Solution of time harmonic Maxwell's equations (using phasors)

Review of plane waves (source free or source is far away)

Starting from

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = -jkZ\mathbf{H} \qquad \left(Z = \sqrt{\frac{\mu}{\epsilon}}\text{, impedance}\right)$$

$$\nabla \times \mathbf{H} = +j\omega\epsilon\mathbf{E} = +jkY\mathbf{E} \qquad \left(Y = \sqrt{\frac{\epsilon}{\mu}} = \frac{1}{Z}\text{, admittance}\right)$$

$$k = \omega\sqrt{\mu\epsilon} = k' - jk'' = \beta - j\alpha$$

$$\beta = \mathrm{Re}(k)$$

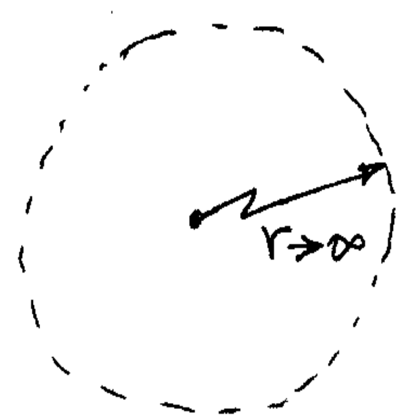
in which ϵ is complex and α is an attenuation constant. As before, taking the curl of the first Maxwell's equation and substituting into the second yields

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0 \qquad \text{vector wave equation}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$$

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \qquad \text{wave equation } (\nabla \cdot \mathbf{E} = 0)$$

$$\nabla^2 E_{\substack{x\\y\\z}} + k^2 E_{\substack{x\\y\\z}} = 0$$



We like to solve this latter equation subject to some boundary conditions. Since we assume free space, there are no conditions or interfaces to invoke $\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$ or $\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$. Instead the only boundary condition of consideration is the one at infinity. That is, our domain ends at $r \to \infty$. The only reasonable condition to apply is therefore that the field decays at infinity.

One-dimensional plane wave solution

To begin with, let us consider a one-dimensional solution to the wave equation. In this case, the field can be rewritten as

$$\mathbf{E} = \mathbf{E}_0 \; f(x) = \hat{e} \; f(x)$$

in which $\hat{e}$ represents the polarization of the wave such that $\nabla \cdot \mathbf{E} = 0$. When used in the wave equation, this field gives

$$\frac{\partial^2 f}{\partial x^2} + k^2 f = 0$$

This is a second order ordinary differential equation, and has the well-known possible solutions:

$$e^{\pm jkx}, \quad \cos kx, \quad \sin kx$$

Note also that

$$e^{\pm jkx} \longrightarrow e^{+jk'x} e^{-k''x}$$

with $e^{-k''x}$ vanishing to 0 when $x \to \infty$ ($k'' > 0$). Hereon, we will assume $k'' = 0$. Thus ($k = k' - jk'' = \beta$)

$$f(x) = C_1 e^{-j\beta x} + C_2 e^{+j\beta x}$$

is the appropriate solution. In time domain

$$\begin{aligned} f(x,t) = |C_1| \cos(\omega t - \beta x + \phi_1) \quad &\longleftarrow \mathrm{Re}\{C_1 e^{-j\beta x} e^{j\omega t}\}, \quad C_1 = |C_1| e^{j\phi_1} \\ + |C_2| \cos(\omega t + \beta x + \phi_2) & \end{aligned}$$

We can rewrite the above in a more general form as

$$\mathbf{E} = \hat{e}(C_1 e^{-j\mathbf{k}_1 \cdot \hat{x}x} + C_2 e^{-j\mathbf{k}_2 \cdot \hat{x}x})$$

with

$$\nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

If we choose

$$\hat{e} = \left\{ \hat{y}, \; \hat{z}, \; \text{or} \; \frac{\hat{y} + \hat{z}}{\sqrt{2}}, \; \text{etc.} \right\}$$

then $\nabla \cdot \mathbf{E} = 0$, as required.

