Waves in lossy media

Plane waves in lossy media

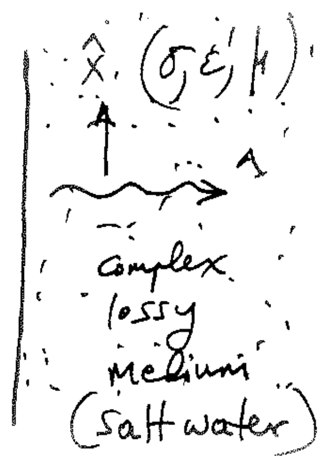
Consider the field $\mathbf{E} = \hat{x} E_0^+ e^{-\gamma z}$. Since $\epsilon = \epsilon' - j\frac{\sigma}{\omega}$,

$$\gamma = jk = j\omega\sqrt{\mu\epsilon} = \alpha + j\beta$$

or $k = \beta - j\alpha$. By substituting for ϵ, we can simplify $k = \omega\sqrt{\mu\epsilon}$ to get

$$\alpha = \frac{\omega\sqrt{\mu\epsilon'}}{\sqrt{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon'}\right)^2} - 1 \right\}^{1/2}$$

$$\beta = \frac{\omega\sqrt{\mu\epsilon'}}{\sqrt{2}} \left\{ \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon'}\right)^2} + 1 \right\}^{1/2}$$



Power density carried by a plane wave

$$\mathbf{S} = \frac{1}{2}\,\mathrm{Re}\{\mathbf{E} \times \mathbf{H}^*\} = \frac{1}{2}\,\mathrm{Re}\left\{\mathbf{E} \times \frac{\hat{k} \times \mathbf{E}}{Z}\right\}$$

from

$$\begin{aligned}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ \mathbf{E} \times (\hat{k} \times \mathbf{E}^*) &= \mathbf{E} \cdot \mathbf{E}^* \hat{k} - (\mathbf{E} \cdot \hat{k}) E^*\end{aligned}$$

and since $\hat{k} \cdot \mathbf{E} = 0$, we get

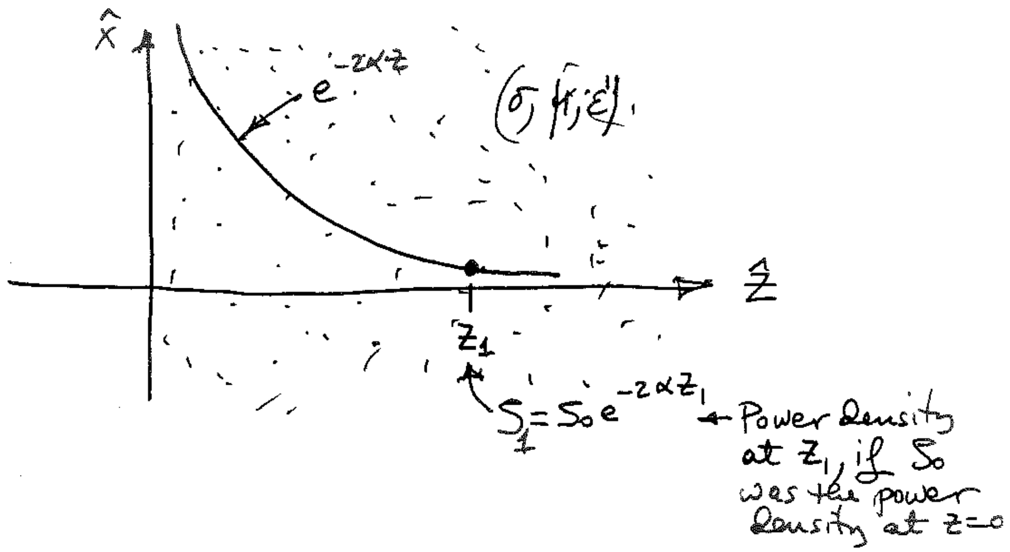
$$\begin{aligned}\mathbf{S} = \hat{k}\,\frac{1}{2}\,\mathrm{Re}\left\{\frac{\mathbf{E} \cdot \mathbf{E}^*}{Z}\right\} &= \frac{1}{2}\,\hat{k}|\mathbf{E}|^2\,\mathrm{Re}\left\{\frac{1}{Z}\right\} \\ &= \frac{1}{2}\,\hat{k}\,\frac{|\mathbf{E}|^2}{|Z|^2}\,\mathrm{Re}\{Z^*\} = \frac{1}{2}\,\hat{k}\,\frac{|\mathbf{E}|^2}{Z^2}\,\mathrm{Re}(z)\end{aligned}$$

