Theorems

So far, we have talked about several theorems used in electromagnetics. Among them are

1) Image theory

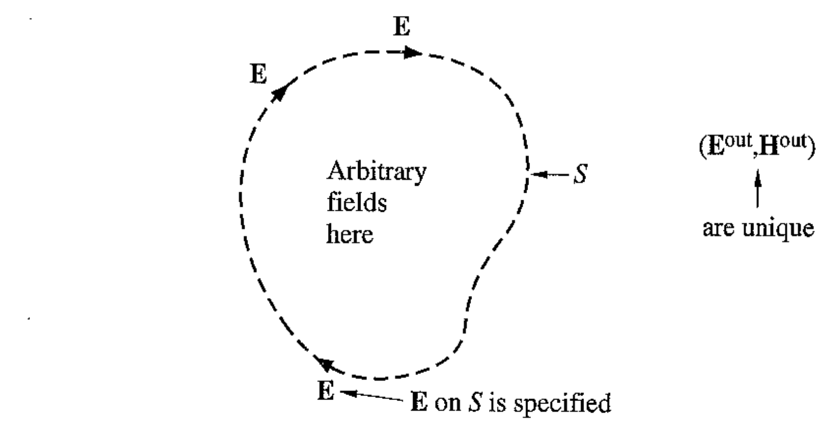
2) Duality

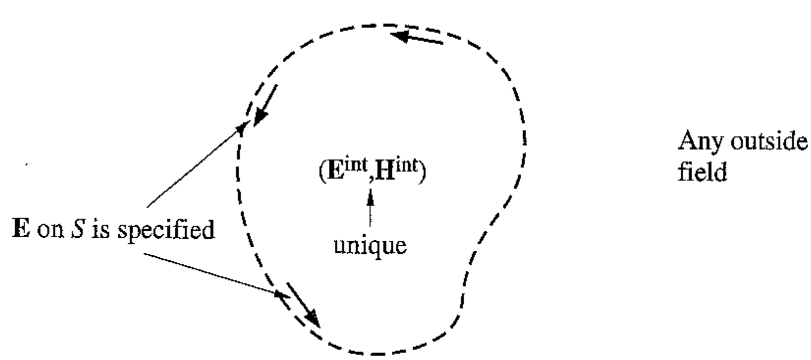
3) Superposition: if $J_1 \to \mathbf{E}_1$ and $J_2 \to \mathbf{E}_2$ in isolation $\Longrightarrow (\mathbf{J}_1 + \mathbf{J}_2) \to \mathbf{E} + \mathbf{E}_2$.

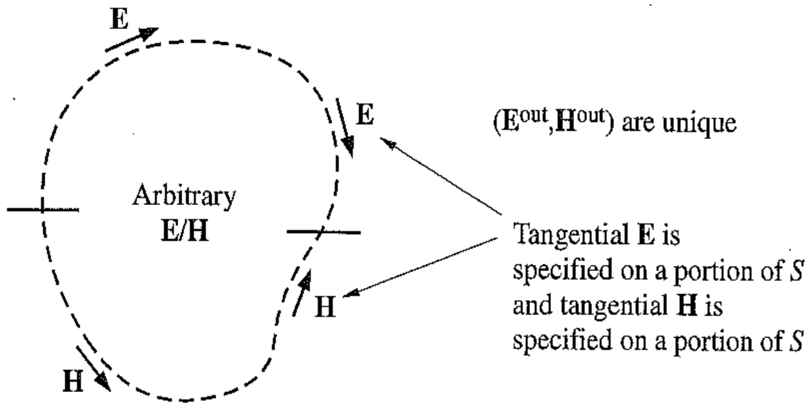
A fourth very important theorem is the *equivalence theorem.* The reason we have not discussed the equivalence theorem so far is because this theorem is of value when certain applications in radiation and scattering are considered. Having introduced radiation and scattering, we now proceed to discuss the equivalence theorem and its basic role in EM problems. I should state that almost all formulations leading to a numerical solution, aperture theory etc. employ the equivalence theorem to simplify and carry out the analysis. The equivalence theorem has its "roots" in the uniqueness theorem: This theorem states that if we find "a solution" (that satisfies the boundary conditions and wave equation) to a problem, then, this solution is "the solution" to the subject problem. A consequence of the uniqueness theorem is the following:

If tangential fields **E** or **H** on a closed surface are known, then the fields exterior to the surface are uniquely defined.