We identify

$$\mathbf{k}_1 = \hat{x} k_0, \qquad \mathbf{k}_2 = -\hat{x} k_0,$$
$$k_0 = \beta_0 = \omega \sqrt{\mu_0 \epsilon_0}$$

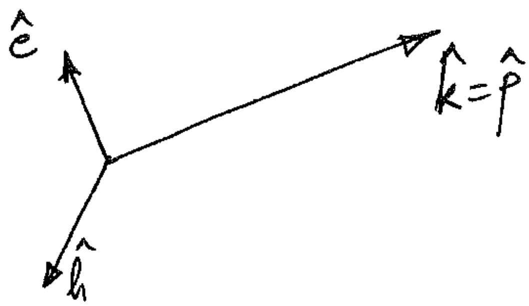
so that $\mathbf{k} \cdot \hat{x}x = k_0 x$ as required. The unit vector

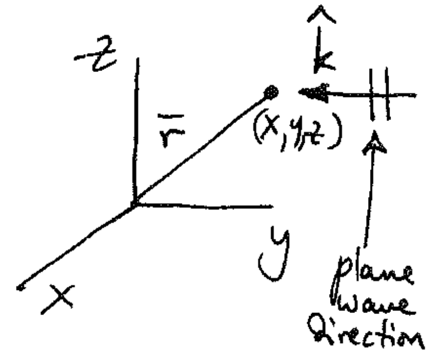
$$\hat{p} = \hat{k} = \frac{\mathbf{k}}{k_0}$$

represents the direction of the wave.

$\bar{E} = \hat{e} E$

$\bar{H} = \hat{h} H$





Using this identification, we can write an expression for the general plane wave as

$$\mathbf{E}(\mathbf{r}) = \hat{e}\{C_1 e^{-j\mathbf{k}\cdot\mathbf{r}} + C_2 e^{+j\mathbf{k}\cdot\mathbf{r}}\}, \qquad \text{in phasor form}$$

$$\overline{\epsilon}(\mathbf{r}, t) = \hat{e}|C_1|\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi_1) + \hat{e}\cos(\omega t + \mathbf{k}\cdot\mathbf{r} + \phi_2), \qquad \text{in time domain form}$$

$$\mathbf{H} = Y(\hat{p} \times \mathbf{E})$$

$$\mathbf{k} = (\hat{x}k_x + \hat{y}k_y + \hat{z}k_z) \qquad \text{direction of the wave}$$

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

where $\mathbf{r}$ is the position vector at which location we measure the phase of the wave. Clearly, the equations $\omega t - \mathbf{k}\cdot\mathbf{r} =$ constant, or $\mathbf{k}\cdot\mathbf{r} =$ constant refer to equations of a plane (the plane on which the phase of the wave is constant). This is the reason why the fields of this form are called "plane waves," i.e., at a specific time, the constant phase surfaces of the wave are planes.

Plane waves in spherical coordinates

Set

$$\mathbf{k} = k_0(\hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta) = k_0\hat{r}$$

$$k_x = k_0\cos\phi\sin\theta$$

$$k_y = k_0\sin\phi\sin\theta$$

$$k_z = k_0\cos\theta$$

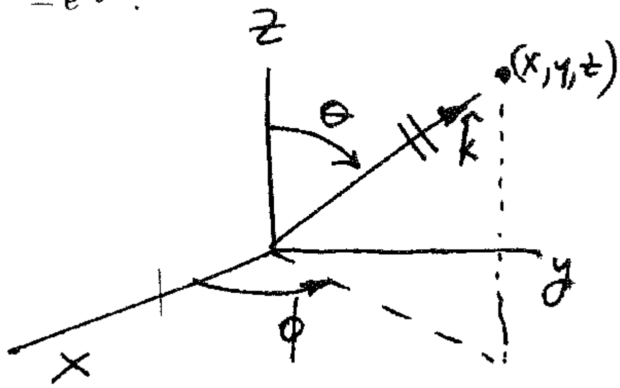
Thus, we write $\mathbf{E}$ as

$$\mathbf{E} = \hat{e}e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{e}e^{-jk_0(x\cos\phi\sin\theta + y\sin\phi\sin\theta + z\cos\theta)} \qquad \text{(outgoing wave)}$$

For constant $\hat{k} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta$, the exponent $\Phi = x\cos\phi\sin\theta + y\sin\phi\sin\theta + z\cos\theta =$ const. represents a plane, and therefore the wave is referred to as a *plane wave.*

Spherical waves

Consider, $e^{-j\mathbf{k}\cdot\hat{r}} = e^{-jkr}$.



