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Chapter 3

Integral Equations and Other Field Representations

The field integral representations given in the previous chapter, although of sufficient generality are often inconvenient and possibly inefficient for specific applications. Also, their integrands are highly singular requiring special treatment, when the observation point is in the source region. This difficulty cannot be eliminated but any reduction in the integrand's singularity is desirable for achieving higher accuracies in numerical computations involving such integrals. Obviously, there are a variety of field representations, integral equations and formal solutions that could be derived, many of which can only be applicable to a specific situation. Below, we shall consider some alternative field representations to construct integral equations that are among the most frequently used. First, we shall develop three dimensional representations. Many of the two-dimensional representations can then be reduced from the three dimensional ones. However, for scattering applications a larger variety of two-dimensional representations are available primarily because the topic has been extensively studied.

3.1 Three-Dimensional Integral Equations

3.1.1 Kirchhoff's Integral Equation

Perhaps the simplest integral equation can be derived by considering the wave equations (2.86) in conjunction with Green's second identity (see (2.42)). To proceed, we assume the existance of certain structures whose surfaces will be denoted by $S_1, S_2, \ldots, S_N$. The collection of these surfaces, henceforth referred to as S_Ω (enclosing the volume V_Ω), are illuminated by sources which are enclosed within the volume V_{is}. The volume region exterior to S_Ω shall be denoted by V_∞ which, as seen, is also bounded by the surface S_∞ placed at infinity.

Without loss of generality, let us consider one of the electric field component, say E_a. Then from (2.86)

$$\nabla^2 E_a + k^2 E_a = \begin{cases} F_a(\mathbf{r}) & r \in V_{is} \\ 0 & r \notin V_{is} \end{cases} \tag{3.1}$$

in which $F_a(\mathbf{r}) = \mathbf{F}(r) \cdot \hat{a}$ represents the source terms in the right hand side of (2.86). Multiplying this by the free space Green's function and integrating yields

$$\int\int\int_{V_\infty} G(\mathbf{r}, \mathbf{r}')\left[\nabla^2 E_a(\mathbf{r}) + k^2 E_a(\mathbf{r})\right] dv = \begin{cases} \int\int\int_{V_{is}} F_a(\mathbf{r}) G(\mathbf{r}, \mathbf{r}') dv & \mathbf{r} \in V_{is} \\ 0 & \mathbf{r} \notin V_{is} \end{cases} \tag{3.2}$$

and we remark that V_∞ includes the source volume V_{is}. Also, from Green's second identity (see (2.42)) we have

$$\begin{aligned} &\int\int\int_{V_\infty} \left[E_a(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla^2 E_a(\mathbf{r})\right] dv \\ &= -\oiint_{S_\Omega} \left[E_a(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} - G(\mathbf{r}, \mathbf{r}')\frac{\partial E_a(\mathbf{r})}{\partial n}\right] ds \\ &+ \oiint_{S_\infty} \left[E_a(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} - G(\mathbf{r}, \mathbf{r}')\frac{\partial E_a(\mathbf{r}, \mathbf{r}')}{\partial n}\right] ds \end{aligned} \tag{3.3}$$

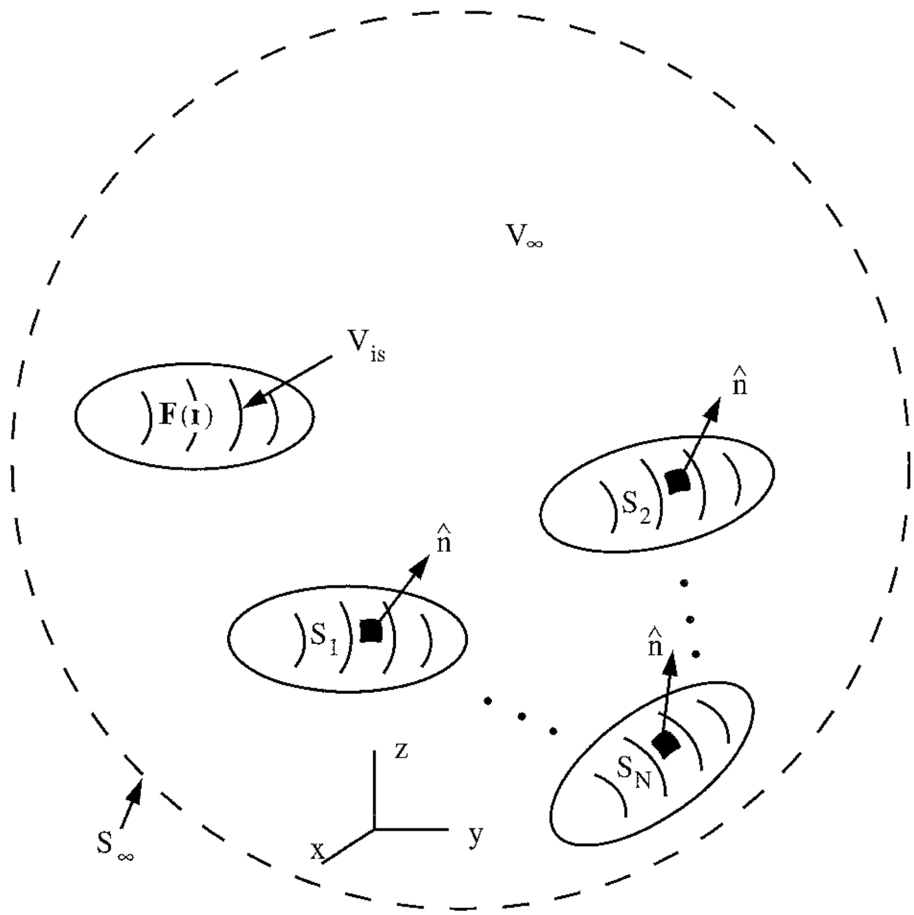

Figure 3.1: Geometry for the application of Green's second identity.

where $\frac{\partial}{\partial n} = \hat{n} \cdot \nabla$ and we remark that the negative sign in front of the integral over S_Ω was introduced because the unit normal $\hat{n}$ points toward the interior of V_∞. Further, by noting that $G(\mathbf{r}, \mathbf{r}')$ and $E_a(\mathbf{r})$ satisfy the radiation condition (2.39), it follows that the integral over S_∞ in (3.3a) vanishes. Thus we have

$$\int\int\int_{V_\infty} G(\mathbf{r}, \mathbf{r}')\nabla^2 E_a(\mathbf{r}) dv = \int\int\int_{V_\infty} E_a(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}') dv + \oiint_{S_\Omega} \left[E_a(\mathbf{r}) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} - G(\mathbf{r}, \mathbf{r}') \frac{\partial E_a(\mathbf{r})}{\partial n} \right] ds \qquad (3.4)$$

and when this is combined with (3.2) we obtain

$$\int\int\int_{V_\infty} E_a(\mathbf{r})\left[\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}')\right] dv = -\oint\!\!\oint_{S_\Omega}\left[E_a(\mathbf{r})\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\right.$$

$$\left. -G(\mathbf{r},\mathbf{r}')\frac{\partial E_a(\mathbf{r})}{\partial n}\right] + \int\int\int_{V_{is}} F_a(\mathbf{r})G(\mathbf{r},\mathbf{r}')dv \qquad (3.5)$$

We now recall the differential equation (2.38) satisfied by the Green's function and when this is introduced into (3.5), upon interchanging $\mathbf{r}$ and $\mathbf{r}'$ we obtain

$$\oint\!\!\oint_{S_\Omega}\left[E_a(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial E_a(\mathbf{r}')}{\partial n'}\right] ds' - \int\int\int_{V_{is}} F_a(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dv'$$

$$= \begin{cases} E_a(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \\ 0 & \mathbf{r} \text{ not in } V_\infty \end{cases} \qquad (3.6)$$

in which the differentiation is on the primed coordinates and is taken along the normal directed away from $S_1, S_2, \ldots, S_N$.