If $\mathbf{E} = \mathbf{E}_0^+ e^{-\gamma z}$ ($\gamma = \alpha + j\beta$), then $|\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E}^* = |\mathbf{E}_0^+|^2 e^{-2\alpha z}$. Thus,

$$\mathbf{S} = \hat{k}\,\frac{1}{2}\,\frac{|\mathbf{E}_0^+|^2}{|Z|^2}\,\mathrm{Re}(Z)\,e^{-2\alpha z}$$

For lossless media and real Z, we have

$$\mathbf{S} = \hat{k}\,\frac{1}{2}\,\frac{|\mathbf{E}|^2}{Z}$$



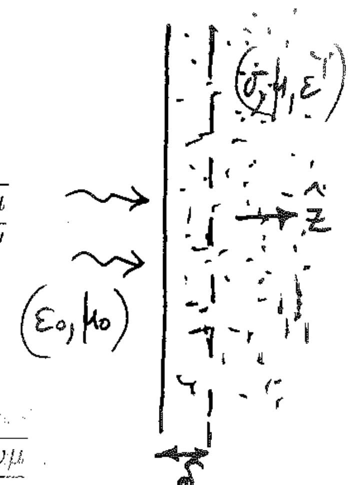
Note that since we have a plane wave,

$$\mathbf{H} = Y\hat{k}\times\mathbf{E} = \frac{\gamma}{j\omega\mu}\cdot\hat{k}\times\mathbf{E} = \sqrt{\frac{\sigma + j\omega\epsilon'}{j\omega\mu}}\,\hat{k}\times\mathbf{E} = \frac{1}{Z_{\text{TEM}}}\hat{k}\times\mathbf{E}$$

where

$$Z_{\text{TEM}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon'}}$$

Approximations for Z and γ in lossy media



1) α good dielectric: $\frac{\sigma}{\omega\epsilon} \ll 1 < \frac{1}{100}$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \qquad \beta \approx \omega\sqrt{\mu\epsilon} \qquad Z = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon'}} = \sqrt{\frac{\mu}{\epsilon'}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon'}}, \qquad v = \frac{\omega}{\beta}$$

2) good conductors: $\frac{\sigma}{\omega\epsilon} > 100$

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \qquad Z = \eta = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) = e^{j\pi/4}\sqrt{\frac{\omega\mu}{2\sigma}}$$

$$\lambda = \frac{2\pi}{\beta} = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}, \qquad v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Skin depth: distance in conductor when the transmitted wave loses 60% → (63.2%) of its strength.

$$e^{-\alpha z} = e^{-1} = 0.368 \quad \Longrightarrow$$

$$\boxed{z = \delta = \frac{1}{\alpha}}$$

Potentials and Green's functions

- Goal is to solve $\mathbf{E}$ in terms of potentials.
- Potentials are intermediate quantities.
- We avoid "dyadic Green's functions" tensors this way.

$$\textbf{Potentials:} \qquad \underbrace{\Phi_\mathrm{e}, \Phi_\mathrm{m},}_{\text{scalar}} \underbrace{\mathbf{A}, \mathbf{F}}_{\text{vector}}$$

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} - \mathbf{M} \\ \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} + \mathbf{J} \\ \nabla \cdot (\mu\mathbf{H}) &= \nabla \cdot \mathbf{B} = \rho_\mathrm{m} \\ \nabla \cdot \mathbf{D} &= \rho \end{aligned}$$

which can be decomposed into contributions from $\mathbf{J}$ or $\mathbf{M}$ as follows:

$$\begin{aligned} \nabla \times \mathbf{E}_\mathrm{e} &= -j\omega\mu\mathbf{H}_\mathrm{e} \\ \nabla \times \mathbf{H}_\mathrm{e} &= +j\omega\epsilon\mathbf{E}_\mathrm{e} + J \\ \nabla \cdot (\mu\mathbf{H}_\mathrm{e}) &= 0 \\ \nabla \cdot \mathbf{D}_\mathrm{e} &= \rho \end{aligned}$$