1)

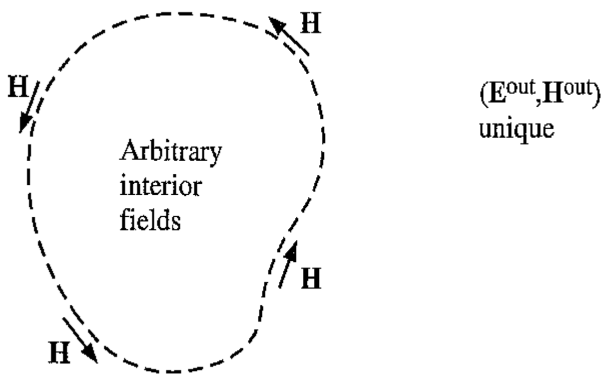






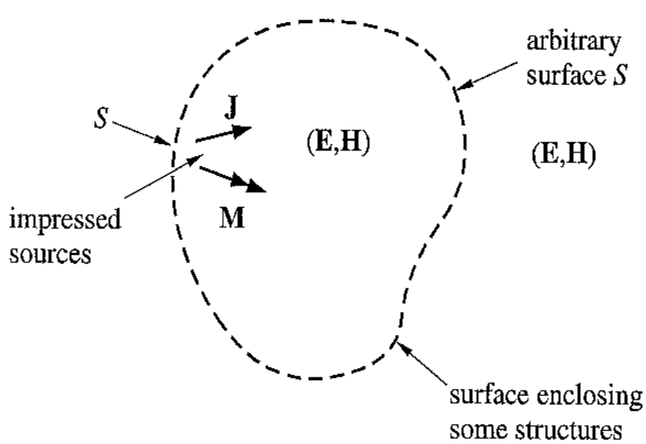
Similarly for the interior fields.

3)

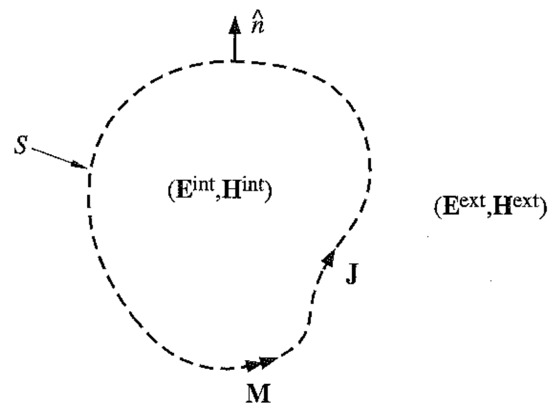


There are various forms of the equivalence principle to be considered.

First equivalence



For convenience, let us rename the fields inside S as $(E^{\text{int}}, H^{\text{int}})$ and those outside S as $(E^{\text{ext}}, H^{\text{ext}})$.



Then, in accordance with boundary conditions it is necessary that currents

$$\mathbf{J} = \hat{n} \times (\mathbf{H}^{\text{ext}} - \mathbf{H}^{\text{int}})$$
$$\mathbf{M} = -\hat{n} \times (\mathbf{E}^{\text{ext}} - \mathbf{E}^{\text{int}})$$

exist on S. The significance of this set-up is as follows:

If we arbitrarily set $(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}}) = 0$, then

$$\mathbf{J} = \hat{n} \times \mathbf{H}^{\text{ext}}$$
$$\mathbf{M} = -\hat{n} \times \mathbf{E}^{\text{ext}}$$

This reduction is called (*Love's equivalence theorem*, also referred to the *induction theorem*). Based on the uniqueness theorem, these currents are guaranteed to produce the fields $(\mathbf{E}^{\text{ext}}, \mathbf{H}^{\text{ext}})$. Given $\mathbf{J}$ and $\mathbf{M}$, the actual $(\mathbf{E}^{\text{ext}}, \mathbf{H}^{\text{ext}})$ are obtained from

$$\begin{aligned}\mathbf{E}^{\text{ext}} &= -j\omega\mathbf{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{A} - \frac{1}{\epsilon}\nabla\times\mathbf{F} \\ &= \iint \left\{-\nabla\times[\mathbf{M}G] - jk_0Z_0\mathbf{J}G - \frac{jZ_0}{k_0}\left(\nabla'\cdot\mathbf{J}\right)\cdot\nabla G\right\}\, ds'\end{aligned}$$

or

$$\mathbf{E}^{\text{ext}} = \iint \left\{(\mathbf{E}^{\text{ext}}\times\hat{n}')\times\nabla G - jk_0Z_0(\hat{n}'\times\mathbf{H}^{\text{ext}})G - \frac{jZ_0}{k_0}[\nabla'\cdot(\hat{n}'\times\mathbf{H}^{\text{ext}})]\cdot\nabla G\right\}$$