Since we know that waves have a decay of $\frac{1}{r}$, a proposed solution in 3D is

$$\frac{e^{-jkr}}{r}$$

Spherical waves decay in accordance with the $\frac{1}{r}$ factor and have constant phase on $r = r_i$ spheres.

But let's not take the above generalization for granted. Let us instead proceed to solve the wave equation in 3D using the method of separation of variables (uniqueness theorem helps here).

General solution (separation of variables)

$$\left.\begin{array}{r} \nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \\ \text{with } \nabla\cdot\mathbf{E} = 0 \end{array}\right\} \quad \Leftarrow \quad \begin{array}{c} \nabla\times\mathbf{E} = -jkZ\mathbf{H} \\ \nabla\times\mathbf{H} = +jkY\mathbf{E} \end{array}$$

$$jkY = j\omega\epsilon, \qquad jkZ = j\omega\mu$$

$$k = \omega\sqrt{\mu\epsilon} = k' - jk'' = \beta - j\alpha$$

where α is the attenuation constant. Consider the solution

$$\mathbf{E}(x,y,z) = \hat{e}\,X(x)\,Y(y)\,Z(z) = \hat{e}\,\psi(x,y,z)$$

Then

$$\nabla^2\psi + k^2\psi = 0 \qquad \text{(Helmholtz equation)}$$

Substituting $\psi = XYZ$, we get

$$YZ\,\frac{\partial^2 X}{\partial x^2} + XZ\,\frac{\partial^2 Y}{\partial y^2} + XY\,\frac{\partial^2 Z}{\partial z^2} = 0$$

$$\frac{\partial X}{\partial x} = X'$$

or (divide by XYZ)

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}+k^2=0$$

Since X, Y and Z are independent of each other, each must be a constant if their sum is a constant. Thus, we conclude that

$$\frac{X''}{X}=-k_x^2, \qquad \frac{Y''}{Y}=-k_y^2, \qquad \frac{Z''}{Z}=-k_z^2$$

$\Longrightarrow$

$$X''+k_x^2X=0, \qquad Y''+k_y^2Y=0, \qquad Z''+k_z^2Z=0$$

and

$$k_x^2+k_y^2+k_z^2=k^2 \qquad \text{(consistency condition or characteristic equation)}$$

General solution is (see pp. 85 and 143 of Harrington)

$$X(x)=\begin{cases} A_1e^{-jk_xx}+B_1e^{jk_xx} & \text{more convenient} \\ A_1'\cos k_xx+B_1'\sin k_xx & \text{not convenient}\end{cases}$$

$$Y(y)=\begin{cases} A_2e^{-jk_yy}+B_2e^{jk_yy} \\ A_2'\cos k_yy+B_2'\sin k_yy\end{cases}$$

$$Z(z)=\begin{cases} A_3e^{-jk_zz}+B_3e^{jk_zz} \\ A_3'\cos k_zz+B_3'\sin k_zz\end{cases}$$

We remark that each of these solutions satisfies the so-called radiation condition at infinity ($\xi\to\infty$):

$$\text{1D:} \quad \frac{\partial E_t}{\partial \xi}\pm jk_0E_t=0$$

where $\xi=\{\,x,y,\text{ or } z\,\}$.