and

$$\begin{aligned} \nabla \times \mathbf{E}_\mathrm{m} &= -j\omega\mu\mathbf{H}_\mathrm{m} - \mathbf{M} \\ \nabla \times \mathbf{H}_\mathrm{m} &= +j\omega\epsilon\mathbf{E}_\mathrm{m} \\ \nabla \cdot (\mu\mathbf{H}_\mathrm{m}) &= \rho_\mathrm{m} \\ \nabla \cdot \mathbf{D} &= 0 \end{aligned}$$

where $\mathbf{E}_\mathrm{total} = \mathbf{E}_\mathrm{e} + \mathbf{E}_\mathrm{m}$. Since

$$\nabla \cdot (\mu\mathbf{H}_\mathrm{e}) = 0 \qquad \text{and} \qquad \nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$$
$$\Rightarrow \qquad \mu\mathbf{H}_\mathrm{e} = \nabla \times \mathbf{A}$$

then

$$\nabla \times \mathbf{E}_\mathrm{e} = -j\omega(\nabla \times \mathbf{A}) \qquad \Longrightarrow \qquad \nabla \times (\mathbf{E}_\mathrm{e} + j\omega\mathbf{A}) = 0$$

and since $\nabla \times \nabla\Phi_\mathrm{e} \equiv 0$

$$\boxed{\mathbf{E}_\mathrm{e} = -\nabla\Phi_\mathrm{e} - j\omega\mathbf{A}}$$

Now, we can solve for $\mathbf{A}$ and Φ_e instead! The equation for $\mathbf{A}$ is (from $\nabla \times \mathbf{H}_\mathrm{e} = +j\omega\mathbf{E}$)

$$\nabla \times \left(\frac{\nabla \times \mathbf{A}}{\mu}\right) = j\omega\epsilon(-j\omega\mathbf{A} - \nabla\Phi_\mathrm{e}) + \mathbf{J}$$

Introducting the identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$

gives

$$\begin{aligned} \nabla^2\mathbf{A} + \beta^2\mathbf{A} &= \nabla(\nabla \cdot \mathbf{A}) + j\omega\mu\epsilon\nabla\Phi_\mathrm{e} - \mu\mathbf{J} \\ \beta^2 &= \omega^2\mu\epsilon \end{aligned} \tag{1}$$

We can simplify this by noting that we still have freedom in specifying $\mathbf{A}$. We already specified $\nabla \times \mathbf{A} = \mu\mathbf{H}$, but this specifies only a few derivatives of $\mathbf{A}$. As is the case with the definition of any function, we need all derivatives of the function to specify $\mathbf{A}$ precisely. For example, in a Taylor series

$$f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial x\, \partial y}(x - x_0)(y - y_0) + \text{etc.}$$

The $\nabla \cdot \mathbf{A}$ has different derivatives than $\nabla \times \mathbf{A}$, and this implies that we can specify $\nabla \cdot \mathbf{A}$ without affecting $\nabla \times \mathbf{A}$. So we choose to define it conveniently, viz., set

$$\nabla(\nabla \cdot \mathbf{A}) = -j\omega\mu\epsilon\nabla\Phi_\mathrm{e}$$

implying that (after cancelling the dels)

$$\boxed{\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon\Phi_\mathrm{e}} \tag{2}$$

Thus, (1) becomes (recall $\nabla^2\mathbf{E} + \beta^2\mathbf{E} = 0$)

$$\boxed{\nabla^2\mathbf{A} + \beta^2\mathbf{A} = -\mu\mathbf{J}}$$

This is a standard inhomogeneous wave equation to be solved subject to boundary conditions.

We proceed to find an expression for $\mathbf{E}$ by combining (1) and (2) to get

$$\mathbf{E}_\mathrm{e} = -j\omega\mathbf{A} - \nabla\Phi_\mathrm{e} = -j\omega\left(\mathbf{A} + \frac{1}{\omega^2\mu\epsilon}\nabla(\nabla \cdot \mathbf{A})\right)$$

$$\boxed{\mathbf{E}_\mathrm{e} = -\frac{j\beta Z}{\mu}\left(\mathbf{A} + \frac{1}{\beta^2}\nabla\nabla \cdot \mathbf{A}\right)}$$