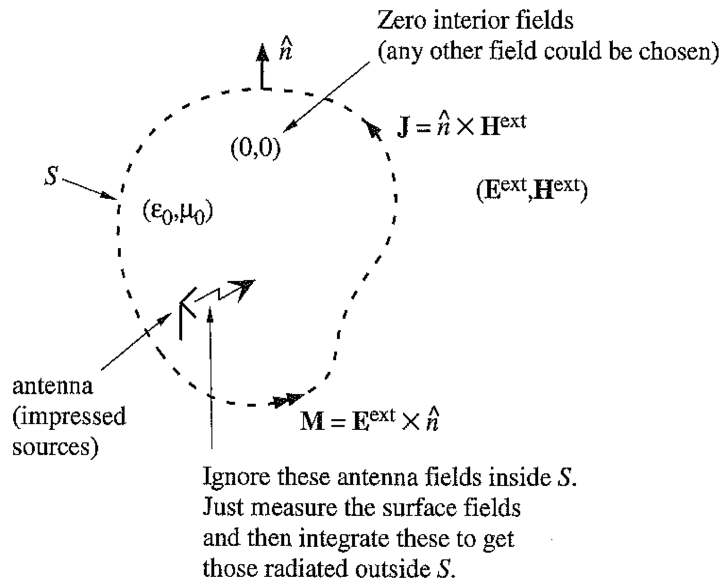
where we also note that $\nabla\times\mathbf{M}G = -\mathbf{M}\times\nabla G\cdot\mathbf{H}^{\text{ext}}$ is obtained via duality and

$$\hat{n}'\cdot\mathbf{E}^{\text{ext}} = -\frac{jZ_0}{k_0}\left[\nabla'\cdot(\hat{n}'\times\mathbf{H}^{\text{ext}})\right]$$

an expression commonly referred to as the *Stratton-Chu integral representation.*

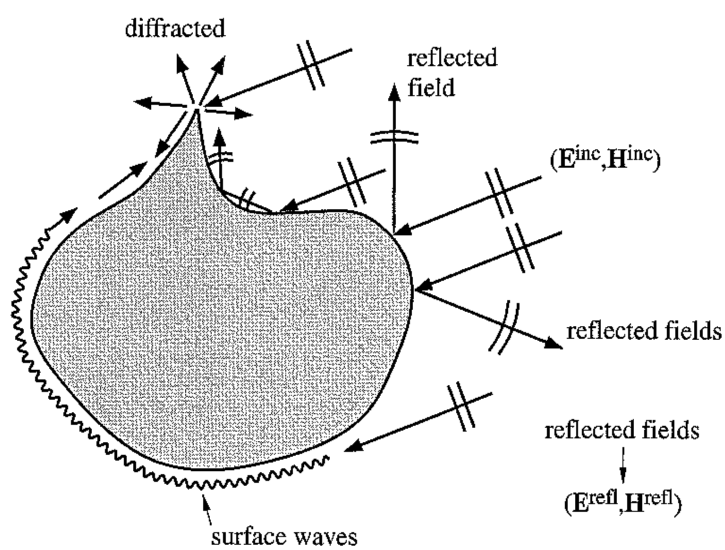
The significance of the above is that we can avoid a need for getting close to the antenna or scatterer, since measuring its surface fields/currents is very difficult if not impossible.

Instead, we select a surface S enclosing the antenna or the scatterer. Measure the fields around it and then use these surface fields inside the Stratton-Chu integral to get the fields in the entire space including far zone:



The above formulation is commonly used for solving radiation and scattering of multimedia problems.