In 3D, the general form of the radiation condition is

$$\lim_{r\to\infty} r(\nabla\times\mathbf{E}+jk_0\hat{r}\times\mathbf{E})=0$$

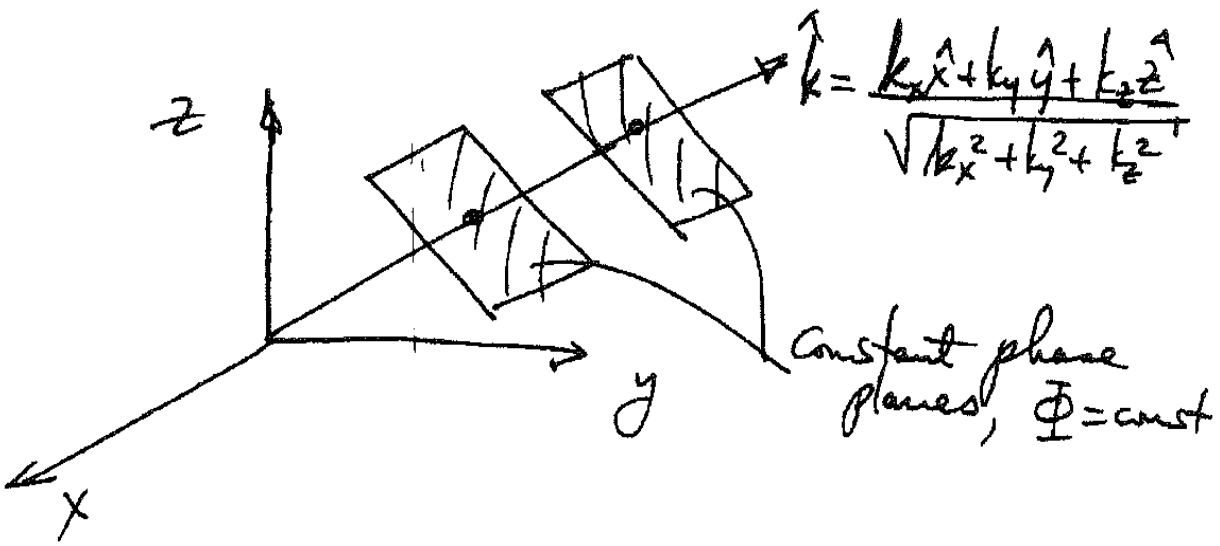
(in unbounded media). Fields must satisfy Maxwell's equations and the radiation condition as stated above. A specific example of the solutions $E=X(x)\,Y(y)\,Z(z)$ is

$$\mathbf{E}(x,y,z)=\hat{e}E_0e^{-jk_xx}e^{-jk_yy}e^{-jk_zz}=\hat{e}E_0e^{-j\,\Phi(x,y,z)}$$

$$\text{In time domain, } \mathcal{E}=\hat{e}E_0\cos(\omega t-\Phi)$$

$$\hat{k}=\frac{k_x\hat{x}+k_y\hat{y}+k_z\hat{z}}{\sqrt{k_x^2+k_y^2+k_z^2}}$$

$$\Phi(x,y,z)=\text{constant}$$



We remark that this wave propagates along $\hat{k}$, where $\hat{k}$ is normal to the surface

$$-\Phi(x, y, z) = \text{constant}$$

In fact, the "phase front" (the plane on which the wave has constant phase) of the wave is on one of these surfaces (say $\Phi = \Phi_0$) at some specific time $t = t_0$. Since

$$\Phi = k_x x + k_y y + k_z z = \text{constant}$$

is the equation of a plane, the wave is called "plane wave" (i.e., its constant phase surfaces are planes).

Spherical and cylindrical waves

If

$$\Phi = k\sqrt{x^2 + y^2 + z^2} = kr$$

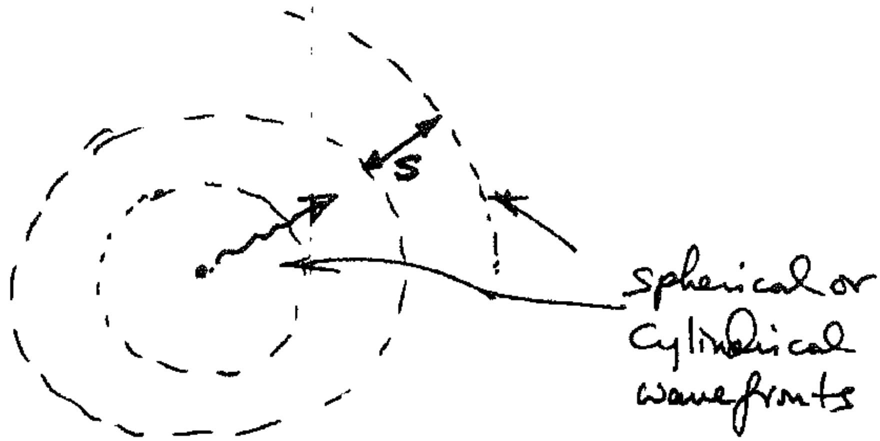
then we will refer to the wave as "spherical." If

$$\Phi = k\sqrt{x^2 + y^2} = k\rho$$

we will refer to the wave as "cylindrical," and so on.

