MAXWELL'S TIME DEPENDENT EQUATIONS

Law	Integral Form	Differential Form
Ampere	$\oint \mathbf{H} \cdot d\mathbf{L} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot ds = I_{\text{total}}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
Faraday	$\oint \mathbf{E} \cdot d\mathbf{L} = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot ds = V$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
Gauss for electric fields	$\oint_s \mathbf{D} \cdot ds = \int_v \rho\, dv = Q_{\text{encl.}}$	$\nabla \cdot \mathbf{D} = \rho$
Gauss for magnetic fields	$\oint_s \mathbf{B} \cdot ds = 0$	$\nabla \cdot \mathbf{B} = 0$

Electric and magnetic field equations

Electric fields	
Voltage and field	$V = \oint \mathbf{E} \cdot d\mathbf{L}$
Coulomb's force law	$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}$
Gauss's law	$\oiint \mathbf{D} \cdot ds = \iiint \rho\, dv = Q$
Constitutive relation	$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}$ where $\varepsilon_r = \varepsilon/\varepsilon_0$
Capacitance	$C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$
Capacitor energy	$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$
Energy density	$w = \frac{1}{2}\varepsilon E^2$
Magnetic fields	
Ampere's law	$I = \oint \mathbf{H} \cdot d\mathbf{L} = \iint \mathbf{J} \cdot ds$
Lorentz motor law	$\mathbf{F} = (\mathbf{I} \times \mathbf{B})L$
Force between wires of two-wire transmission line	$F = \frac{\mu_0 I_1 I_2}{2\pi d} L$
Gauss's law	$\oiint \mathbf{B} \cdot ds = 0$
Faraday's law	$V = \oint \mathbf{E} \cdot d\mathbf{L} = -\iint \frac{\partial \mathbf{B}}{\partial t} \cdot ds$
Constitutive relation	$\mathbf{B} = \mu_r \mu_0 \mathbf{H}$ where $\mu_r = \mu/\mu_0$
Inductance	$\mathcal{L} = \frac{\Lambda}{I} = \frac{\mu_0 N^2 A}{l}$
Inductor energy	$W = \frac{1}{2}\mathcal{L}I^2 = \frac{1}{2}\Lambda I = \frac{1}{2}\frac{\Lambda^2}{\mathcal{L}}$
Energy density	$w = \frac{1}{2}\mu H^2$

Maxwell's Equations for Time Harmonic Fields

$$\mathcal{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}(x, y, z)e^{j\omega t}]$$
$$= \hat{x}E_{x0}\cos(\omega t + \phi_x) + \hat{y}E_{y0}\cos(\omega t + \phi_y) + \hat{z}E_{z0}\cos(\omega t + \phi_z)$$

where the complex vector

$$\mathbf{E}(x, y, z) = \hat{x}E_{x0}e^{j\phi_x} + \hat{y}E_{y0}e^{j\phi_y} + \hat{z}E_{z0}e^{j\phi_z}$$

Differential Forms

$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}$ Maxwell-Ampere Law

$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega\mu\mathbf{H}$ Faraday's Law

$\nabla \cdot (\mu\mathbf{H}) = \rho_m$

$\nabla \cdot (\epsilon\mathbf{E}) = \rho$

Continuity equations

$\nabla \cdot \mathbf{J} + j\omega\rho = 0$

$\nabla \cdot \mathbf{M} + j\omega\rho_m = 0$

Duality Relations

$\mathbf{J} \to \mathbf{M}$

$\mathbf{M} \to -\mathbf{J}$

$\mathbf{E} \to \mathbf{H}$

$\mathbf{H} \to -\mathbf{E}$

$\mu \to \epsilon$

$\epsilon \to \mu$

For Isotropic Media

$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\epsilon_r\mathbf{E}$

$\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H}$

$\mathbf{J} = \sigma\mathbf{E}$

$\mathbf{M} = \sigma_m\mathbf{H}$

$$\dot{\epsilon}_r = \epsilon_r - j\frac{\sigma}{\omega\epsilon_0} = \epsilon' - j\epsilon'' = \epsilon'(1 - j\tan\delta)$$

$$\dot{\mu}_r = \mu_r - j\frac{\sigma_m}{\omega\mu_0} = \mu' - j\mu'' = \mu'(1 - j\tan\delta_m)$$