The equation satisfied by Φ_e can be obtained from

$$\nabla \cdot (\epsilon\mathbf{E}_\mathrm{e}) = \rho$$

or

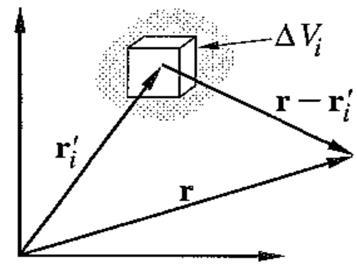
$$\nabla \cdot (-j\omega\mathbf{A} - \nabla\Phi_\mathrm{e}) = \rho \quad \Longrightarrow \quad \underbrace{-j\omega\epsilon(\nabla \cdot \mathbf{A})}_{-j\omega\epsilon(-j\omega\mu\epsilon\Phi_\mathrm{e})} - \epsilon\nabla \cdot (\nabla\Phi_\mathrm{e}) = \rho$$

$$\Longrightarrow \quad \omega^2\mu\epsilon\Phi_\mathrm{e} - \nabla^2\Phi_\mathrm{e} = \rho \quad \Longrightarrow \quad \boxed{\nabla^2\Phi_\mathrm{e} + \beta^2\Phi_\mathrm{e} = -\frac{\rho}{\epsilon}}$$

Recall that the solution of Poisson's equation is

$$\nabla^2 V \text{ (or } \Phi_e) = -\frac{\rho}{\epsilon}$$

$$\Phi_e = V = \sum_i \frac{\rho_i \, \Delta V_i}{4\pi\epsilon|\mathbf{r} - \mathbf{r}_i|}$$



or more generally

$$\Phi_e = V = \iiint \frac{\rho(\mathbf{r}')}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|} \, dv' \qquad \text{with } \mathbf{E} = -\nabla\Phi_e$$

For

$$\nabla^2\Phi_e + \beta^2\Phi = -\frac{\rho}{\epsilon}$$

it turns out that

$$\Phi_e = \iiint \frac{\rho(\mathbf{r}') \, e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|}$$
$$= \iiint \frac{\rho(\mathbf{r}')}{\epsilon} \, G(\mathbf{r}, \mathbf{r}') \, dv'$$

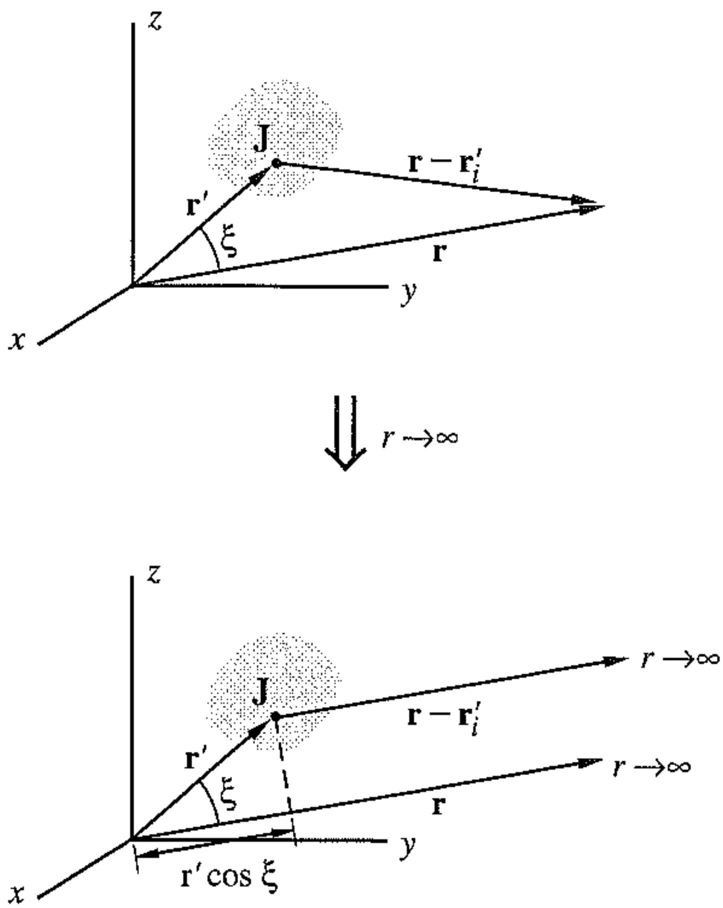
where $G(\mathbf{r}, \mathbf{r}')$ is the free space Green's function. Likewise

$$\mathbf{A} = \mu \iiint \mathbf{J}(\mathbf{r}') \, \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|}$$

In the far zone ($r \to \infty$) these are not difficult to evaluate.