We remark that the solution of the wave equation in spherical coordinates is

$$\frac{e^{\pm jkr}}{r}$$



Power decays as $1/r^2$.

Also, the solution of the wave equation in cylindrical coordinates is

$$\frac{e^{\pm jkr}}{\sqrt{\rho}}$$

Power decays as $1/\rho$.

Velocity of the plane wave

The velocity of the wave represents the speed at which the wave moves from one phase front Φ to another. Alternatively, the wave velocity is that seen by an observer who is situated on a specific phase front Φ as the wave passes by. To find the wave velocity, let us try to "track" the crest of the wave. For a sinusoidal wave $\cos(\omega t - \Phi)$, the wave crest satisfies the condition

$$d(\omega t - \Phi) = 0 \quad \Rightarrow \quad \omega\, dt - d\Phi = 0$$

where $d\Phi =$ the derivative along direction of the wave. By chain rule

$$d\Phi = \frac{\partial \Phi}{\partial x}\, dx + \frac{\partial \Phi}{\partial y}\, dy + \frac{\partial \Phi}{\partial y}\, dz$$

it follows that

$$d\Phi = \nabla\Phi \cdot d\mathbf{s} = (\nabla\Phi \cdot \hat{k})\, ds$$

Thus

$$\omega\, dt - d\Phi = \omega\, dt - (\nabla\Phi \cdot \hat{k})\, ds = 0 \quad \Rightarrow$$

$$\boxed{\frac{ds}{dt} = \text{Velocity} = \frac{+\omega}{\nabla\Phi \cdot \hat{k}}}$$

For a plane wave,

$$\frac{ds}{dt} = v_{\text{phase}} = v_{\text{p}} = \frac{+\omega}{k}$$

since for a plane wave

$$\nabla\Phi \cdot \hat{k} = (\hat{x}k_x + \hat{y}k_y + \hat{z}k_z) \cdot \hat{k} = k$$

Also, the velocity along the x direction is

$$v_{\text{p}_x} = \frac{\omega}{\nabla\Phi \cdot \hat{x}} = \frac{\omega}{k_x} \qquad \text{etc.}$$

Other forms of plane waves

We can rewrite $\mathbf{E}$ as

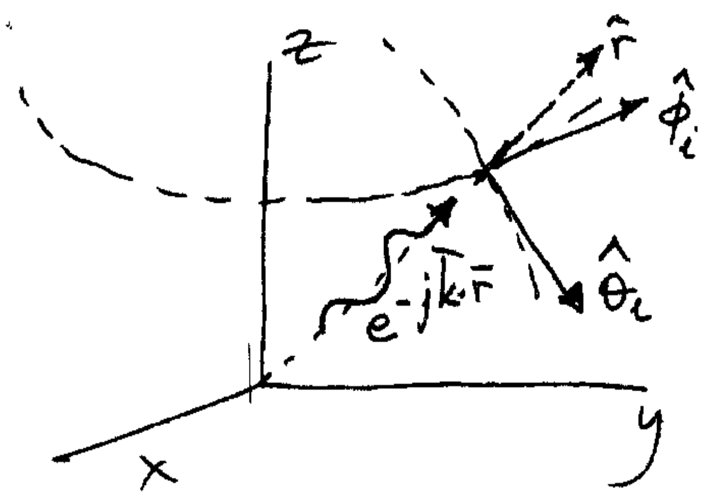
$$\mathbf{E} = \hat{e}E_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{e}E_0 e^{-jk_0\hat{k}\cdot\mathbf{r}}$$