Loss tangents

$$\tan\delta = \frac{\epsilon''}{\epsilon'}$$

$$\tan\delta_m = \frac{\mu''}{\mu'}$$

For Anisotropic Media: $\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E}$, $\bar{B} = \bar{\bar{\mu}} \cdot \bar{H}$

For Bianisotropic Media: $\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E} + \bar{\bar{\zeta}} \cdot \bar{B}$, $\bar{B} = \bar{\bar{\mu}} \cdot \bar{H} + \bar{\bar{\xi}} \cdot \bar{E}$

Integral Forms

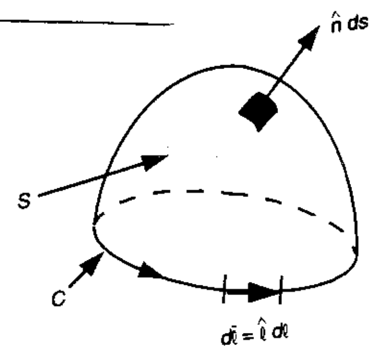
$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S (\mathbf{J}_i + j\omega\epsilon\mathbf{E}) \cdot d\mathbf{S}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S (\mathbf{M}_i + j\omega\mu\mathbf{H}) \cdot d\mathbf{S}$$

$$\oiint_{S_c} \mu\mathbf{H} \cdot d\mathbf{S} = -\iiint_V \frac{\nabla \cdot \mathbf{M}_i}{j\omega}\, dV = \iiint_V \rho_m\, dV$$

$$\oiint_{S_c} \epsilon\mathbf{E} \cdot d\mathbf{S} = -\iiint_V \frac{\nabla \cdot \mathbf{J}_i}{j\omega}\, dV = \iiint_V \rho\, dV$$

S_c encloses V



UNITS...UNITS

$\mathbf{E}$ = electric field intensity in volts/meter (V/m)

$\mathbf{H}$ = magnetic field intensity in amperes/meter (A/m)

$\mathbf{J}$ = electric current density in amperes/meter^2 (A/m^2)

$\mathbf{M}$ = magnetic current density in volts/meter^2 (V/m^2)

$\mathbf{D}$ = electric flux density in coulombs/meter^2 (C/m^2)

$\mathbf{B}$ = magnetic field density in webers/meter^2 (Wb/m^2)

ρ = electric charge density in coulombs/meter^3 (C/m^3)

ρ_m = magnetic charge density in webers/meter^3 (Wb/m^3)

ϵ_0 = free space permittivity = 8.854×10^{-12} farads/meter (F/m)

μ_0 = free space permeability = $4\pi \times 10^{-7}$ henrys/meter (H/m)

ϵ_r = medium's relative permittivity constant

μ_r = medium's relative permeability constant

σ = electric current conductivity in mhos/m ($\frac{1}{\Omega}$/m) or S/m

σ_m = magnetic current conductivity in ohms/m (Ω/m)

RADIATION CONDITION

$$\lim_{r\to\infty} r\left[\nabla \times \begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix} + jk_0\hat{r} \times \begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix}\right] = 0$$

$$k_0 = \frac{2\pi}{\lambda_0} = \omega\sqrt{\mu_0\epsilon_0}$$

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Balanis
pp. 925-926

VECTOR DIFFERENTIAL OPERATORS

The differential operators of gradient of a scalar ($\nabla\psi$), divergence of a vector ($\nabla \cdot \mathbf{A}$), curl of a vector ($\nabla \times \mathbf{A}$), Laplacian of a scalar ($\nabla^2\psi$), and Laplacian of a vector ($\nabla^2\mathbf{A}$) frequently encountered in electromagnetic field analysis will be listed in the rectangular, cylindrical, and spherical coordinate systems.