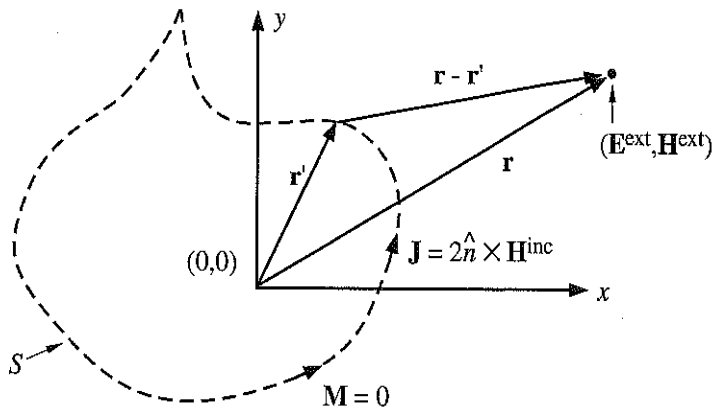
An interesting application of the above equivalence therem is in scattering by PEC structures. In this case a source field is impinging from the radar as shown below.



Thus, on the surface S,

$$\begin{aligned}\mathbf{H}^{\text{ext}} &= \mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{scat}} \\ &\approx \mathbf{H}^{\text{inc}} + R\mathbf{H}^{\text{refl}}\end{aligned}$$

If the target is PEC, then $\hat{n} \times \mathbf{E}^{\text{ext}} = 0$ and $\hat{n} \times \mathbf{H}^{\text{ext}} \approx 2\hat{n} \times \mathbf{H}^{\text{inc}}$. We then get the following set-up for finding the radiated fields



This approximation is referred to as the *P.O. field approximation.*

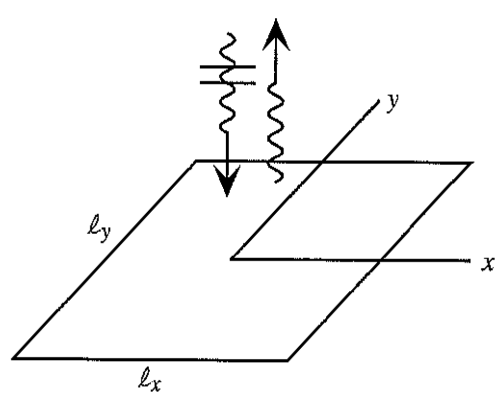
The far zone field is

$$\mathbf{E}^{\text{ext}} = \mathbf{E}^{\text{scat}} \approx \mathbf{E}^{\text{PO}} = \frac{-jk_0 Z_0}{4\pi} \frac{e^{-jkr}}{r} \iint\limits_{\text{Area}} 2\hat{n} \times \mathbf{H}^{\text{i}} e^{+jk\hat{r}\cdot\mathbf{r}'}\, ds'$$

and a quantity of interest is the radar cross section (RCS) given by

$$\begin{aligned}\sigma &= 4\pi r^2_{r\to\infty} \left|\frac{\mathbf{E}^{\text{scat}}}{\mathbf{E}^{\text{inc}}}\right|^2 = 4\pi r^2_{r\to\infty} \left|\frac{\mathbf{H}^{\text{scat}}}{\mathbf{H}^{\text{inc}}}\right|^2 \\ &= \frac{(k_0 Z_0)^2}{|\mathbf{E}^{\text{i}}|^2} \left| \iint 2(\hat{n} \times H^{\text{i}}) e^{jk\hat{r}\cdot\mathbf{r}'}\, ds' \right|^2\end{aligned}$$

As an example for a normal incidence on a flat plate



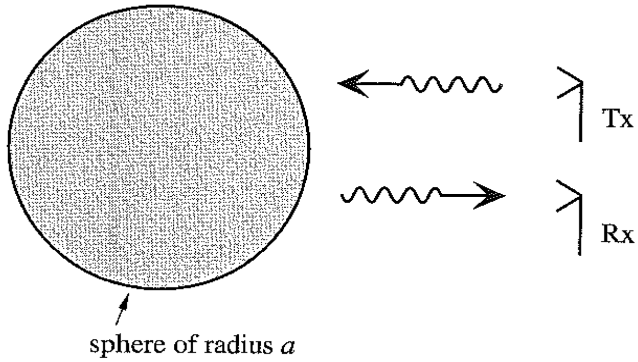
we get

$$\sigma = \frac{4\pi(\ell_x \ell_y)^2}{\lambda^2} = \frac{4\pi A^2}{\lambda^2}$$

