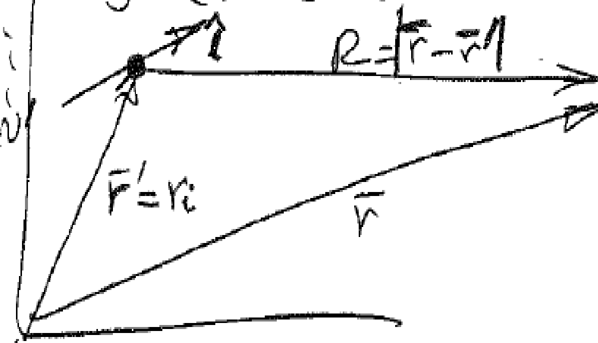
Far Zone Fields for Dipoles

$$\bar{J} = (I\ell)\hat{\ell}\,\delta(\bar{r}_i - \bar{r}) = I\ell\,\hat{\ell}\,\delta(\bar{r}-\bar{r}_i)$$



$$\bar{A} = \mu \iiint \bar{J}(\bar{r}')\,\frac{e^{-jk_0|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}\,dv'$$

line
volume
or surface

$$G(\bar{r},\bar{r}')$$

$$J(\bar{r}') = \hat{\ell}(I\ell)\,\delta(\bar{r}'-\bar{r}_i)$$

$$\Rightarrow \bar{A} = \mu\,\hat{\ell}(I\ell)\,\frac{e^{-jk_0|\bar{r}-\bar{r}_i|}}{4\pi|\bar{r}-\bar{r}_i|}$$

$$R = |\bar{r}-\bar{r}_i|$$

From

$$\boxed{\begin{aligned}\bar{E} &= -j\omega\bar{A} - \nabla\Phi_e \\ &= -j\omega\bar{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\bar{A}\end{aligned}}$$

$$\mathbf{E} = -jk\eta(I\ell)\left[1-\frac{j}{kR}-\frac{1}{(kR)^2}\right]\frac{e^{-jkR}}{4\pi R}\,\ell$$
$$+\,jk\eta(I\ell)\left[1-\frac{3j}{kR}-\frac{3}{(kR)^2}\right]\frac{e^{-jkR}}{4\pi R}\,(\hat{\ell}\cdot\hat{R})\hat{R}$$
$$\mathbf{H} = jk\eta(I\ell)\left[1+\frac{1}{jkR}\right]\frac{e^{-jkR}}{4\pi R}\,\hat{\ell}\times\hat{R}$$

Note:
$$\omega\mu\epsilon = k\sqrt{\mu\epsilon} = k/v$$

Note that these are exact expressions, something rare in electromagnetics since the integrations cannot be performed except for the simplest of the current excitations.