Rectangular Coordinates

$$\nabla\psi = \hat{a}_x \frac{\partial\psi}{\partial x} + \hat{a}_y \frac{\partial\psi}{\partial y} + \hat{a}_z \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \hat{a}_x\nabla^2 A_x + \hat{a}_y\nabla^2 A_y + \hat{a}_z\nabla^2 A_z$$

In class,

$\hat{a}_z = \hat{z}$

$\hat{a}_y = \hat{y}$

$\hat{a}_x = \hat{x}$

$\hat{a}_\rho = \hat{\rho}$

$\hat{a}_\phi = \hat{\phi}$

$\hat{a}_r = \hat{r}$

$\hat{a}_\theta = \hat{\theta}$

Cylindrical Coordinates

$$\nabla\psi = \hat{a}_\rho \frac{\partial\psi}{\partial\rho} + \hat{a}_\phi \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{a}_z \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \hat{a}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{a}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)$$
$$+ \hat{a}_z\left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

or in an expanded form

$$\nabla^2\mathbf{A} = \hat{a}_\rho\left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial\rho} - \frac{A_\rho}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\phi^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial^2 A_\rho}{\partial z^2}\right)$$
$$+ \hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\rho} - \frac{A_\phi}{\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi} + \frac{\partial^2 A_\phi}{\partial z^2}\right)$$
$$+ \hat{a}_z\left(\frac{\partial^2 A_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial A_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \quad \text{(II-23a)}$$

In the cylindrical coordinate system $\nabla^2\mathbf{A} \neq \hat{a}_\rho\nabla^2 A_\rho + \hat{a}_\phi\nabla^2 A_\phi + \hat{a}_z\nabla^2 A_z$ because the orientation of the unit vectors $\hat{a}_\rho$ and $\hat{a}_\phi$ varies with the ρ and ϕ coordinates.

Spherical Coordinates

$$\nabla \psi = \hat{a}_r \frac{\partial \psi}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{a}_\phi \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right] + \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

or in an expanded form

$$\begin{aligned}
\nabla^2 \mathbf{A} = {} & \hat{a}_r\left(\frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}A_r + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_r}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_r}{\partial\phi^2}\right. \\
& \left. - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
& + \hat{a}_\theta\left(\frac{\partial^2 A_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial A_\theta}{\partial r} - \frac{A_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\theta}{\partial\theta}\right. \\
& \left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
& + \hat{a}_\phi\left(\frac{\partial^2 A_\phi}{\partial r^2} + \frac{2}{r}\frac{\partial A_\phi}{\partial r} - \frac{1}{r^2\sin^2\theta}A_\phi + \frac{1}{r^2}\frac{\partial^2 A_\phi}{\partial\theta^2}\right. \\
& + \frac{\cot\theta}{r^2}\frac{\partial A_\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} \\
& \left. + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right)
\end{aligned}$$

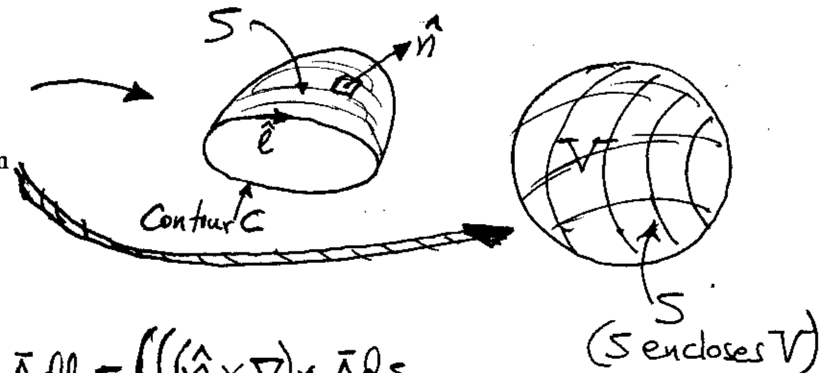
IDENTITIES $\left(\begin{array}{l}\bar{A},\bar{B},\bar{C},\bar{D} : \text{Vectors} \\ \psi,\phi : \text{Scalars}\end{array}\right)$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times \nabla\psi = 0$$
$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$$
$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$
$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$
$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$
$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A}$$
$$\nabla \times (\psi\mathbf{A}) = \nabla\psi \times \mathbf{A} + \psi\nabla \times \mathbf{A}$$
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$$
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$
$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla \cdot \nabla\psi = \nabla^2\psi$$
$$\nabla[\phi(J)] = \phi'(J) \cdot \nabla J, \quad J \text{ is a variable}$$