The above result given by (3.6) is often referred to as the *extinction or Kirchhoff's integral equation* and is valid for all field components provided these satisfy the radiation condition. No other boundary condition is required to be satisfied by the field and since $E_a(\mathbf{r})$ is completely arbitrary we can generalize it to the case of vector fields. We have

$$\oint\!\!\oint_{S_\Omega}\left[\mathbf{E}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' - \int\int\int_{V_{is}} \mathbf{F}_E(\mathbf{r})G(\mathbf{r},\mathbf{r}')dv'$$

$$= \begin{cases} \mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \\ 0 & \mathbf{r} \text{ not in } V_\infty \end{cases} \qquad (3.7a)$$

and by duality

$$\oint\!\!\oint_{S_\Omega}\left[\mathbf{H}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' - \int\int\int_{V_{is}} \mathbf{F}_H(\mathbf{r})G(\mathbf{r},\mathbf{r}')dv'$$

$$= \begin{cases} \mathbf{H}(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \\ 0 & \mathbf{r} \text{ not in } V_\infty \end{cases} \qquad (3.7b)$$

in which

$$\mathbf{F}_E(\mathbf{r}) = j\omega\mu\mathbf{J}(\mathbf{r}) - \frac{\nabla\nabla\cdot\mathbf{J}(\mathbf{r}')}{j\omega\epsilon} + \nabla\times\mathbf{M}(\mathbf{r}).$$

and

$$\mathbf{F}_H(\mathbf{r}) = j\omega\epsilon\mathbf{M}(\mathbf{r}) - \frac{\nabla\nabla\cdot\mathbf{M}(\mathbf{r}')}{j\omega\mu} - \nabla\times\mathbf{J}(\mathbf{r}).$$

If no external sources are present or if all sources are located on or within the surfaces S_1, S_2, etc, then $\mathbf{F}_E = \mathbf{F}_M = 0$. In that case $\mathbf{E}, \mathbf{H}$, $\frac{\partial\mathbf{E}}{\partial n'}$, or $\frac{\partial\mathbf{H}}{\partial n'}$ when integrated over S_Ω play the role of equivalent sources of the same type as $\mathbf{F}_E$ and $\mathbf{F}_H$. This will become apparent in later applications. Alternatively, the integrals associated with $\mathbf{F}_E$ and $\mathbf{F}_M$ can be recognized to yield the fields radiated by the sources within V_∞ and we may thus set

$$\begin{aligned} -\int\int\int \mathbf{F}_E(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dv' &\rightarrow \mathbf{E}^i(\mathbf{r}) \\ -\int\int\int \mathbf{F}_H(\mathbf{r}')G(\mathbf{r},\mathbf{r}')dv' &\rightarrow \mathbf{H}^i(\mathbf{r}) \end{aligned}$$

where $(\mathbf{E}^i, \mathbf{H}^i)$ denote the excitation or incident fields. For scattering computations these are usually plane waves whose source is at infinity.

In practice, additional boundary conditions would be imposed on the fields at the surfaces S_1, S_2, etc. This leads to the construction of integral equations for a unique solution of the fields. However, in their present form, (3.6) and (3.7) are not applicable to the case where $\mathbf{r}$ is on S_Ω, i.e. at the boundary of V_Ω coinciding with S_Ω. To make them applicable to this case we shall consider the limit as the observation point P at $\mathbf{r} = \mathbf{r}_o$ approaches the surface from outside or inside S_Ω. In order to simulate the last situation, we distort the surface S_Ω about the observation point P as shown in Fig. 3.2, i.e. by adding a hemispherical surface to S_Ω of radius $R_o \rightarrow 0$ which has its center at the observation point P. Accordingly, from (3.7a)

$$\begin{aligned} &\int\int_{S_\Omega - S_o} \left[\mathbf{E}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial\mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' \\ &\quad = -\int\int_{S_o} \left[\mathbf{E}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial\mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' \end{aligned}$$

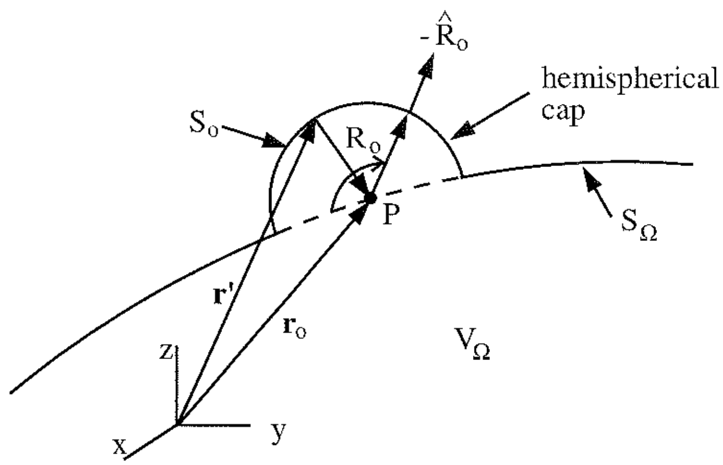

Figure 3.2: Geometry for evaluating the field on S_Ω.

For the integral over the hemispherical surface S_o we have $ds' = R_o^2 \sin\theta_o d\phi_o d\theta_o$,

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jkR_o}}{4\pi R_o}$$

and

$$\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} = -\hat{R}_o \cdot \nabla' G(\mathbf{r}, \mathbf{r}') = \hat{R}_o \cdot \nabla G(\mathbf{r}, \mathbf{r}') = \frac{\partial G}{\partial R_o} = -\left(jk + \frac{1}{R_o}\right) \frac{e^{-jkR_o}}{4\pi R_o}$$

Substituting these into the integral gives

$$-\int\int_{S_o \to 0} [\ \ ]\, ds' = \mathbf{E}(\mathbf{r}_o) \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(jk + \frac{1}{R_o}\right) \frac{e^{-jkR_o}}{4\pi R_o} R_o^2 \sin\theta_o d\phi_o d\theta_o$$
$$- \int_0^{\pi} \int_0^{2\pi} \frac{\partial \mathbf{E}}{\partial n'} \frac{e^{-jkR_o}}{4\pi R_o} R_o^2 \sin\theta_o d\phi_o d\theta_o$$

and it is seen that the last integral vanishes as $R_o \to 0$. Also,

$$\mathbf{E}(\mathbf{r}_o) \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left(jk + \frac{1}{R_o}\right) \frac{e^{-jkR_o}}{4\pi R_o} R_o^2 \sin\theta_o d\phi_o d\theta_o = \frac{1}{2}\mathbf{E}(\mathbf{r}_o) \qquad (3.8)$$

and thus we can write [Kellogg, 1929]

$$\oiint_{S_\Omega}\left[\mathbf{E}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' = \frac{1}{2}\mathbf{E}(\mathbf{r}) \tag{3.9}$$

for $\mathbf{r}$ on S_Ω. This simply states that the field on the surface S_Ω is obtained by averaging its values just inside and just outside S_Ω. Using (3.9) we can now revise the integral expressions (3.7) to read

$$\oiint_{S_\Omega}\left[\mathbf{E}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'}\right] ds' + \mathbf{E}^i(\mathbf{r})$$
$$= \begin{cases} \mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \frac{1}{2}\mathbf{E}(\mathbf{r}) & \mathbf{r} \text{ on } S_\Omega \\ 0 & \mathbf{r} \text{ within } S_\Omega \end{cases} \tag{3.10a}$$