Far zone approximation:

$$\mathbf{A} = \mu \iint \frac{\mathbf{J}(\mathbf{r}') \, e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi\epsilon|\mathbf{r} - \mathbf{r}'|}$$



From the figure

$$\left.\begin{array}{rcl}|\mathbf{r}-\mathbf{r}'| & \approx & r - r'\cos\xi \\ \cos\xi & = & \hat{r}'\cdot\hat{r}\end{array}\right\} \Longrightarrow |\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}'\cdot\hat{r}$$

Duality gives us $\mathbf{E}_{\mathrm{m}}$.

In this case

$$\mathbf{J} \to \mathbf{M} \qquad \text{and } \rho_{\mathrm{e}} \to \rho_{\mathrm{m}}$$

$$\mathbf{E}_{\mathrm{e}} \to \mathbf{H}_{\mathrm{m}}$$

$$\mathbf{H}_{\mathrm{e}} \to -\mathbf{E}_{\mathrm{m}}$$

$$\mathbf{E}_{\mathrm{m}} = -\frac{1}{\epsilon}\nabla \times F$$

in which F is the potential due to M.

Fields in the presence of electric and magnetic sources

$$
\begin{aligned}
\mathbf{E} &= \mathbf{E}_{\mathrm{e}} + \mathbf{E}_{\mathrm{m}} \\
&= -j\omega\mathbf{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{A} - \frac{1}{\epsilon}\nabla\times\mathbf{F} \\
\mathbf{E} &= -\frac{j\beta Z}{\mu}\left(\mathbf{A} + \frac{1}{\beta^2}\nabla\nabla\cdot\mathbf{A}\right) - \frac{1}{\epsilon}\nabla\times\mathbf{F} \\
&= \nabla\nabla\cdot\mathbf{\Pi}_{\mathrm{e}} + \beta^2\mathbf{\Pi}_{\mathrm{e}} - j\omega\mu\nabla\times\mathbf{\Pi}_{\mathrm{m}} \\
\mathbf{H} &= \mathbf{H}_{\mathrm{e}} + \mathbf{H}_{\mathrm{m}} \\
&= -j\omega\mathbf{F} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{F} + \frac{1}{\mu}\nabla\times\mathbf{A} \\
\mathbf{H} &= -\frac{j\beta Y}{\epsilon}\left(\mathbf{F} + \frac{1}{\beta^2}\nabla\nabla\cdot\mathbf{F}\right) + \frac{1}{\mu}\nabla\times\mathbf{A} \\
\mathbf{H} &= \nabla\nabla\cdot\mathbf{\Pi}_{\mathrm{m}} + \beta^2\mathbf{\Pi}_{\mathrm{m}} + j\omega\epsilon\nabla\times\mathbf{\Pi}_{\mathrm{m}}
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbf{J} &\to \mathbf{M} \\
\mathbf{M} &\to -\mathbf{J}
\end{aligned}
$$

also

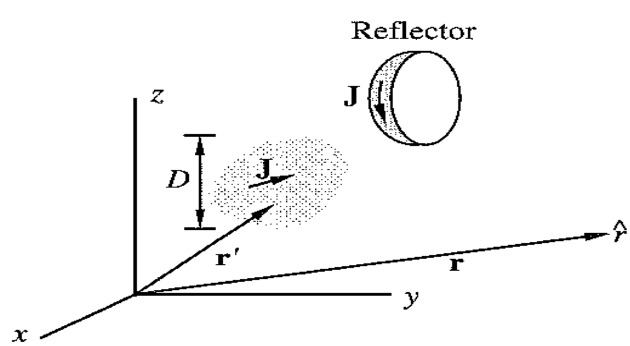
$$
\begin{aligned}
\mathbf{A} &\to \mathbf{F} \\
\mathbf{F} &\to -\mathbf{A} \\
\mathbf{\Pi}_{\mathrm{e}} &\to \mathbf{\Pi}_{\mathrm{m}} \\
\mathbf{\Pi}_{\mathrm{m}} &\to -\mathbf{\Pi}_{\mathrm{e}}
\end{aligned}
$$

Note

$$
\begin{aligned}
\nabla^2\mathbf{\Pi}_{\mathrm{e}} + \beta^2\mathbf{\Pi}_{\mathrm{e}} &= -\frac{1}{j\omega\epsilon}\mathbf{J} \\
\nabla^2\mathbf{\Pi}_{\mathrm{m}} + \beta^2\mathbf{\Pi}_{\mathrm{m}} &= -\frac{1}{j\omega\mu}\mathbf{M}
\end{aligned}
$$

Sources

3-Dimensional



$$\mathbf{A} = \mu \iint \mathbf{J}\; G(\mathbf{r}, \mathbf{r}')$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

in the far zone, and $r \gg \lambda$ (typically $r > 2D^2/\lambda$)

$$\mathbf{A} = \mu \frac{e^{-jkr}}{4\pi r} \iint \mathbf{J}(\mathbf{r}')\; e^{jk\mathbf{r}'\cdot\hat{r}}$$

in the far zone.

