates is greater than υ_{0p} for a propagating mode (say TE_1), a cylindrical wave (or a beam) may exit as a plane wave on the right side.

Whether a mode propagates or attenuates as it travels along the x direction, this depends on the value of the propagation constant

$$\beta_{gn} = \begin{cases} \sqrt{\beta_0^2 - \beta_n^2} & \beta_0 > \beta_n \\ -j\sqrt{\beta_n^2 - \beta_0^2} & \beta_0 < \beta_n \end{cases}$$

For $\beta_0 > \beta_n$ the propagation factor is

$$e^{\pm j\sqrt{\beta_0^2-\beta_n^2}\,x} \qquad \text{(no attenuation)}$$

and for $\beta_n > \beta_0$ we have

$$e^{-\sqrt{\beta_n^2-\beta_0^2}\,|x|} \qquad \text{(attenuating)}$$

The transition from a propagating to an attenuating mode occurs when

$$\sqrt{\beta_0^2 - \beta_n^2} = 0 \qquad \omega_{0c}^2\mu\varepsilon = \left(\frac{n\pi}{b}\right)^2 \quad \Longrightarrow$$

$$\omega_{0c} = \frac{1}{\sqrt{\mu\varepsilon}}\left(\frac{n\pi}{b}\right) = 2\pi f_{0c} \quad \Longrightarrow$$

$$f_{0c} = \frac{n}{2b\sqrt{\mu\varepsilon}} = \text{cutoff frequency for the } n\text{th mode}$$

That is, if the excitation frequency is below f_{0c} ($f < f_{0c}$) the nth mode will not propagate. The associated cutoff wavelength is

$$\lambda_{0c} = \frac{\upsilon_{0p}}{f_{0c}} = \frac{1}{\sqrt{\mu\varepsilon}\, f_{0c}} = \frac{2b}{n}$$

The wavelength of the mode in the guide is

$$\lambda_g = \frac{2\pi}{\beta_{gn}} = \frac{2\pi}{\beta_0\sqrt{1-\left(\frac{\omega_{0c}}{\omega_0}\right)^2}} = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_{0c}}\right)^2}}$$

$$\boxed{\lambda_0 = \frac{\upsilon_{0p}}{f_0} = \frac{2\pi\upsilon_{0p}}{\omega_0}}$$

Group velocity

Each mode in the guide can be rewritten as

$$C\left(\frac{e^{j(n\pi/b)y}+e^{-j(n\pi/b)y}}{2j}\right)e^{\pm j\beta_{gn}x}$$

This constitutes the superposition of two plane waves ϕ_n^+ and ϕ_n^-, where

$$\phi_n^{\pm}=e^{-j[\beta_{gn}x\pm(n\pi/b)y]}\qquad \beta_{gn}=\sqrt{\beta_0^2-\left(\frac{n\pi}{b}\right)^2}$$

Recalling the typical plane wave expression

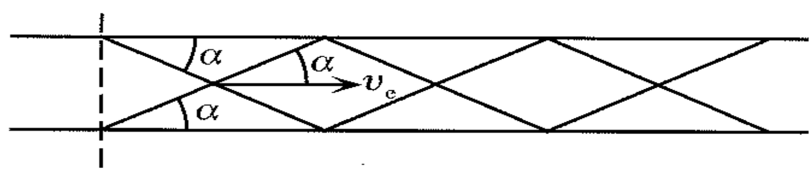
$$\phi\sim e^{-j\beta_0(x\cos\phi\pm y\sin\phi)}$$

we introduce the substitutions

$$\frac{n\pi}{b}=\beta_0\sin\alpha\quad\Longrightarrow\quad\beta_{gn}=\sqrt{\beta_0^2-\beta_0^2\sin^2\alpha}=\beta_0\cos\alpha$$