$$\oiint_{S_\Omega}\left[\mathbf{H}(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - G(\mathbf{r},\mathbf{r}')\frac{\partial \mathbf{H}(\mathbf{r}')}{\partial n'}\right] ds' + \mathbf{H}^i(\mathbf{r})$$
$$= \begin{cases} \mathbf{H}(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \frac{1}{2}\mathbf{H}(\mathbf{r}) & \mathbf{r} \text{ on } S_\Omega \\ 0 & \mathbf{r} \text{ within } S_\Omega \end{cases} \tag{3.10b}$$

and it should be noted that these are valid provided the observation point is not at a corner or an edge formed by S_Ω. They are evocative of Huygens' principle which states that the fields caused by the presence of the volume enclosed by the surface(s) S_Ω can be determined uniquely everywhere from a knowledge of that field and its normal derivative on S_Ω. Alternatively, it will be shown in the next section that a knowledge of the tangential electric and magnetic fields on S_Ω is sufficient to uniquely determine the fields exterior to S_Ω regardless of the volume composition enclosed by S_Ω. These statements are valid even if $(\mathbf{E}^i, \mathbf{H}^i)$ are zero and sources exist within S_Ω. In that case we can state that the fields exterior to S_Ω can be determined uniquely from a knowledge of the surface tangential electric and magnetic fields or a knowledge of the electric/magnetic field and its normal derivative. By referring to chapter 1, one concludes that the surface equivalence principle can be thought as another statement of Huygens' principle [Baker and Compson, 1939].

Equations (3.10) can be referred to as the vector form of Kirchhoff's equations who first employed (a scalar form of) these for computing diffraction by apertures. To obtain the standard Kirchhoff's scalar equations the vector field in (3.10) is replaced by a scalar function or a component of the field. Because of their simplicity, Kirchhoff's [1882] (also Rubinoqicz, 1917) equations are widely used for obtaining the diffraction by apertures or the scattering by closed surfaces whose surface fields are known or can be reasonably approximated (using physical optics, for example).

3.1.2 Stratton-Chu Integral Equations

The Stratton-Chu [1939, 1940] integral formulae for field representations are among the most popular in scattering and antenna related problems. Perhaps a primary reason for their popularity is their reduced kernel singularity in comparison to the representations (2.52) or (2.102), which integrate the current sources directly over the volume. The main feature of the Stratton-Chu representations is the transferring of one of the del operators from the Green's function to the current reducing the kernel singularity from R^{-3} to R^{-2} (see (2.63)). There are several ways to derive the Stratton-Chu equations but it is instructive to begin their derivation by considering one of the integral expansions given earlier. Let us for example begin with equation (2.52a) where our goal is to reduce the singularity of the integrand (or kernel) associated with the last right hand side term of this equation. This term can be written as

$$\begin{aligned}-\frac{jZ}{k}\int\int\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla\nabla G(\mathbf{r},\mathbf{r}')dv' &= -\frac{jZ}{k}\nabla\left\{\int\int\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla G(\mathbf{r},\mathbf{r}')dv'\right\}\\ &= \frac{jZ}{k}\nabla\left\{\int\int\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla' G(\mathbf{r},\mathbf{r}')dv'\right\}\end{aligned}$$

and by invoking the identity (2.50) we have

$$\begin{aligned}\int\int\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla' G(\mathbf{r},\mathbf{r}')dv' &= \int\int\int \nabla'\cdot\left\{\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\right\}dv'\\ &\quad -\int\int\int_V \left[\nabla'\cdot\mathbf{J}(\mathbf{r}')\right]G(\mathbf{r},\mathbf{r}')dv'.\end{aligned}$$

Next, by employing the divergence theorem we obtain

$$\int\int\int_V \nabla'\cdot\left\{\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\right\}dv' = \oiint_{S_c} G(\mathbf{r},\mathbf{r}')\left[\mathbf{J}(\mathbf{r}')\cdot\hat{n}(\mathbf{r}')\right]ds' \tag{3.11}$$

where $\hat{n}$ is the unit normal pointing outward of the surface S_c enclosing the volume V containing the source $\mathbf{J}(\mathbf{r}')$. A natural boundary condition is that the current be confined within the volume V implying that the component of $\mathbf{J}$ normal to the surface S_c must be zero. Thus, the integral in (3.11) vanishes and we can then write

$$\frac{-jZ}{k}\int\int\int_V \mathbf{J}(\mathbf{r}')\cdot\nabla\nabla G(\mathbf{r},\mathbf{r}')dv' = \frac{-jZ}{k}\int\int\int_V \left[\nabla'\cdot\mathbf{J}(\mathbf{r}')\right]\nabla G(\mathbf{r},\mathbf{r}')dv' \tag{3.12}$$

When this identity and its dual is used in (2.52) we obtain the equations

$$\begin{aligned}\mathbf{E}(\mathbf{r}) = \int\int\int_V \Big[&\mathbf{M}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}') - jkZ\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\\ &-\frac{jZ}{k}\nabla'\cdot\mathbf{J}(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}')\Big]dv'\end{aligned} \tag{3.13a}$$

$$\begin{aligned}\mathbf{H}(\mathbf{r}) = \int\int\int_V \Big[&-\mathbf{J}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}') - jkY\mathbf{M}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\\ &-\frac{jY}{k}\nabla'\cdot\mathbf{M}(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}')\Big]dv'\end{aligned} \tag{3.13b}$$

Alternative representations can be obtained by invoking the continuity equations (1.38) and (1.39) to replace the divergence of the current quantities with volume charges. Doing so yields

$$\begin{aligned}\mathbf{E}(\mathbf{r}) = \int\int\int_V \Big[&\mathbf{M}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}') - jkZ\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\\ &-\frac{\rho(\mathbf{r}')}{\epsilon}\nabla G(\mathbf{r},\mathbf{r}')\Big]dv'\end{aligned} \tag{3.14a}$$

$$\begin{aligned}\mathbf{H}(\mathbf{r}) = \int\int\int_V \Big[&-\mathbf{J}(\mathbf{r}')\times\nabla G(\mathbf{r},\mathbf{r}') - jkY\mathbf{M}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\\ &-\frac{\rho_m(\mathbf{r}')}{\mu}\nabla G(\mathbf{r},\mathbf{r}')\Big]dv'\end{aligned} \tag{3.14b}$$

which are the natural equations that result if we introduce the scalar potentials Φ_e and Φ_m in equations (2.19).

When the above expressions (3.13) and (3.14) are applied to an antenna or scattering configuration such as that shown in Figure 2.10 it is convenient to employ Love's equivalence principle (see Chapter 1). This allows one to replace the presence of the volume enclosed by the surfaces $S_1, S_2, \ldots, S_N$ (comprising the surface S_Ω) by a set of equivalent sources

$$\mathbf{J} = \hat{n} \times \mathbf{H}, \quad \mathbf{M} = \mathbf{E} \times \hat{n} \tag{3.15}$$

placed on the surfaces $S_1, S_2, \ldots, S_N$. Also, in accordance with the boundary conditions (1.62) and (1.63) we may set

$$\rho_s = \epsilon(\hat{n} \cdot \mathbf{E}), \quad \rho_{ms} = \mu(\hat{n} \cdot \mathbf{H}) \tag{3.16}$$

Introducing these into (3.14) yields

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^i + \oiint_{S_\Omega} \Big\{ [\mathbf{E}(\mathbf{r}') \times \hat{n}'] \times \nabla G(\mathbf{r}, \mathbf{r}') - jkZ\, [\hat{n}' \times \mathbf{H}(\mathbf{r}')]\, G(\mathbf{r}, \mathbf{r}') - \hat{n}' \cdot \mathbf{E}(\mathbf{r}') \nabla G(\mathbf{r}, \mathbf{r}') \Big\} ds' \tag{3.17a}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^i + \oiint_{S_\Omega} \Big\{ [\mathbf{H}(\mathbf{r}') \times \hat{n}'] \times \nabla G(\mathbf{r}, \mathbf{r}') - jkY\, [\mathbf{E}(\mathbf{r}') \times \hat{n}']\, G(\mathbf{r}, \mathbf{r}') - \hat{n}' \cdot \mathbf{H}(\mathbf{r}') \nabla G(\mathbf{r}, \mathbf{r}') \Big\} ds' \tag{3.17b}$$

in which $\hat{n}' = \hat{n}(\mathbf{r}')$, where $\hat{n}(\mathbf{r}')$ denotes the outward unit normal outward to S_Ω at $\mathbf{r}$. We have also included the incident fields $(\mathbf{E}^i, \mathbf{H}^i)$ to account for any source exterior to S_Ω. We remark that (3.17) give the most common form of the Stratton-Chu equations.