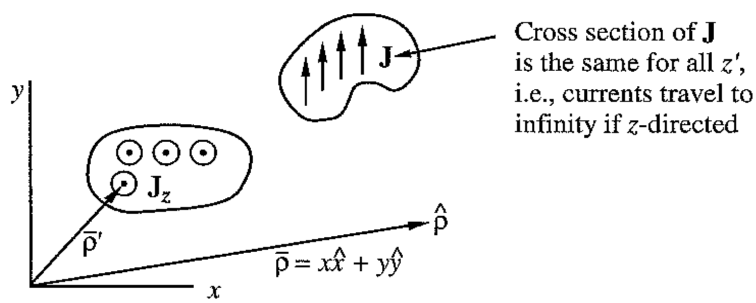
$$\mathbf{E} = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{A}$$

See the "Holy Grail of EM Radiation" handout.

$$\mathbf{E}_{\text{ff}}|_{r\to\infty} = -j\omega\mathbf{A}_{\text{trans}} = j\omega\hat{r}\times(\hat{r}\times\mathbf{A})$$

$$\boxed{\mathbf{A}_{\text{trans}} = \mathbf{A} - \hat{r}(\hat{r}\cdot\mathbf{A})}$$

2D sources



For 2D, the potential expression is the same, but $\partial/\partial z$ terms vanish, viz., $\partial/\partial z \to 0$.

$$\mathbf{A} = \mu \iint \mathbf{J}(\boldsymbol{\rho}')\; G_{\text{2D}}(\boldsymbol{\rho}, \boldsymbol{\rho}')$$

where

$$G_{\text{2D}} = \frac{-j}{4}\; H_0^{(2)}(k_0|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$$

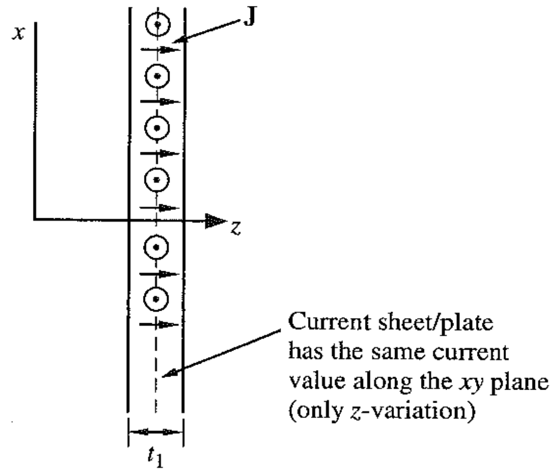
In far zone

$$\mathbf{A}_{\text{ff}} = \mu\left(\frac{-j}{4}\right)\frac{e^{-jk\rho}}{\sqrt{\rho}} \iint_{\text{cross section}} J(x', y')\; e^{+jk\boldsymbol{\rho}'\cdot\hat{\rho}}\; dx'\; dy'$$

Note that $H_0^{(2)}(k_0|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ was reduced from