Thus

$$\phi_n^{\pm}=e^{-j\beta_0(x\cos\alpha\pm y\sin\alpha)}$$



The energy or group velocity for this mode/wave is given by

$$\upsilon_e=\upsilon_{0p}\cos\alpha=\underbrace{\left(\frac{c_0}{N}\right)}_{\upsilon_{0p}}\underbrace{\left(\frac{\beta_{gn}}{\beta_0}\right)}_{\cos\alpha}=\frac{c_0}{N}\sqrt{1-\left(\frac{n\pi}{\beta_0 b}\right)^2}\le\upsilon_{0p}$$

which is also a function of the excitation frequency. Note that in general

$$\upsilon_e=\frac{d\omega_0}{d\beta_g}=\frac{1}{\left(\dfrac{d\beta_g}{d\omega_0}\right)}$$

where β_g or β_{gn} is the propagation constant of the guided wave. Since

$$\begin{aligned}\frac{d\beta_g}{d\omega_0}=\frac{d}{d\omega_0}\left(\sqrt{\beta_0^2-\left(\frac{n\pi}{b}\right)^2}\right)&=\frac{d}{d\omega_0}\left(\sqrt{\omega_0^2\mu\varepsilon-\left(\frac{n\pi}{b}\right)^2}\right)\\&=\frac{\omega_0\mu\varepsilon}{\sqrt{\omega_0^2\mu\varepsilon-\left(\frac{n\pi}{b}\right)^2}}\\&=\frac{N}{c_0}\frac{1}{\sqrt{1-\left(\frac{n\pi}{\beta_0 b}\right)^2}}=\frac{1}{\upsilon_e}\end{aligned}$$

This expression agrees with the above, which was derived geometrically.

If $N = \sqrt{\mu_r \varepsilon_r}$ is also a function of ω (material dispersion), then (note $\omega_0^2 \mu\varepsilon = \beta_0^2$ and $\mu\varepsilon = N^2/c_0^2$)

$$\frac{d\beta_g}{d\omega_0} = \frac{d}{d\omega_0}\left(\sqrt{\beta_0^2 - \frac{n\pi}{b}}\right) = \frac{d}{d\omega_0}\left(\sqrt{\omega_0^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2}\right)$$

$$= \frac{\beta_0^2 c_0^2}{N^2}\frac{1}{2}\frac{d}{d\omega_0}(\omega_0^2\mu\varepsilon)\left[\sqrt{\quad}\right]^{-1} = \frac{1}{2}\frac{(d/d\omega_0)(\omega_0^2\mu\varepsilon)}{\sqrt{\omega_0^2\mu\varepsilon - \left(\frac{n\pi}{b}\right)^2}}$$

$$\frac{d}{d\omega_0}\left(\omega_0^2\mu\varepsilon\right) = 2\omega_0\mu\varepsilon + \frac{2\omega_0^2}{c_0^2}\frac{dN}{d\omega_0}$$

$$= 2\omega_0\mu\varepsilon + \frac{2\beta_0^2 c_0^2}{N^2 c_0^2}\frac{dN}{d\omega_0} = 2\omega_0\mu\varepsilon + \frac{2\beta_0^2}{N^2}\frac{dN}{d\omega_0}$$

Thus,

$$\frac{d\beta_g}{d\omega_0} = \left(\frac{N}{c_0} + \frac{\omega_0}{c_0}\frac{dN}{d\omega_0}\right)\frac{1}{\sqrt{1 - \left(\frac{n\pi}{\beta_0 b}\right)^2}} = \frac{N}{c_0}\left(1 + \frac{\beta_0 c_0}{N^2}\frac{dN}{d\omega_0}\right)\frac{1}{\sqrt{1 - \left(\frac{n\pi}{\beta_0 b}\right)^2}}$$

and

$$\tilde{\upsilon}_e = \frac{1}{d\beta_g/d\omega_0} = \frac{c_0}{N}\left(1 - \frac{\omega_0}{N}\frac{dN}{d\omega_0}\right)\frac{1}{\sqrt{1 - \left(\frac{n\pi}{\beta_0 b}\right)^2}} = \upsilon_e\left(1 - \frac{\omega_0}{N}\frac{dN}{d\omega_0}\right)$$