An alternative field representation in terms of the dyadic Green's function can be obtained by substituting (3.15) into (2.102). Since the equivalent sources are only over the surface(s) S_Ω we have

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^i + \int\!\!\int_{S_\Omega} \Big\{ \left[\nabla \times \overline{\Gamma}(\mathbf{r}, \mathbf{r}')\right] \cdot [\mathbf{E}(\mathbf{r}') \times \hat{n}'] + jkZ\overline{\Gamma}(\mathbf{r}, \mathbf{r}') \cdot [\hat{n}' \times \mathbf{H}(\mathbf{r}')] \Big\} ds' \tag{3.18a}$$

$$\mathbf{H}(\mathbf{r}') = \mathbf{H}^i \int\!\!\int_{S_\Omega} \Big\{ jkY\overline{\Gamma}(\mathbf{r}, \mathbf{r}') \cdot [\mathbf{E}(\mathbf{r}') \times \hat{n}'] - \left[\nabla \times \overline{\Gamma}(\mathbf{r}, \mathbf{r}')\right] \cdot [\hat{n}' \times \mathbf{H}(\mathbf{r}')] \Big\} ds' \tag{3.18b}$$

and this is equivalent to (3.17) only for closed surfaces in which case the identity [Van Bladel, p. 503]

$$\oiint_{S_\Omega} \nabla \cdot [\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')]\, ds' = -\oiint_{S_\Omega} \left[\frac{1}{R_1} + \frac{1}{R_2}\right] G(\mathbf{r},\mathbf{r}')\, [\mathbf{J}(\mathbf{r}') \cdot \hat{n}(\mathbf{r}')]\, ds' = 0 \tag{3.19}$$

holds when $\mathbf{J}$ is replaced by $\hat{n}' \times \mathbf{H}(\mathbf{r}')$ or $\mathbf{E}(\mathbf{r}') \times \hat{n}(\mathbf{r}')$. In (3.19), R_1 and R_2 denote the principle radii of curvature at the surface point $\mathbf{r}'$ but in the event S_Ω is not closed (i.e. S_Ω is the surface of a flat or curved conducting sheet as shown in Fig. 1.1) then this identity must be replaced by [Van Bladel, p. 502]

$$\begin{aligned} \int\int_{S_\Omega} \nabla \cdot [\mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')]\, ds' &= +\oint_C \hat{b}' \cdot \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\ell' \\ &\quad - \int\int_{S_\Omega} \left[\frac{1}{R_1} + \frac{1}{R_2}\right] G(\mathbf{r},\mathbf{r}')\, [\mathbf{J}(\mathbf{r}') \cdot \hat{n}(\mathbf{r}')]\, ds' \\ &= \oint_C \hat{b}' \cdot \mathbf{J}(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\ell' \end{aligned} \tag{3.20}$$

in which C denotes the contour defining the outer perimeter of S_Ω and $\hat{b}' = \hat{\ell}' \times \hat{n}'$ where $\hat{\ell}'$ is the unit tangent to C at $\mathbf{r}'$. Thus, one cannot specialize (3.13) to open surfaces such as curved plates (see Fig. 3.3) by simply changing

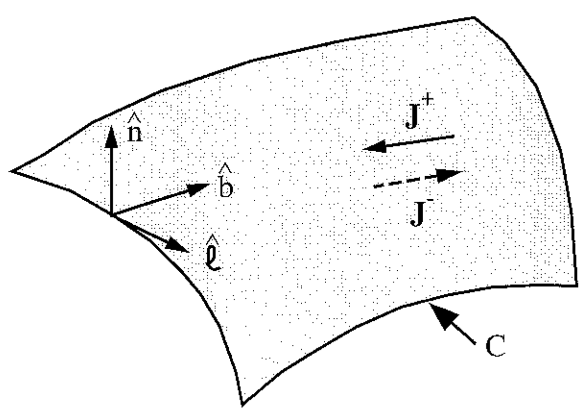

Figure 3.3: Geometry of a curved plate representing an open surface.

the volume integral to one over the boundary domain of $\mathbf{J}$ and $\mathbf{M}$. Such an interchange of the volume and surface integral is permitted for the dyadic representations (2.102) or (2.42) which are similar to Franz's integral formulas whose precise form is given in one of the problems. However, in the case of (3.13) it is permitted only if the contour integral in (3.20) also vanishes. The last is often referred to as Kottler's boundary line integral and its presence is necessary to ensure the divergenceless of the field for all $\mathbf{r}$. If, however, one thinks of $\mathbf{J}$ and $\mathbf{M}$ as representing the net currents on the open surface, then $\mathbf{J}\cdot\hat{b} = (\mathbf{J}^+ - \mathbf{J}^+)\cdot\hat{b} = 0$ and $\mathbf{M}\cdot\hat{b} = (\mathbf{M}^+ - \mathbf{M}^-)\cdot\hat{b} = 0$ at the boundary line C. Consequently, the Kottler integral in (3.20) again vanishes implying that with this interpretation of $\mathbf{J}$ and $\mathbf{M}$, (3.13) remains valid when the volume integrals are replaced by ones over the surface of the curved plate.

We remark that (3.18) are again evocative of Huygens' principle as discussed in the previous section in connection with Kirchhoff's integral equation. In practice, however, the Stratton-Chu equations are more attractive than (3.18) because of the lower singularity of their kernel leading to a more accurate numerical implementation.

Integral equations such as those in (3.17) can be used for solving the fields on S_Ω by enforcing the specific boundary conditions associated with the surfaces comprising S_Ω. To enforce these boundary conditions it is necessary to have the observation point directly on S_Ω leading to singular kernels which must be carefully integrated as done in the previous section. As before, we refer to figure 3.2 and rewrite (3.17a) as

$$\begin{aligned}
\mathbf{E}(\mathbf{r}_o) &= \mathbf{E}^i(\mathbf{r}_o) + \int\!\!\int_{S_\Omega - S_o} \Big\{ [\mathbf{E}(\mathbf{r}')\times\hat{n}']\times\nabla G(\mathbf{r}_o,\mathbf{r}') \\
&- jkZ\,[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\,G(\mathbf{r}_o,\mathbf{r}') - \hat{n}'\cdot\mathbf{E}(\mathbf{r}')\nabla G(\mathbf{r}_o,\mathbf{r}') \Big\} ds' \\
&+ \int\!\!\int_{S_o} \Big\{ [\mathbf{E}(\mathbf{r}')\times\hat{n}']\times\nabla G(\mathbf{r}_o,\mathbf{r}') - jkZ\,[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\,G(\mathbf{r}_o,\mathbf{r}') \\
&- \hat{n}'\cdot\mathbf{E}(\mathbf{r}')\nabla G(\mathbf{r}_o,\mathbf{r}') \Big\} ds' \qquad (3.21)
\end{aligned}$$

in which S_o is a vanishingly small hemispherical surface. Noting the identities

$$(\mathbf{E}\times\hat{n}')\times\nabla G = \nabla G\times(\hat{n}'\times\mathbf{E})$$

$$= \hat{n}'(\mathbf{E}\cdot\nabla G) - \mathbf{E}(\hat{n}'\cdot\nabla G) \tag{3.22}$$

$$-(\hat{n}'\cdot\mathbf{E})\nabla G = -\hat{n}'(\mathbf{E}\cdot\nabla G) + \mathbf{E}\times(\hat{n}'\times\nabla G) \tag{3.23}$$

it follows that

$$(\mathbf{E}\times\hat{n}')\times\nabla G - (\hat{n}'\cdot\mathbf{E})\nabla G = \mathbf{E}\times(\hat{n}'\times\nabla G) - \mathbf{E}(\hat{n}'\cdot\nabla G) \tag{3.24}$$