If

$$\hat{k} = \hat{x}\cos\phi_{\text{i}}\sin\theta_{\text{i}} + \hat{y}\sin\phi_{\text{i}}\sin\theta_{\text{i}} + \hat{z}\cos\theta_{\text{i}}$$

where $(\theta_{\text{i}}, \phi_{\text{i}})$ is the direction angles of the wave, then

$$\mathbf{E} = \hat{e}E_0 e^{-jk_0(x\cos\phi_{\text{i}}\sin\theta_{\text{i}} + y\sin\phi_{\text{i}}\sin\theta_{\text{i}} + z\cos\theta_{\text{i}})}$$



Usually

$$\hat{e} = \left\{\phi_{\text{i}}E_0 \text{ or } \hat{\theta}_{\text{i}}E_0\right\}$$

(no $\hat{r}$ component exists for a plane wave since $\nabla \cdot \mathbf{E} \neq 0$) when defined in spherical system or free space.

Case 1: $\theta_\mathrm{i} = \frac{\pi}{2}$

$$\begin{aligned}\mathbf{E} &= \left\{\begin{array}{c}\hat{\phi}_\mathrm{i}\\ \hat{\theta}_\mathrm{i}\end{array}\right\} E_0 e^{-jk(x\cos\phi_\mathrm{i} + y\sin\phi_\mathrm{i})}\\ &= \left\{\begin{array}{c}\hat{\phi}_\mathrm{i}\\ \hat{\theta}_\mathrm{i}\end{array}\right\} E_0 e^{-jk\hat{\rho}^\mathrm{i}\cdot\boldsymbol{\rho}}\end{aligned}$$

$$\hat{\phi}_\mathrm{i} = -\hat{x}\sin\phi_\mathrm{i} + \hat{y}\cos\phi_\mathrm{i}$$

$$\hat{\theta}_\mathrm{i} = \hat{x}\cos\theta_\mathrm{i}\cos\phi_\mathrm{i} + \hat{y}\cos\theta_\mathrm{i}\sin\phi_\mathrm{i} - \hat{z}\sin\theta_\mathrm{i}\big|_{\theta_\mathrm{i}=\pi/2}$$

Case 2: $\phi_\mathrm{i} = \frac{\pi}{2}$ and $\theta_\mathrm{i} = \frac{\pi}{2}$

$$\mathbf{E} = \left\{\begin{array}{c}-\hat{x}\\ -\hat{z}\end{array}\right\} E_0 e^{-jky} \qquad \leftarrow \text{ as before}$$

Note that for plane waves

$$\mathbf{H} = Y\hat{k}\times\mathbf{E}, \qquad \mathbf{E} = Z\mathbf{H}\times\hat{k}$$

i.e., $\nabla \to \hat{k}$ for Maxwell's equations for plane waves.