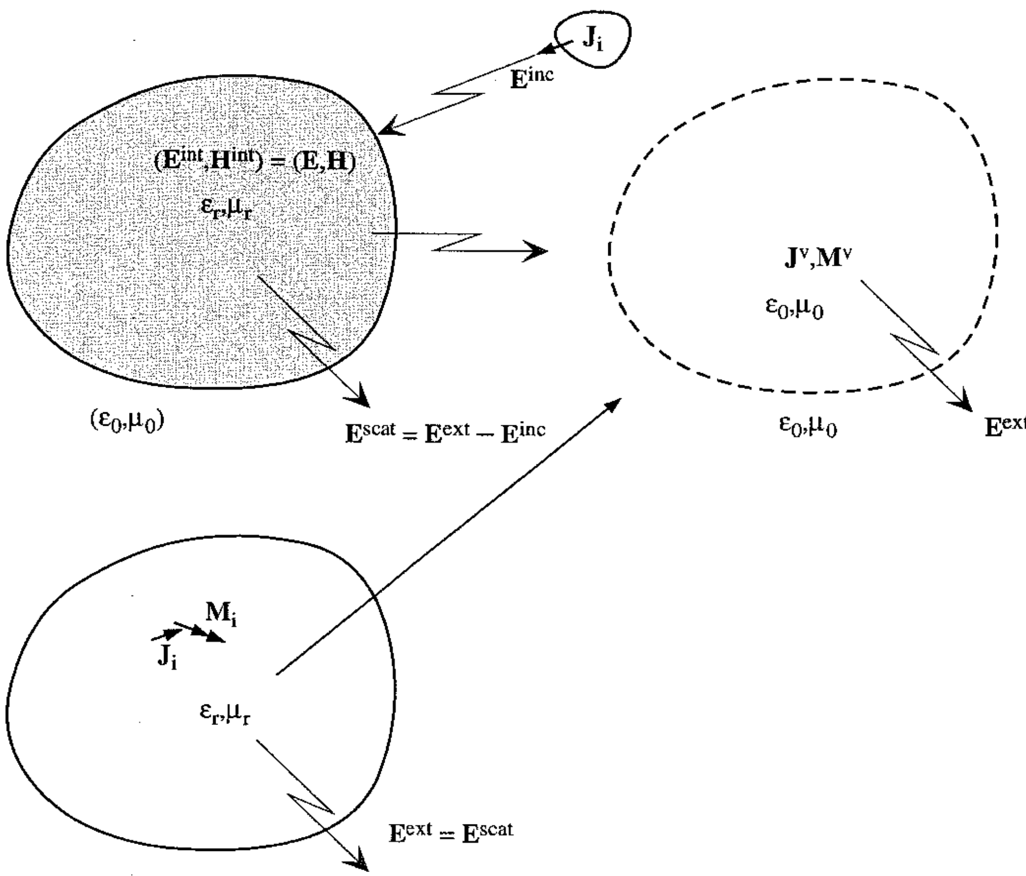
$$\frac{-j}{4}\; H_0^{(2)}(k_0|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \int_{-\infty}^{\infty} \frac{e^{-jk|\boldsymbol{\rho}-\boldsymbol{\rho}'|}}{4\pi|\boldsymbol{\rho} - \boldsymbol{\rho}'|}\; d\rho$$

1D sources



$$\mathbf{A} = \mu \int_{t_1}^{t} J(z') \, \frac{e^{-jk_0|z-z'|}}{2jk_0} \, dz'$$

Volume equivalence



The above $(\mathbf{J}_v, \mathbf{M}_v)$ are equivalent volume currents which generate the same fields $(\mathbf{E}^{ext}, \mathbf{H}^{ext})$ as if the dielectric was there. The derivation in terms of $(\mathbf{E}^{int}, \mathbf{H}^{int})$ is straightforward.

Summary of Equations for Potentials

see: Balanis : pp. 256-266; Harrington: p. 77, 127

Magnetic Vector potential

$$\bar{H}_e = \frac{1}{\mu}\nabla\times\bar{A} \quad (\text{Harrington uses } \bar{H}_e = \nabla\times\bar{A})$$

$$\bar{E}_e = -j\omega\bar{A} - \nabla\Phi_e = -j\omega\bar{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\bar{A} \quad (\nabla\cdot\bar{A} = -j\omega\mu\varepsilon\Phi_e)$$

In terms of Hertz Potentials

$$\bar{\Pi}_e = \frac{\bar{A}}{j\omega\mu\varepsilon}$$

$$\bar{E}_e = k^2\bar{\Pi}_e + \nabla\nabla\cdot\bar{\Pi}_e$$

Wave equations satisfied by $\bar{A}$ & Φ_e

$$\nabla^2\bar{A} + k^2\bar{A} = -\mu\bar{J} \qquad \nabla^2\bar{\Pi}_e + k^2\bar{\Pi}_e = -\bar{J}/j\omega\varepsilon$$

$$\nabla^2\Phi_e + k^2\Phi_e = -\rho/\varepsilon \qquad \rho = -\frac{1}{j\omega}\nabla\cdot\bar{J}$$