When the last is substituted into (3.21) the surface integral over S_o becomes

$$\int\int_{S_o}\Big\{\mathbf{E}(\mathbf{r}')\times[\hat{n}'\times\nabla G(\mathbf{r}_o,\mathbf{r}')] - \mathbf{E}(\mathbf{r}')\,[\hat{n}'\cdot\nabla G(\mathbf{r}_o,\mathbf{r}')]$$
$$-jkZ\,[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\,G(\mathbf{r}_o,\mathbf{r}')\Big\}ds'$$

For this integral $\hat{n}' = \hat{R}_o$, $ds' = R_o^2\sin\theta_o d\phi_o d\theta_o$ and since $S_o \to 0$ we may set $\mathbf{E}(\mathbf{r}') \approx \mathbf{E}(\mathbf{r}_o)$ and $\mathbf{H}(\mathbf{r}') = \mathbf{H}(\mathbf{r}_o)$. When we substitute for ∇G as given in (2.55) with $R = R_o$, we find that the first term of the integral vanishes because $\nabla G = \hat{R}_o|\nabla G|$ and $\hat{n}\times\hat{R}_o = 0$. Also the third term goes to zero as $R_o \to 0$. The second term (see (3.8)) when integrated gives $-\frac{1}{2}\mathbf{E}(\mathbf{r}_o)$ and thus we can rewrite (3.21) as

$$\mathbf{E}(\mathbf{r}_o) = 2\mathbf{E}^i(r_o) + 2\oiint_{S_\Omega}\Big\{[\mathbf{E}(\mathbf{r}')\times\hat{n}']\times\nabla G(\mathbf{r}_o,\mathbf{r}')$$
$$-jkZ\,[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\,G(\mathbf{r}_o,\mathbf{r}') - \hat{n}'\cdot\mathbf{E}(\mathbf{r}')\nabla G(\mathbf{r}_o,\mathbf{r}')\Big\}ds' \tag{3.25}$$

Incorporating this result into (3.17) we have

$$\oiint_{S_\Omega}\Big\{[\mathbf{E}(\mathbf{r}')\times\hat{n}']\times\nabla G(\mathbf{r},\mathbf{r}') - jkZ\,[\hat{n}'\times\mathbf{H}(\mathbf{r})]\,G(\mathbf{r},\mathbf{r}')$$
$$-\hat{n}'\cdot\mathbf{E}(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}')\Big\}ds' + \mathbf{E}^i(\mathbf{r}) = \begin{cases}\mathbf{E}(\mathbf{r}) & \mathbf{r}\text{ in }V_\infty\\ \frac{1}{2}\mathbf{E}(\mathbf{r}) & \mathbf{r}\text{ on }S_\Omega\\ 0 & \mathbf{r}\text{within }S_\Omega\end{cases} \tag{3.26a}$$

$$\oiint_{S_\Omega}\Big\{[\mathbf{H}(\mathbf{r}')\times\hat{n}']\times\nabla G(\mathbf{r},\mathbf{r}') - jkY\,[\mathbf{E}(\mathbf{r})\times\hat{n}']\,G(\mathbf{r},\mathbf{r}')$$

$$-\hat{n}'\cdot\mathbf{H}(\mathbf{r}')\nabla G(\mathbf{r},\mathbf{r}')\Big\}ds' + \mathbf{H}^i(\mathbf{r}) = \begin{cases} \mathbf{H}(\mathbf{r}) & \mathbf{r} \text{ in } V_\infty \\ \frac{1}{2}\mathbf{H}(\mathbf{r}) & \mathbf{r} \text{ on } S_\Omega \\ 0 & \text{within } S_\Omega \end{cases} \qquad (3.26b)$$

For completeness, we note that the impressed fields $(\mathbf{E}^i, \mathbf{H}^i)$ may be replaced by their volume integral representations (3.13) or (2.52), However, when the observation point is within the volume of the impressed or equivalent volume sources $(\mathbf{J}, \mathbf{M})$, we must then revert to the principle-value integral representation given in (2.69). It should also be noted that the Stratton-Chu equations are completely equivalent to the vector Kirchhoff equations (3.10). Notably, both sets of integral equations involve the normal and tangential field components on the surface S_Ω but Kirchhoff's equations decouple each field component from the others. However, these are unavoidably coupled upon application of the boundary conditions on S_Ω. Nevertheless, in the case of two-dimensional applications where only a z-directed electric or magnetic field exists, Kirchhoff's equations are the most simple to use. By setting $\mathbf{E} = \hat{z}E_z$ or $\mathbf{H} = \hat{z}H_z$ in (3.10), a scalar equation is obtained instead of the vector integral equation resulting from (3.26). Consequently, the extinction or Kirchhoff's integral equations are, generally, the preferred choice in formulating two-dimensional problems.

3.1.3 Integral Equations for Homogeneous Dielectrics

Man-made structures such as vehicles made of composites and microstrip antennas are typically composed of piecewise homogeneous dielectrics. The effects of these materials must therefore be accounted for in computing the radiated or scattered fields. So far, field representations were given which apply in the presence of structures enclosed within a surface S_Ω by invoking the equivalence theorem. In this section we will specialize these expressions to the case where the surface S_Ω encloses a piecewise homogeneous dielectric body. We shall first consider the simplest case, i.e. that pertinent to a homogeneous dielectric body.

Consider the homogeneous dielectric body enclosed by the surface $S_\Omega = S_d$ as shown in Fig. 3.4. The dielectric is immersed in some excitation field $(\mathbf{E}^i, \mathbf{H}^i)$ generated by the sources $(\mathbf{J}^i, \mathbf{M}^i)$ which are exterior to S_d and we are interested in finding a representation of the field in the exterior region (region