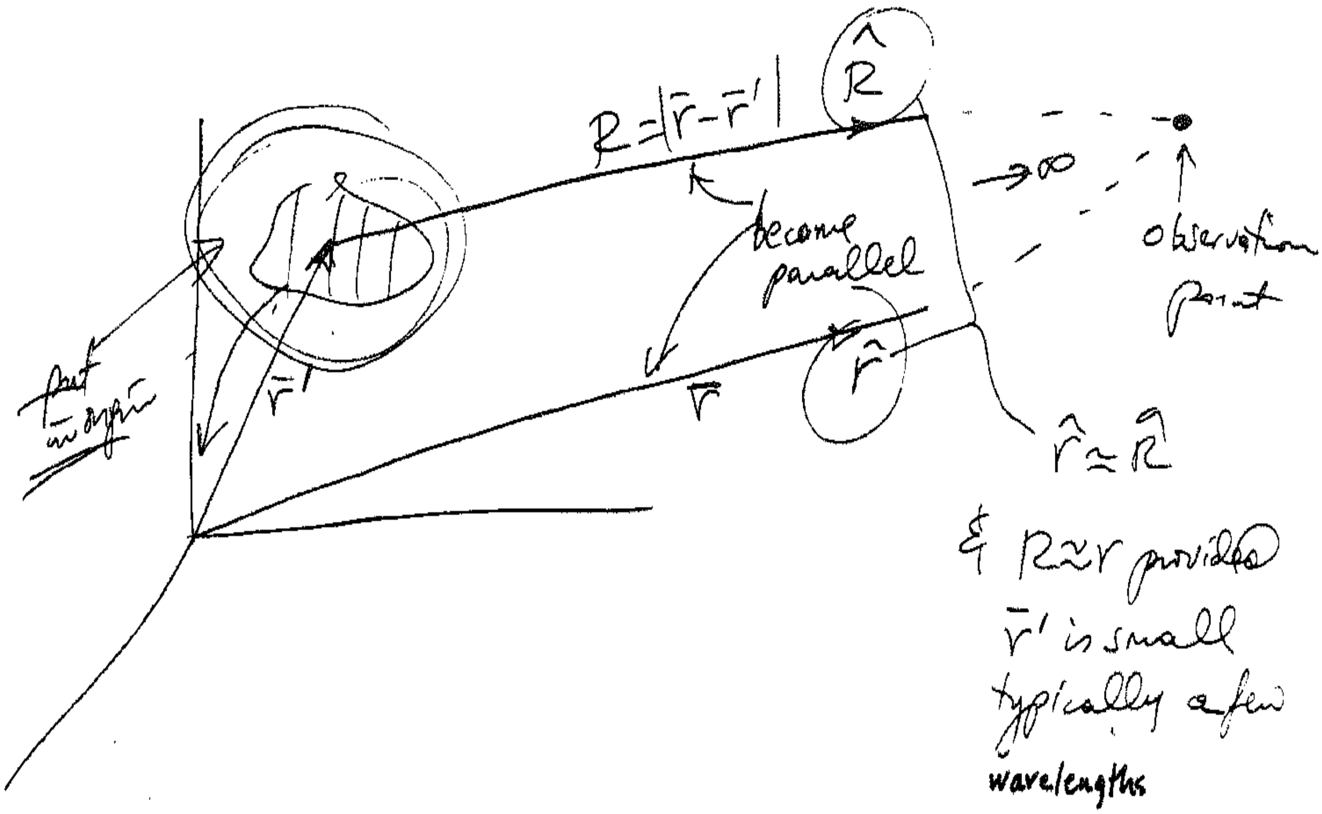
Of particular importance to us in antenna analysis and design are the field expressions when $R \to \infty$. In this case, the integral expressions simplify substantially. For example, we can drop all terms of the integrals that decay as $1/R^2, 1/R^3, \ldots$, giving

$$\mathbf{E} \approx -jk\eta \int_{\text{line, surface, or volume}} \left[\mathbf{J}(\mathbf{r}') - \hat{R}(\hat{R}\cdot\mathbf{J}(\mathbf{r}'))\right]\frac{e^{-jkR}}{4\pi R}\,dd$$
$$\mathbf{H} \approx +jk\eta \int_{\text{line, surface, or volume}} \left[\mathbf{J}(\mathbf{r}') \times \hat{R}\right]\frac{e^{-jkR}}{4\pi R}\,dd$$

where $dd = \{dl, ds, dv\}$. We can do further approximations, by noting that as $R \to \infty$,

$$\hat{R} \to \hat{r}$$
$$R \approx r \quad \text{(for amplitude terms)}$$

These approximations are apparent when we view them geometrically:



Clearly, when talking about amplitude,

$$\frac{1}{10010\lambda} \approx \frac{1}{10000\lambda}$$

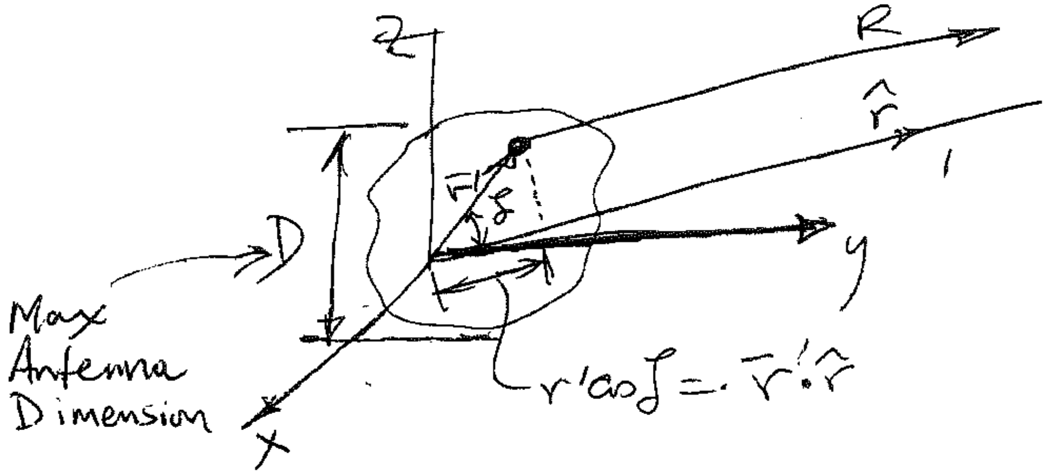
However, for phase computations, a substantial difference between

$$\begin{array}{ccc} e^{+jk(10000\lambda)} & \text{and} & e^{jk(10000.5\lambda)} \\ \downarrow & & \downarrow \\ \cos(2\pi \times 10000) & & \cos(2\pi \times 10000.5) \\ \downarrow & & \downarrow \\ 1 & & -1 \end{array}$$

exists. This implies that for the phase, R needs to be carefully considered.