Solution (Free space, $k\to k_0$)

$$\bar{A} = \mu\int_{\substack{\text{line, area}\\ \text{or volume}}} \bar{J}(\bar{r}')\,G_0(\bar{r},\bar{r}')\,d\bar{r}'$$

$$G_0(\bar{r},\bar{r}') = \begin{cases} \dfrac{e^{-jk_0|z-z'|}}{2jk_0} & 1D \\[2mm] -\dfrac{j}{4}H_0^{(2)}(k_0|\bar{\rho}-\bar{\rho}'|) & 2D \\[2mm] \dfrac{e^{-jk_0|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} & 3D \end{cases}$$

Radiation Conditions satisfied by G_0

1D: $\dfrac{\partial G_0}{\partial z} + jk_0G_0 = 0, \quad z\to\infty$

2D: $\sqrt{\rho}\left(\dfrac{\partial G_0}{\partial \rho} + jk_0G_0\right) = 0, \quad \rho\to\infty$

3D: $r\left(\dfrac{\partial G_0}{\partial r} + jk_0G_0\right) = 0, \quad r\to\infty$

$$\Phi_e(\bar{r}) = \int_{\substack{\text{line, area}\\ \text{or volume}}} \frac{\rho(\bar{r}')}{\varepsilon}G_0(\bar{r},\bar{r}')\,d\bar{r}', \qquad \bar{\Pi}_e(\bar{r}) = \int \frac{\bar{J}(\bar{r}')}{j\omega\varepsilon}G(\bar{r},\bar{r}')\,d\bar{r}'$$

2) Electric Vector Potential

$$\overline{E}_m = -\frac{1}{\varepsilon}\nabla\times\overline{F}$$

$$\overline{H}_M = -j\omega\overline{F} - \nabla\Phi_M = -j\omega\overline{F} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\overline{F} \quad (\nabla\cdot\overline{F} = -j\omega\mu\varepsilon\Phi_M)$$

In terms of Hertz Potentials

$$\overline{\Pi}_m = \overline{F}/j\omega\mu\varepsilon$$

$$\overline{H}_M = k^2\overline{\Pi}_M + \nabla\nabla\cdot\overline{\Pi}_M$$

Wave equations satisfied by $\overline{F}$ & Φ_M

$$\nabla^2\overline{F} + k^2\overline{F} = -\varepsilon\overline{M} \qquad \nabla^2\overline{\Pi}_m + k^2\overline{\Pi}_m = -\overline{M}/j\omega\mu$$

$$\nabla^2\Phi_m + k^2\Phi_M = -\rho_m/\mu \qquad \rho_m = -\frac{1}{j\omega}\nabla\cdot\overline{M}$$

Solutions (free space)

$$\overline{F}(\overline{r}) = \varepsilon\int_{\substack{\text{line, area}\\ \text{or Volume}}} \overline{M}(\overline{r}')\,G_0(\overline{r},\overline{r}')\,d\overline{r}'$$

$$\Phi_M(\overline{r}) = \int_{\substack{\text{line, area}\\ \text{or Volume}}} \frac{\rho_m(\overline{r}')}{\mu}\,G_0(\overline{r},\overline{r}')\,d\overline{r}'$$

Duality: If $\overline{J}\to\overline{M} \Rightarrow \overline{A}\to\overline{F}$ (provided $\varepsilon\leftrightarrow\mu$)

$\Phi_e\to\Phi_M$ (provided $\varepsilon\leftrightarrow\mu$)

If $\overline{M}\to-\overline{J} \Rightarrow \overline{F}\to-\overline{A}$

$\Phi_M\to-\Phi_e$

Superposition of solutions from $\overline{J}$ and $\overline{M}$.

$$\overline{E} = \overline{E}_e + \overline{E}_M = -j\frac{kZ}{\mu}\left(\overline{A} + \frac{1}{k^2}\nabla\nabla\cdot\overline{A}\right) - \frac{1}{\varepsilon}\nabla\times\overline{F}$$

$$= k^2\overline{\Pi}_e + \nabla\nabla\cdot\overline{\Pi}_e - j\omega\mu\nabla\times\overline{\Pi}_M$$

$\updownarrow$ Dual

$$\overline{H} = \overline{H}_e + \overline{H}_M = -\frac{jkY}{\varepsilon}\left(\overline{F} + \frac{1}{k^2}\nabla\nabla\cdot\overline{F}\right) + \frac{1}{\mu}\nabla\times\overline{A}$$

$$= k^2\overline{\Pi}_m + \nabla\nabla\cdot\overline{\Pi}_m + j\omega\varepsilon\nabla\times\overline{\Pi}_e$$