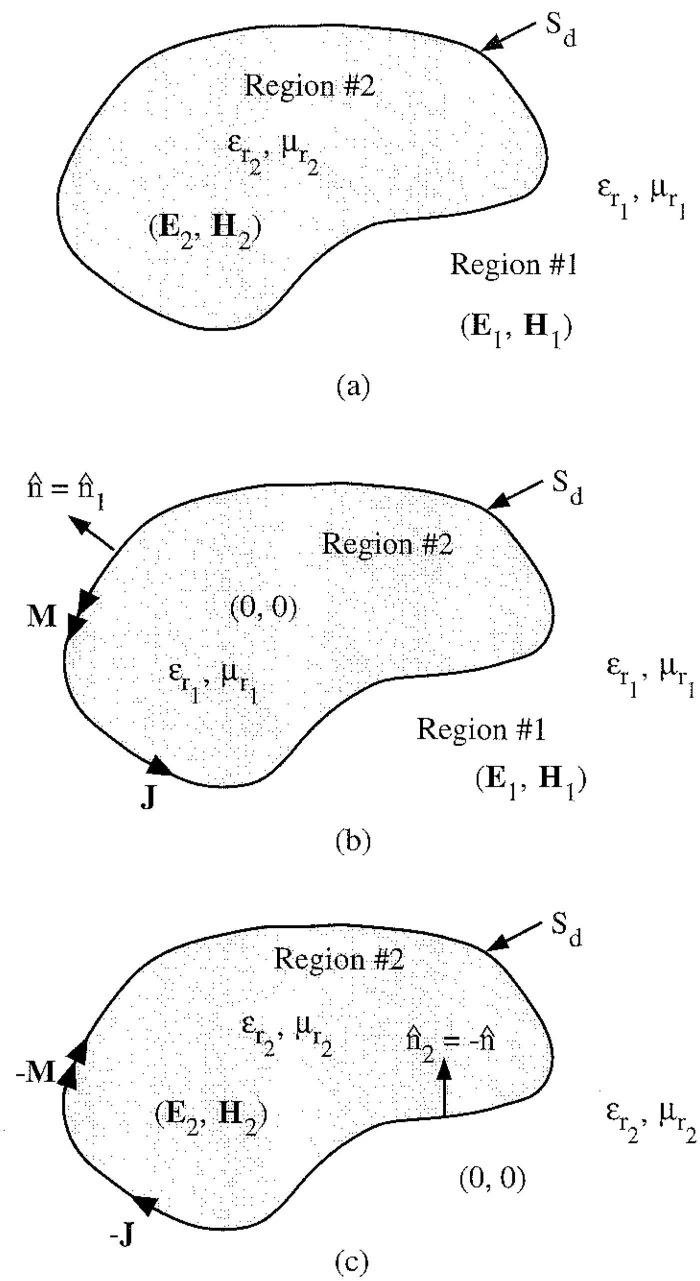

Figure 3.4: Application of the equivalence principle for a dielectric. (a) original problem (b) equivalent problem for Region #1, (c) equivalent problem for Region #2.

#1) and perhaps interior to S_d (region #2). One of the simplest integral expressions in this case is obtained by invoking the surface equivalence theorem and with this in mind we set up the two problems illustrated in Figure 3.4. The set-up in Figure 3.4(a) assumes zero field interior to S_d and thus the equivalent current $(\mathbf{J}_1, \mathbf{M}_1)$ can be used for computing the fields $(\mathbf{E}_1, \mathbf{H}_1)$ exterior to S_d. In contrast, the set-up in Figure 3.4(b) assumes zero exterior fields and thus the equivalent currents $(\mathbf{J}_2, \mathbf{M}_2)$ can be used for computing the interior fields $(\mathbf{E}_2, \mathbf{H}_2)$.

It should be remarked that the set-up assumed here, where the fields are set to zero in the indicated region, is not unique. Any other non-zero field could have been used and this would result in a different, albeit equivalent, formulation. In fact, certain judicious choices for the interior fields of the set-up in figure 4(b) or the exterior fields in figure 4(c) lead to formulations which may involve a single surface equivalent current [Glisson, 1984]. An alternative approach will be to eliminate the introduction of the equivalent surface currents altogether and express the scattered fields in terms of the tangential electric and magnetic fields at the dielectric interface. In this case, the representation (3.18) may be used (or some other equivalent expression) to set-up integral equations for the tangential fields upon invoking field continuity at the interface. Nevertheless, below we shall consider the solution of the scattered/radiated fields in the presence of a dielectric via the set-up in fig. 4 since this appears to be one of the most often used approaches.

The introduced equivalent current illustrated in fig. 4 can be substituted into (3.13) to obtain integral expressions for the exterior and interior fields upon changing the volume integral to one over the closed surface S_d. However before doing so, it is important to note that by enforcing the tangential field continuity equations

$$\hat{n}_1 \times \mathbf{E}_1 = \hat{n}_1 \times \mathbf{E}_2, \quad \hat{n}_1 \times \mathbf{H}_1 = \hat{n}_1 \times \mathbf{H}_2 \tag{3.27}$$

($\hat{n}_1$ denotes the unit normal pointing away from S_d) across the surface S_d, it follows that

$$\mathbf{J}_1 = -\mathbf{J}_2 = \mathbf{J}, \quad \mathbf{M}_1 = -\mathbf{M}_2 = -\mathbf{M} \tag{3.28}$$

In arriving at (3.28) we could have also implied that (see section 1.10)

$$\mathbf{J}_1 \;=\; \hat{n}_1 \times \mathbf{H}_1, \qquad \mathbf{M}_1 = -\hat{n}_1 \times \mathbf{E}_1$$

$$\mathbf{J}_2 \;=\; \hat{n}_2 \times \mathbf{H}_2 = -\hat{n}_1 \times \mathbf{H}_2 \qquad \mathbf{M}_2 = -\hat{n}_2 \times \mathbf{E}_2 = \hat{n}_1 \times \mathbf{E}_2 \tag{3.29}$$

However, it is not necessary to introduce these expressions since the surface fields are unknown and it is thus more convenient to retain $(\mathbf{J}, \mathbf{M})$ as the variable functions to be determined by enforcing the boundary conditions associated with problems defined in Figure 3.4. From Figure 3.4a, since the interior fields have been set to zero, we have that on S_d (actually just inside S_d)

$$\begin{aligned} \hat{n}_1 \times \mathbf{E}_1 &= 0 \\ \hat{n}_1 \times \mathbf{H}_1 &= 0 \end{aligned} \tag{3.30}$$

By defining the total fields $(\mathbf{E}_1, \mathbf{H}_1)$ to be the sum of the source fields and those radiated by $(\mathbf{J}, \mathbf{M})$ we may rewrite (3.30) as

$$\begin{aligned} \hat{n}_1 \times \mathbf{E}^i &= -\hat{n}_1 \times \mathbf{E}_1^s \\ \hat{n}_1 \times \mathbf{H}^i &= -\hat{n}_1 \times \mathbf{H}_1^s \end{aligned} \tag{3.31a}$$

where

$$\mathbf{E}_1^s = \oiint_{S_d} \Big[\mathbf{M}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') - jk_o Z_o \mu_{r_1} \mathbf{J}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') - j\frac{Z_o}{k_o \epsilon_{r_1}} \nabla_s' \cdot \mathbf{J}(\mathbf{r}') \nabla G_1(\mathbf{r}, \mathbf{r}') \Big] ds' \tag{3.31b}$$

$$\mathbf{H}_1^s = \oiint_{S_d} \Big[-\mathbf{J}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') - jk_o Y_o \epsilon_{r_1} \mathbf{M}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') - j\frac{Y_o}{k_o \mu_{r_1}} \nabla_s' \cdot \mathbf{M}(\mathbf{r}') \nabla G_1(\mathbf{r}, \mathbf{r}') \Big] ds' \tag{3.31c}$$

In these k_o, Z_o, Y_o are the free space wavenumber impedance and admittance, respectively, whereas ϵ_{r_1} and μ_{r_1} are the relative material constants of the exterior medium and are usually unity. Also,

$$G_1(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_1|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{e^{-jk_o\sqrt{\mu_{r_1}\epsilon_{r_1}}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{3.32}$$

is the Green's function associated with the exterior region. The fields $(\mathbf{E}_1^s, \mathbf{H}_1^s)$ are customarily referred to as those scattered by the dielectric body due to the excitation $(\mathbf{E}^i, \mathbf{H}^i)$. Instead of repeatedly using the explicit integral representation (3.31) it is convenient to define the operators

$$L_{1E_m}(\mathbf{M}) = \oiint_{S_d} \mathbf{M}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') ds' \tag{3.33}$$

$$L_{1H_m}(\mathbf{M}) = -\oiint_{S_d} \left[jk_o Y_o \epsilon_{r1} \mathbf{M}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') + \frac{jY_o}{k_o \mu_{r1}} \nabla_s' \cdot \mathbf{M}(\mathbf{r}') \nabla G_1(\mathbf{r}, \mathbf{r}') \right] ds' \tag{3.34}$$

$$L_{1E_e}(\mathbf{J}) = -\oiint_{S_d} \left[jk_o Z_o \mu_{r1} \mathbf{J}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') + \frac{jZ_o}{k_o \epsilon_{r1}} \nabla_s' \cdot \mathbf{J}(\mathbf{r}') \nabla G_1(\mathbf{r}, \mathbf{r}') \right] ds' \tag{3.35}$$

$$L_{1H_e}(\mathbf{J}) = -\oiint_{S_d} \mathbf{J}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') ds' \tag{3.36}$$

