Chapter 7

CURVILINEAR COORDINATES

TRANSFORMATION OF COORDINATES. Let the rectangular coordinates (x,y,z) of any point be expressed as functions of (u_1, u_2, u_3) so that

$$(1) \qquad x = x(u_1, u_2, u_3), \qquad y = y(u_1, u_2, u_3), \qquad z = z(u_1, u_2, u_3)$$

Suppose that *(1)* can be solved for u_1, u_2, u_3 in terms of x, y, z, i.e.,

$$(2) \qquad u_1 = u_1(x,y,z), \qquad u_2 = u_2(x,y,z), \qquad u_3 = u_3(x,y,z)$$

The functions in *(1)* and *(2)* are assumed to be single-valued and to have continuous derivatives so that the correspondence between (x,y,z) and (u_1, u_2, u_3) is unique. In practice this assumption may not apply at certain points and special consideration is required.

Given a point P with rectangular coordinates (x,y,z) we can, from *(2)* associate a unique set of coordinates (u_1, u_2, u_3) called the *curvilinear coordinates* of P. The sets of equations *(1)* or *(2)* define a *transformation of coordinates*.

ORTHOGONAL CURVILINEAR COORDINATES.

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$, where c_1, c_2, c_3 are constants, are called *coordinate surfaces* and each pair of these surfaces intersect in curves called *coordinate curves or lines* (see Fig.1). If the coordinate surfaces intersect at right angles the curvilinear coordinate system is called *orthogonal*. The u_1, u_2 and u_3 coordinate curves of a curvilinear system are analogous to the x, y and z coordinate axes of a rectangular system.

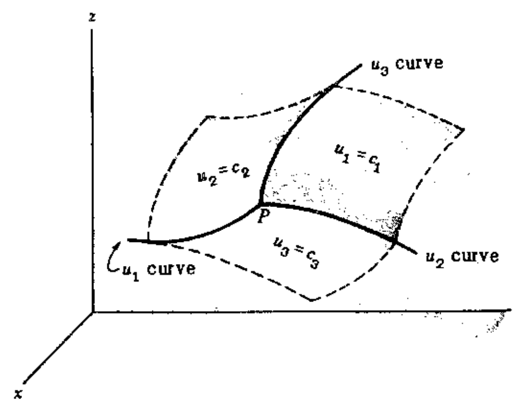

Fig. 1

UNIT VECTORS IN CURVILINEAR SYSTEMS. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P. Then *(1)* can be written $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$. A tangent vector to the u_1 curve at P (for which u_2 and u_3 are constants) is $\dfrac{\partial \mathbf{r}}{\partial u_1}$. Then a unit tangent vector in this direction is $\mathbf{e}_1 = \dfrac{\partial \mathbf{r}}{\partial u_1} \Big/ \left|\dfrac{\partial \mathbf{r}}{\partial u_1}\right|$ so that $\dfrac{\partial \mathbf{r}}{\partial u_1} = h_1 \mathbf{e}_1$ where $h_1 = \left|\dfrac{\partial \mathbf{r}}{\partial u_1}\right|$. Similarly, if $\mathbf{e}_2$ and $\mathbf{e}_3$ are unit tangent vectors to the u_2 and u_3 curves at P respectively, then $\dfrac{\partial \mathbf{r}}{\partial u_2} = h_2 \mathbf{e}_2$ and $\dfrac{\partial \mathbf{r}}{\partial u_3} = h_3 \mathbf{e}_3$ where $h_2 = \left|\dfrac{\partial \mathbf{r}}{\partial u_2}\right|$ and $h_3 = \left|\dfrac{\partial \mathbf{r}}{\partial u_3}\right|$. The quantities h_1, h_2, h_3 are called *scale factors*. The unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are in the directions of increasing u_1, u_2, u_3, respectively.

Since ∇u_1 is a vector at P normal to the surface $u_1 = c_1$, a unit vector in this direction is given by $\mathbf{E}_1 = \nabla u_1 / |\nabla u_1|$. Similarly, the unit vectors $\mathbf{E}_2 = \nabla u_2 / |\nabla u_2|$ and $\mathbf{E}_3 = \nabla u_3 / |\nabla u_3|$ at P are normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively.

Thus at each point P of a curvilinear system there exist, in general, two sets of unit vectors, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ tangent to the coordinate curves and $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ normal to the coordinate surfaces (see Fig.2). The sets become identical if and only if the curvilinear coordinate system is orthogonal (see Problem 19). Both sets are analogous to the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors in rectangular coordinates but are unlike them in that they may change directions from point to point. It can be shown (see Problem 15) that the sets $\dfrac{\partial \mathbf{r}}{\partial u_1}, \dfrac{\partial \mathbf{r}}{\partial u_2}, \dfrac{\partial \mathbf{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$ constitute reciprocal systems of vectors.

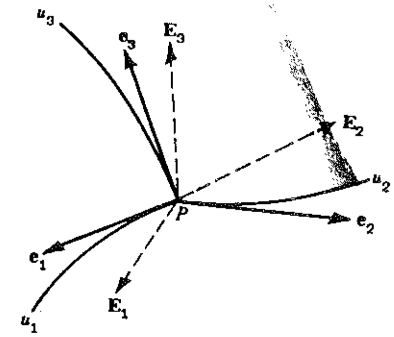

Fig. 2

A vector $\mathbf{A}$ can be represented in terms of the unit base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ or $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ in the form

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 = a_1\mathbf{E}_1 + a_2\mathbf{E}_2 + a_3\mathbf{E}_3$$

where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective *components* of $\mathbf{A}$ in each system.