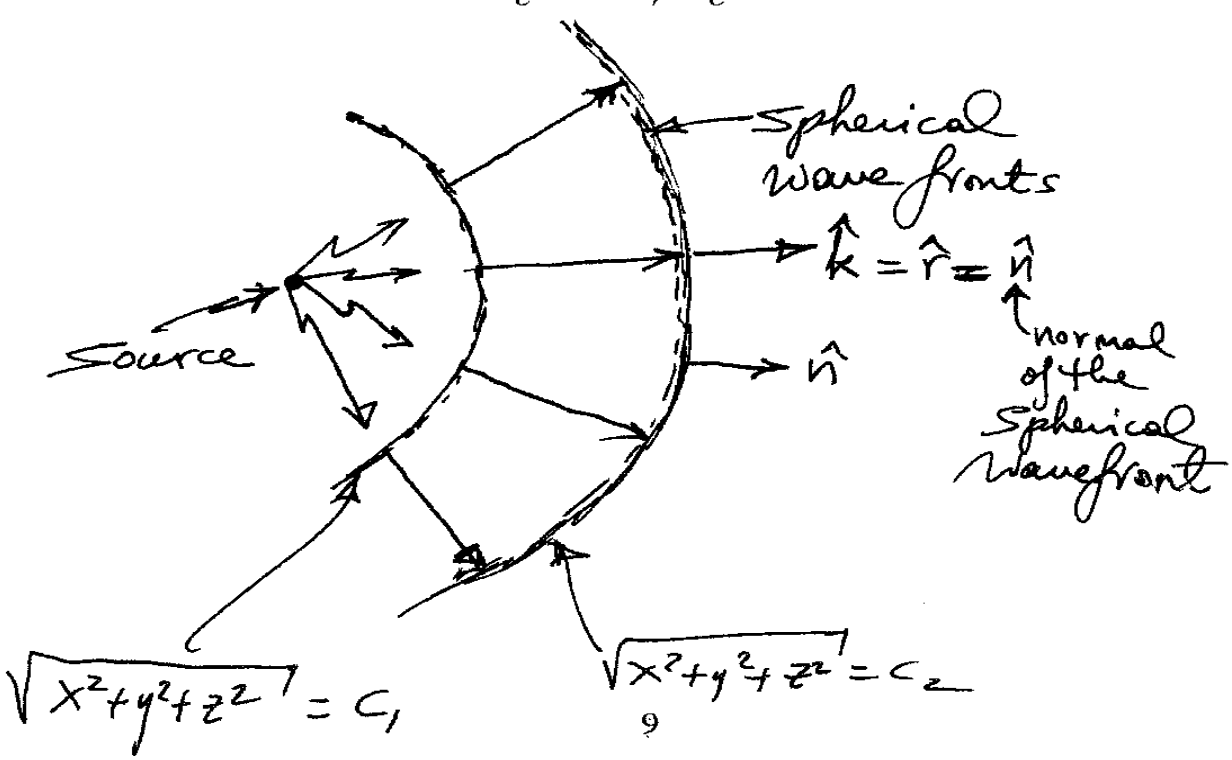
Spherical wave fronts

If

$$\Phi = k\sqrt{x^2+y^2+z^2} = kr$$

then

$$e^{-j\Phi} \quad \to \quad e^{-jkr}$$

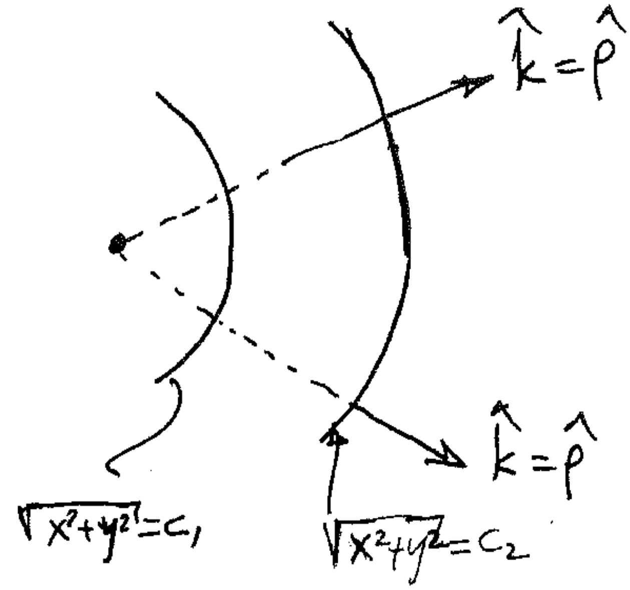


Cylindrical wave fronts

$$\Phi = k\sqrt{x^2 + y^2} \longrightarrow k\rho$$

then

$$e^{-j\Phi} \longrightarrow e^{-jk\rho}$$



Spherical solutions

The following spherical wave is also a solution to Maxwell's equations:

$$\mathbf{E} \sim \hat{e}\,\frac{e^{\pm jkr}}{r}$$

This is the field generated by all 3D sources as $r \to \infty$, subject to the radiation/boundary condition

$$\lim_{r\to\infty} r[\hat{r} \times (\nabla \times \mathbf{E}) \mp jk\mathbf{E}] = 0$$

Cylindrical solutions

The following cylindrical wave is also a solution to Maxwell's equations:

$$\mathbf{E} \sim \hat{z}\,\frac{e^{\pm jk\rho}}{\sqrt{\rho}}$$

This field is generated by all 2D sources (for example, current of an infinite wire) as $\rho \to \infty$ subject to the radiation/boundary condition

$$\lim \sqrt{\rho}\left\{\frac{\partial}{\partial\rho} \mp jk\right\} E_z = 0$$