Then, since [Van Bladel, 1985 (p. 254), Müller, 1969]

$$\begin{aligned} \hat{n}_1 \times \oiint_{S_d} \mathbf{A}(\mathbf{r}') \times \nabla G_1(\mathbf{r}_o^{\pm}, \mathbf{r}') ds' &= \mp \frac{1}{2} \mathbf{A}(\mathbf{r}_o) \\ &+ \hat{n}_1 \times \oiint_{S_d} \mathbf{A}(\mathbf{r}') \times \nabla G_1(\mathbf{r}_o, \mathbf{r}') ds' \end{aligned} \tag{3.37}$$

where $\mathbf{r}_o^{\pm}$ implies that the observation point is just exterior (+) or interior (-) to S_d, we can rewrite (3.31) more explicitly as

$$\begin{aligned} -\frac{1}{2}\mathbf{M}(\mathbf{r}) - \hat{n}_1 \times L_{1E_m}(\mathbf{M}) - \hat{n}_1 \times L_{1E_e}(\mathbf{J}) &= \hat{n}_1 \times \mathbf{E}^i \\ +\frac{1}{2}\mathbf{J}(\mathbf{r}) - \hat{n}_1 \times L_{1H_e}(\mathbf{J}) - \hat{n}_1 \times L_{1H_m}(\mathbf{M}) &= \hat{n}_1 \times \mathbf{H}^i \end{aligned} \tag{3.38}$$

valid for $\mathbf{r}$ on S_d. We note that (3.37) can be proven by following a similar procedure to that employed for the derivation of (3.9).

Another set of equations to be coupled with (3.38) can be obtained by enforcing the boundary conditions on $(\mathbf{E}_2, \mathbf{H}_2)$. From Fig. 3.4(b) we have that on S_d (actually just outside S_d)

$$\hat{n}_1 \times \mathbf{E}_2 = \hat{n}_1 \times \mathbf{E}_2^s = 0$$

$$\hat{n}_1 \times \mathbf{H}_2 = \hat{n}_1 \times \mathbf{H}_2^s = 0 \tag{3.39}$$

and upon making use of (3.37) these can be more explicitly written as

$$\begin{aligned} -\frac{1}{2}\mathbf{M}(\mathbf{r}) - \hat{n}_1 \times L_{2E_m}(-\mathbf{M}) - \hat{n}_1 \times L_{2E_e}(-\mathbf{J}) = 0 \\ +\frac{1}{2}\mathbf{J}(\mathbf{r}) - \hat{n}_1 \times L_{2H_e}(-\mathbf{J}) - \hat{n}_1 \times L_{2H_m}(-\mathbf{M}) = 0 \end{aligned} \tag{3.40a}$$

In these, the integral operators $L_{2E_m}, L_{2E_e}, L_{2H_e}$ and L_{2H_m} are identical to those defined in (3.33) - (3.36) provided ϵ_{r_1} and μ_{r_1} are replaced by ϵ_{r_2} and μ_{r_2}, respectively. By inspection, it is also seen that the minus sign in the argument of the operators can be factored out giving

$$\begin{aligned} \frac{1}{2}\mathbf{M}(\mathbf{r}) - \hat{n}_1 \times L_{2E_m}(\mathbf{M}) - \hat{n}_1 \times L_{2E_e}(\mathbf{J}) = 0 \\ -\frac{1}{2}\mathbf{J}(\mathbf{r}) - \hat{n}_1 \times L_{2H_e}(\mathbf{J}) - \hat{n}_1 \times L_{2H_m}(\mathbf{M}) = 0 \end{aligned} \tag{3.40b}$$

valid for $\mathbf{r}$ on S_d.

It is apparent that (3.38) and (3.40) are four integral equations involving only two unknowns. This is because we had initially enforced the continuity conditions (3.27) to relate the equivalent currents introduced for representing the exterior and interior fields. It is also a consequence of the fact that only the tangential electric or magnetic fields are needed over a closed surface for determining the fields away from S_d. Thus, we are essentially free to use one from each set of equations (3.38) and (3.40) to obtain a pair of them to be solved (usually numerically, and this will be discussed later) for $(\mathbf{J}, \mathbf{M})$. For example, we could select the equation resulting from the pair of conditions

$$\begin{aligned} \hat{n}_1 \times \mathbf{E}^i = -\hat{n}_1 \times \mathbf{E}_1^s \\ \hat{n}_1 \times \mathbf{E}_2^s = 0 \end{aligned} \tag{3.41a}$$

or from

$$\begin{aligned} \hat{n}_1 \times \mathbf{H}^i = -\hat{n}_1 \times \mathbf{H}_1^s \\ \hat{n}_1 \times \mathbf{H}_2^s = 0 \end{aligned} \tag{3.41b}$$

The integral equations resulting from (3.41a) are usually referred to as the *electric field integral equations* (EFIE) whereas those implied by (3.41b) are referred as the *magnetic field integral equations* (MFIE).

3.1.4 Integral Equations for Metallic Bodies

When S_d encloses a conducting surface (i.e. $\epsilon_{r_2} \to 1 - j\infty$) we may then set $\mathbf{M} = 0$ (see sections 1.4 and 1.10) and in that case the first of (3.41a) gives

$$jk_o Z_o \mu_{r1} \hat{n}_1 \times \oiint_{S_d} \left[\mathbf{J}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') + \frac{1}{k_o^2 \mu_{r1} \epsilon_{r1}} \nabla_s' \cdot \mathbf{J}(\mathbf{r}') \nabla G_1(\mathbf{r}, \mathbf{r}') \right] ds'$$
$$= \hat{n}_1 \times \mathbf{E}^i \qquad (3.42a)$$

whereas from the first of (3.41b) we have

$$\frac{1}{2}\mathbf{J}(\mathbf{r}') + \hat{n}_1 \times \oiint_{S_d} \mathbf{J}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') ds' = \hat{n}_1 \times \mathbf{H}^i \qquad (3.42b)$$

These are, respectively, the well known EFIE and MFIE for perfectly conducting surfaces. This MFIE is also known as *Maue's integral equation* and is the most common for solving the fields scattered by a closed conducting surface. It will be shown later that Maue's MFIE leads to a better conditioned matrix than (3.42a), and this is a primary reason for its popularity in simulating closed conducting surfaces. An EFIE which is of the same form as the MFIE (3.42b), can however be derived from (3.41a) by invoking image theory to eliminate the electric currents (since S_d is perfectly conducting). This gives

$$\frac{1}{2}\mathbf{M}(\mathbf{r}) + \hat{n}_1 \times \oiint_{S_d} \mathbf{M}(\mathbf{r}') \times \nabla G_1(\mathbf{r}, \mathbf{r}') ds' = -\hat{n}_1 \times \mathbf{E}^i \qquad (3.42c)$$

which is clearly the dual of (3.42b). Since (3.42c) and (3.42b) simulate the same metallic surface, it is not surprising that one can be derived from the other. Specifically, (3.42b) can be derived from (3.42c) by taking the curl of the last and making use of the equivalence relation (see (1.111))

$$\mathbf{J} = \frac{\nabla \times \mathbf{M}}{j\omega \mu_o \mu_{r1}}.$$

The fact that (3.42b) and (3.42c) are equivalent (i.e. they predict the same scattered fields) is a vivid demonstration that in the case of perfectly conducting surfaces, one could formulate the fields in terms of electric or magnetic currents.