We can also represent $\mathbf{A}$ in terms of the base vectors $\dfrac{\partial \mathbf{r}}{\partial u_1}, \dfrac{\partial \mathbf{r}}{\partial u_2}, \dfrac{\partial \mathbf{r}}{\partial u_3}$ or $\nabla u_1, \nabla u_2, \nabla u_3$ which are called *unitary base vectors* but *are not* unit vectors in general. In this case

$$\mathbf{A} = C_1\frac{\partial \mathbf{r}}{\partial u_1} + C_2\frac{\partial \mathbf{r}}{\partial u_2} + C_3\frac{\partial \mathbf{r}}{\partial u_3} = C_1\boldsymbol{\alpha}_1 + C_2\boldsymbol{\alpha}_2 + C_3\boldsymbol{\alpha}_3$$

and

$$\mathbf{A} = c_1\nabla u_1 + c_2\nabla u_2 + c_3\nabla u_3 = c_1\boldsymbol{\beta}_1 + c_2\boldsymbol{\beta}_2 + c_3\boldsymbol{\beta}_3$$

where C_1, C_2, C_3 are called the *contravariant components* of $\mathbf{A}$ and c_1, c_2, c_3 are called the *covariant components* of $\mathbf{A}$ (see Problems 33 and 34). Note that $\boldsymbol{\alpha}_p = \dfrac{\partial \mathbf{r}}{\partial u_p}$, $\boldsymbol{\beta}_p = \nabla u_p$, $p = 1, 2, 3$.

ARC LENGTH AND VOLUME ELEMENTS. From $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ we have

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 = h_1\,du_1\,\mathbf{e}_1 + h_2\,du_2\,\mathbf{e}_2 + h_3\,du_3\,\mathbf{e}_3$$

Then the differential of arc length ds is determined from $ds^2 = d\mathbf{r}\cdot d\mathbf{r}$. For orthogonal systems, $\mathbf{e}_1\cdot\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{e}_3 = \mathbf{e}_3\cdot\mathbf{e}_1 = 0$ and

$$ds^2 = h_1^2\,du_1^2 + h_2^2\,du_2^2 + h_3^2\,du_3^2$$

For non-orthogonal or general curvilinear systems see Problem 17.

Along a u_1 curve, u_2 and u_3 are constants so that $d\mathbf{r} = h_1\,du_1\,\mathbf{e}_1$. Then the differential of arc length ds_1 along u_1 at P is $h_1\,du_1$. Similarly the differential arc lengths along u_2 and u_3 at P are $ds_2 = h_2\,du_2$, $ds_3 = h_3\,du_3$.

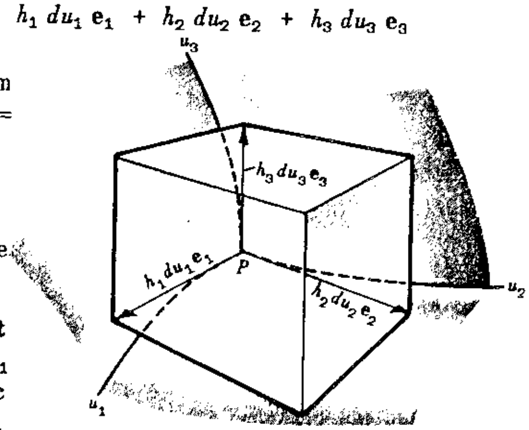

Fig. 3

Referring to Fig.3 the volume element for an orthogonal curvilinear coordinate system is given by

$$dV = |(h_1\,du_1\,\mathbf{e}_1)\cdot(h_2\,du_2\,\mathbf{e}_2)\times(h_3\,du_3\,\mathbf{e}_3)| = h_1h_2h_3\,du_1\,du_2\,du_3$$

since $|\mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3| = 1$.

THE GRADIENT, DIVERGENCE AND CURL can be expressed in terms of curvilinear coordinates. If Φ is a scalar function and $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ a vector function of orthogonal curvilinear coordinates u_1, u_2, u_3, then the following results are valid.

1. $$\nabla\Phi = \text{grad }\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3$$

2. $$\nabla\cdot\mathbf{A} = \text{div }\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

3. $$\nabla\times\mathbf{A} = \text{curl }\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

4. $$\nabla^2\Phi = \text{Laplacian of }\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$

If $h_1 = h_2 = h_3 = 1$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are replaced by $\mathbf{i}, \mathbf{j}, \mathbf{k}$, these reduce to the usual expressions in rectangular coordinates where (u_1, u_2, u_3) is replaced by (x, y, z).

Extensions of the above results are achieved by a more general theory of curvilinear systems using the methods of *tensor analysis* which is considered in Chapter 8.

SPECIAL ORTHOGONAL COORDINATE SYSTEMS.

1. Cylindrical Coordinates (ρ, ϕ, z). See Fig.4 below.

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

$$\text{where} \quad \rho \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad -\infty < z < \infty$$

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

2. Spherical Coordinates (r, θ, ϕ). See Fig.5 below.

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\text{where} \quad r \geqq 0, \quad 0 \leqq \phi < 2\pi, \quad 0 \leqq \theta \leqq \pi$$

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$$