One acceptable approximation is

$$R \approx r - r' \cos\zeta$$
$$\approx r - \mathbf{r}' \cdot \hat{r}$$



This is referred to as the far field approximation, and for $r/\lambda > 2D^2/\lambda^2$ it is associated with a phase error of no more than $\pi/8$, i.e., $kR - kR_{\text{approx}} \leq \pi/8$ or so! Thus we can state:

$$R = \begin{cases} R & \text{amplitude} \\ r - \mathbf{r}' \cdot \hat{r} & \text{phase term} \end{cases}$$

This is the Fraunhofer or far field approximation, valid for $r > 2D^2/\lambda$.

In the handout (see also Balanis, 2nd ed., pp. 145–151) we also show that the improved approximation

$$R_{\text{approx}} \approx r - \mathbf{r}' \cdot \hat{r} + \frac{r'^2[1 - (\mathbf{r}' \cdot \hat{r})^2]}{2r}$$

(Fresnel approximation) makes

$$k(R - R_{\text{approx}}) \leq \frac{\pi}{8} \qquad \text{for } 0.62\left(\frac{D^3}{\lambda}\right)^{1/2} < r$$

Thus, we can write

$$R = \begin{cases} r & \text{amplitude} \\ r - \mathbf{r}' \cdot \hat{r} & \text{far field, } r > \dfrac{2D^2}{\lambda} \\ r - \mathbf{r}' \cdot \hat{r} + \dfrac{r'^2[1 - (\mathbf{r}' \cdot \hat{r})^2]}{2r} & \text{Fresnel, } 0.62\left(\dfrac{D^3}{\lambda}\right)^{1/2} < r < \dfrac{2D^2}{\lambda} \end{cases}$$

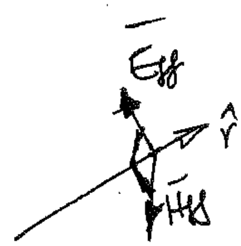
In the latter, using $\frac{1}{2}(D^4/\lambda)^{1/3}$ instead of $\frac{1}{2}(D^3/\lambda)^{1/2}$ is more exact.

With the above approximation, we write now the most simplified expression for the far field (valid for $r > 2D^2/\lambda$):

$$\mathbf{E}_{\text{ff}} = +jk\frac{e^{-jkr}}{4\pi r} \iiint [\hat{r} \times \mathbf{M}(\mathbf{r}') + \eta \hat{r} \times \hat{r} \times \mathbf{J}(\mathbf{r}')] e^{+jk\mathbf{r}' \cdot \hat{r}} \, dv$$

$$\mathbf{H}_{\text{ff}} = \frac{1}{\eta} \hat{r} \times \mathbf{E}_{\text{ff}}$$

since



$$\hat{r}\cdot\mathbf{E}_{\text{ff}} \equiv 0$$
$$\hat{r}\cdot\mathbf{H}_{\text{ff}} \equiv 0$$

i.e., $\mathbf{E}_{\text{ff}}$, $\mathbf{H}_{\text{ff}}$ behave like plane waves in the far field. Also,

$$\mathbf{A}_{\text{ff}} = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int \mathbf{J}(\mathbf{r}')\, e^{+jk\mathbf{r}'\cdot\hat{r}}\, dv \qquad \text{for } \mathbf{F}:\ \begin{matrix}\mu\to\varepsilon\\ \mathbf{J}\to M\end{matrix}$$

Most often, we are interested in the ϕ and θ components of $\mathbf{E}_{\text{ff}}$. We find

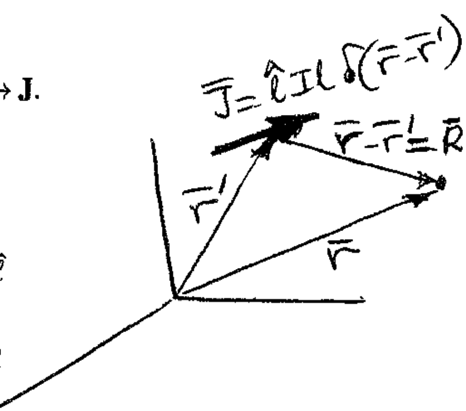
$$E_{\text{ff}_\theta} = -j\omega\eta F_\phi - j\omega A_\theta = \eta H_{\text{ff}_\phi}$$
$$E_{\text{ff}_\phi} = j\omega\eta F_\theta - j\omega A_\phi = -\eta H_{\text{ff}_\theta}$$

See handout for more simplifications.