$$\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi$$
$$\nabla(\nabla \cdot \psi\bar{A}) = \nabla\psi\,\nabla\cdot\bar{A} + \psi\nabla\nabla\cdot\bar{A} + \nabla\psi \times (\nabla \times \bar{A}) + (\bar{A}\cdot\nabla)\nabla\psi + (\nabla\psi\cdot\nabla)\bar{A}$$
$$\nabla\times\nabla\times(\psi\bar{A}) = \nabla\psi\times(\nabla\times\bar{A}) - \bar{A}\,\nabla^2\psi + (\bar{A}\cdot\nabla)\nabla\psi + \psi\nabla\times\nabla\times\bar{A} + \nabla\psi\,\nabla\cdot\bar{A} - (\nabla\psi\cdot\nabla)\bar{A}$$

$$\bar{A}\cdot\bar{B}\times\bar{C} = \bar{B}\cdot\bar{C}\times\bar{A} = \bar{C}\cdot\bar{A}\times\bar{B}$$
$$\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$
$$(\bar{A}\times\bar{B})\cdot(\bar{C}\times\bar{D}) = \bar{A}\cdot\bar{B}\times(\bar{C}\times\bar{D}) = (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D}) - (\bar{A}\cdot\bar{D})(\bar{B}\cdot\bar{C})$$
$$(\bar{A}\times\bar{B})\times(\bar{C}\times\bar{D}) = (\bar{A}\times\bar{B}\cdot\bar{D})\bar{C} - (\bar{A}\times\bar{B}\cdot\bar{C})\bar{D}$$

$$\oint_C \mathbf{A}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{s} \quad \text{Stokes' theorem}$$
$$\oiint_S \mathbf{A}\cdot d\mathbf{s} = \iiint_V (\nabla\cdot\mathbf{A})\,dv \quad \text{divergence theorem}$$



$$\oiint_S (\hat{n}\times\mathbf{A})\,ds = \iiint_V (\nabla\times\mathbf{A})\,dv$$
$$\oiint_S \psi\,d\mathbf{s} = \iiint_V \nabla\psi\,dv$$
$$\oint_C \psi\,d\mathbf{l} = \iint_S \hat{n}\times\nabla\psi\,ds \quad \text{or} \quad \oint_C \hat{\ell}\times\bar{A}\,d\ell = \iint_S (\hat{n}\times\nabla)\times\bar{A}\,ds$$

$$\oint \psi\frac{\partial\phi}{\partial n}\,ds = \iiint (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\,d\tau \quad \text{Green's 1st Identity}$$
$$\oint (\mathbf{A}\times\nabla\times\mathbf{B})\cdot d\mathbf{s} = \iiint (\nabla\times\mathbf{A}\cdot\nabla\times\mathbf{B} - \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B})\,dv \quad \text{Vector Analog of Green's 1st Identity}$$
$$\oint\left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)ds = \iiint (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dv \quad \text{Green's Theorem (or Green's 2nd Identity)}$$
$$\oint (\mathbf{A}\times\nabla\times\mathbf{B} - \mathbf{B}\times\nabla\times\mathbf{A})\cdot d\mathbf{s} = \iiint (\mathbf{B}\cdot\nabla\times\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B})\,dv \quad \text{Vector Analog of Green's theorem}$$