We should remark that neither of (3.42) are valid for open conducting surfaces such as a metallic flat or curved plate (see figure 3.5). This is because

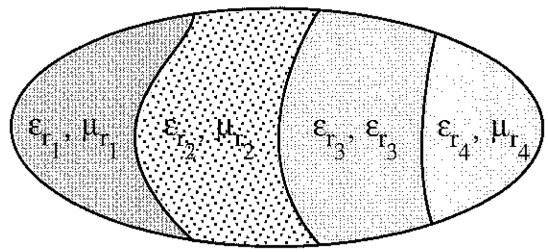

Figure 3.5: Piecewise homogeneous dielectric body.

the surface equivalence principle was used to introduce the equivalent surface currents. To construct an integral equation for the surface currents on a curved plate we may return to the original integral expression (2.52a) or (2.147) and set $\mathbf{M} = 0$. Then upon enforcing the boundary condition $\hat{n} \times (\mathbf{E}^s + \mathbf{E}^i) = 0$, we obtain the integral equation (also an EFIE)

$$+jk_o Z_o \mu_{r1} \hat{n} \times \int\int_S \left[\mathbf{J}(\mathbf{r}') G_1(\mathbf{r}, \mathbf{r}') + \frac{1}{k_o^2 \epsilon_{r1} \mu_{r1}} \mathbf{J}(\mathbf{r}') \cdot \nabla\nabla G_1(\mathbf{r}, \mathbf{r}') \right] ds'$$

$$= \hat{n} \times \mathbf{E}^i \qquad (3.43)$$

In contrast to the current appearing in (3.42), the one in this integral equation should be interpreted to represent the net flow between the top and bottom surfaces of the plate as illustrated in figure 3.3. With this interpretation of $\mathbf{J}$ and from the discussion in section 3.1.2, it is then seen that (3.43) is equivalent to (3.42a). Nevertheless, (3.43) is more difficult to implement than (3.42a) because of its higher kernel singularity.

3.1.5 Combined Field Integral Equations

Returning now to the original integral equation for the dielectric body we must address their uniqueness. Since they were formulated by assuming a null field within certain enclosed volumes, in accordance with the uniqueness theorem (3.41) or (3.42) will fail at those frequencies associated with a resonant mode within S_d. Fortunately, the EFIE is associated with different resonant modes than the MFIE and this has been exploited to construct sets of equations which yield a unique solution. The most obvious approach is to consider various linear combinations of (3.41). For example, we could consider the combination [Mautz and Harrington, 1978]

$$\begin{aligned}\hat{n}_1 \times [\mathbf{E}_1^s + \alpha \mathbf{E}_2^s] &= -\hat{n}_1 \times \mathbf{E}^i \\ \hat{n}_1 \times [\mathbf{H}_1^s + \beta \mathbf{H}_2^s] &= -\hat{n}_1 \times \mathbf{H}^i\end{aligned} \tag{3.44}$$

where α and β are arbitrary non-zero scalars. If we set $\alpha = \beta = 1$ we obtain the PMCHW formulation [Poggio and Miller, 1973] while the choice of $\alpha = -\epsilon_{r_2}/\epsilon_{r_1}$ and $\beta = -\mu_{r_2}/\mu_{r_1}$ leads to the Müeller formulation. Another combination which was proposed [Govind and Wilton, 1979] is

$$\begin{aligned}\hat{n}_1 \times \left[\mathbf{H}_1^s + \frac{\alpha}{Z_1}\mathbf{E}_1^s\right] &= -\hat{n}_1 \times \left[\mathbf{H}_1^i + \frac{\alpha}{Z_1}\mathbf{E}^i\right] \\ \hat{n}_1 \times \left[\mathbf{H}_2^s - \frac{\beta}{Z_2}\mathbf{E}_2^s\right] &= 0\end{aligned} \tag{3.45}$$

in which $Z_1 = Z_0\sqrt{\mu_{r_1}/\epsilon_{r_1}}$, $Z_2 = Z_0\sqrt{\mu_{r_2}/\epsilon_{r_2}}$ whereas α and β are again arbitrary scalars. Finally, a third coupled set of integral was proposed by Yagjian [1981] who noted that the continuity equations are not necessarily satisfied when resonant modes are present. On this basis, the continuity equations can be combined with (3.41) or (3.42) to yield the conditions

$$\begin{aligned}\mathbf{E}_1^s + \hat{n}\frac{\nabla_s \cdot \mathbf{J}}{-j\omega\epsilon_o\epsilon_{r_1}} &= -\mathbf{E}_1^i \\ \mathbf{H}_1^s + \hat{n}\frac{\nabla_s \cdot \mathbf{M}}{-j\omega\mu_o\mu_{r_1}} &= -\mathbf{H}_1^i\end{aligned} \tag{3.46}$$

From these we can readily derive integral equations for $(\mathbf{J}, \mathbf{M})$ upon substituting for the fields as given in (3.31). The integral equations based on (3.44) or (3.45) are generally referred to the literature as the *combined field integral equations* (CFIE) whereas the integral equations resulting from (3.46) is referred to as the *augmented field integral equations* (AFIE). They have all been used primarily for scattering computations and their solution will be considered later. The CFIE have also been used for radiation problems relating to various types of cavity antennas. As can be expected, the CFIE cannot yield unique solutions at those frequencies where the electric and magnetic field integral equations fall concurrently. In addition, for very large structures the spurious resonant modes of S_d are congruent leading to inaccuracies in the solution of CFIE. Further, it has been noted that the AFIE does not ensure the removal of all spurious resonances and later we will discuss other remedies which can ensure uniqueness at the resonant frequencies of the cavity enclosed by S_d.

3.1.6 Integral Equations for Piecewise Homogeneous Dielectrics

The formulation presented in the previous section for treating homogeneous dielectrics can be readily extended to bodies composed of various homogeneous dielectric sections as shown in Figure 3.5. Let us for example consider the structure in Figure 3.6 consisting of a dielectric and a perfectly conducting section. We shall denote the surface of the conducting section which borders the exterior region (region #1) as S_{dc_1} and that which borders the dielectric region of the body (region #2) as S_{dc_2}. Also, the surface of the dielectric which borders the exterior regions will be denoted as S_{dc}. The exterior region has relative dielectric constants $(\epsilon_{r_1}, \mu_{r_1})$ and a characteristic impedance $Z_1 = Z_o\sqrt{\frac{\mu_{r1}}{\epsilon_{r_1}}}$. Correspondingly, the interior dielectric region has relative dielectric constants $(\epsilon_{r_2}, \mu_{r_2})$ and a characteristic impedance $Z_2 = Z_o\sqrt{\frac{\mu_{r2}}{\epsilon_{r_2}}}$. We shall assume that the excitation fields $(\mathbf{E}^i, \mathbf{H}^i)$ will be in the exterior region although they can also be placed within the interior dielectric region as is likely the case with cavity type antennas [Arvas, etc., 1991; Shafai, etc., 1991].

Following the formulation presented in the previous section, we refer to Fig. 3.5 and introduce the equivalent currents $\mathbf{J}_{c_1}$ and $\mathbf{J}_{c_2}$ on the conducting surfaces S_{dc_1} and S_{dc_2}, respectively. Since S_{dc_1} and S_{dc_2} border perfect conductors we choose to retain only the electric equivalent currents although one could also choose to formulate the fields in terms of magnetic currents as discussed in