Summary of handout

$$\mathbf{A} = \frac{\mu}{4\pi}\iiint \frac{\mathbf{J}e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$
$$\mathbf{F} = \frac{1}{\varepsilon}\iiint \frac{\mathbf{M}e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$
$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla\times\mathbf{F} - j\omega\mathbf{A} - j\frac{\nabla(\nabla\cdot\mathbf{A})}{\omega\mu\varepsilon}$$
$$\mathbf{H} = +\frac{1}{\mu}\nabla\times\mathbf{A} - j\omega\mathbf{F} - j\frac{\nabla(\nabla\cdot\mathbf{F})}{\omega\mu\varepsilon}$$
$$\mathbf{E}\to\mathbf{H}$$
$$A\to F$$
$$\mathbf{F}\to-\mathbf{A}$$
$$R = |\mathbf{r}-\mathbf{r}'|$$
$$\hat{R} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

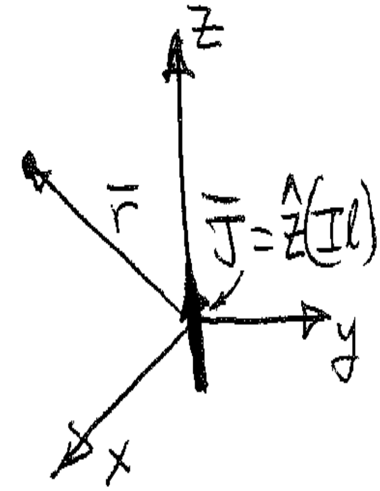
The above can be integrated over line, surface or volume once $(I\ell)\hat{\ell}\to\mathbf{J}$.



Dipole at $\mathbf{r}'$:

$$\mathbf{E} = -jkZ_0(I\ell)\left[1-\frac{j}{kR}-\frac{1}{(kR)^2}\right]\frac{e^{-jkR}}{4\pi R}\hat{\ell}$$
$$\mathbf{H} = \frac{\nabla\times\mathbf{E}}{-j\omega\mu} = jkI\ell\left[1+\frac{1}{jkR}\right]\frac{e^{-jkR}}{4\pi R}\hat{\ell}\times\hat{R}$$

Dipole at $\mathbf{r}' = 0$:



$$E_\theta = jkZ(I\ell)\sin\theta\left[1 - \frac{j}{kr} - \frac{1}{(kr)^2}\right]\frac{e^{-jkr}}{4\pi r}$$

$$E_\phi = 0$$

$$E_r = -jkZ(I\ell)\cos\theta\left[\frac{j}{kr} - \frac{1}{(kr)^2}\right]\frac{e^{-jkr}}{2\pi r}$$

$$H_\phi = jkI\ell\left[1 + \frac{1}{jkr}\right]\sin\theta\,\frac{e^{-jkr}}{4\pi r}$$

$$(H_\theta = H_r = 0)$$

Put $\iint$ for arbitrary source $(I\ell)\hat{\ell} \to \mathbf{J}$

When $r \to \infty$, there is no need to keep $1/r^2$, $1/r^3$, which decay much faster than the $1/r$ terms.

Far field expressions $(r \to \infty)$**:**

$$E_\theta = jkZ(I\ell)\sin\theta\,\frac{e^{-jkr}}{4\pi r} = \overbrace{\frac{jkZ(I\ell)}{4\pi}}^{\text{a constant}}\;\overbrace{\frac{e^{-jkr}}{r}}^{\text{appears in all far fields}}\;\overbrace{\sin\theta}^{\text{field pattern}}$$

$E_r \approx 0, \quad E_\phi = 0 \qquad E_r \cdot \hat{r} = 0$ at $r \to \infty$ for all antennas

$$H_\phi = \frac{jkI\ell}{4\pi}\,\frac{e^{-jkr}}{r}\sin\theta$$

$$\mathbf{E}_{\mathrm{ff}} = -Z\hat{r} \times \mathbf{H}_{\mathrm{ff}}$$

$$\mathbf{H}_{\mathrm{ff}} = +Y\hat{r} \times \mathbf{E}_{\mathrm{ff}}$$

for dipole but this is true for all far zone fields regardless of antenna sources (note $Z = 1/Y = \eta$).