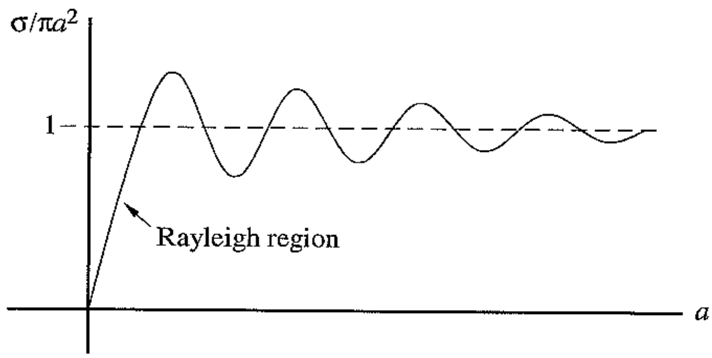
i.e., σ is proportional to the area squared and is measured in units of length squared (λ^2 or m^2). Because of this relation, it is often called the *echo-area* of the target.

RCS of other targets

Sphere

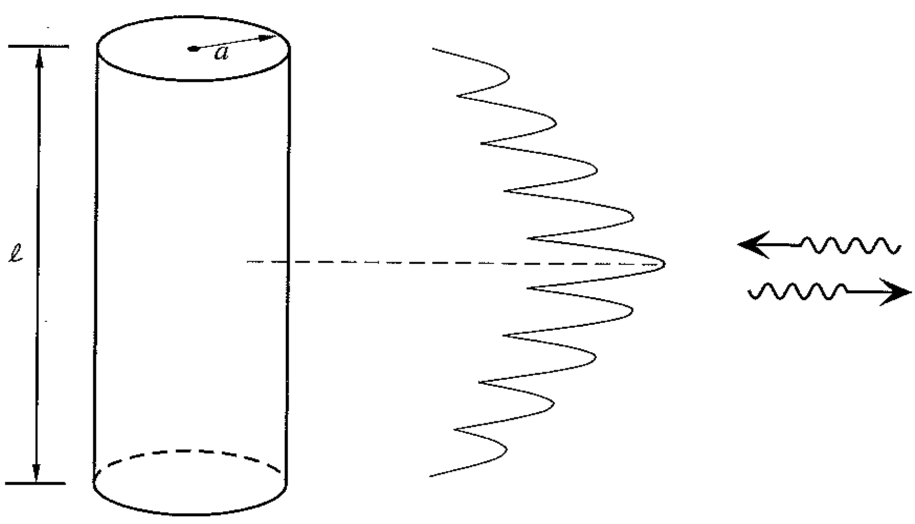


$$\sigma = \pi a^2 \quad \text{for } a \ll \lambda, \quad > 1\lambda$$



Spheres are used for calibration in radar or anechoic chambers since their σ is known exactly.

Cylinder



P.O. approximations of the radar cross section

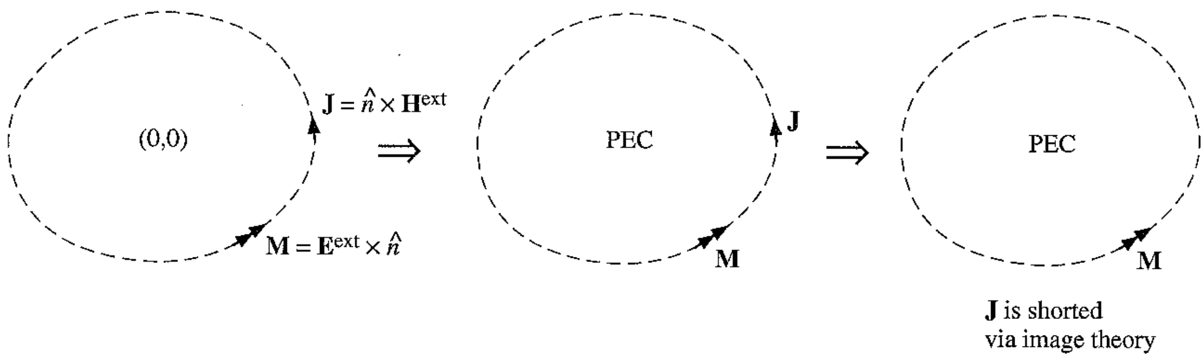
$$\sigma_{\text{backscatter}} = ka\ell^2$$

$$\sigma_{\text{oblique}} = ka\ell^2 \cos\theta \frac{\sin^2(k\ell\sin\theta)}{k^2\ell^2\sin^2\theta}$$

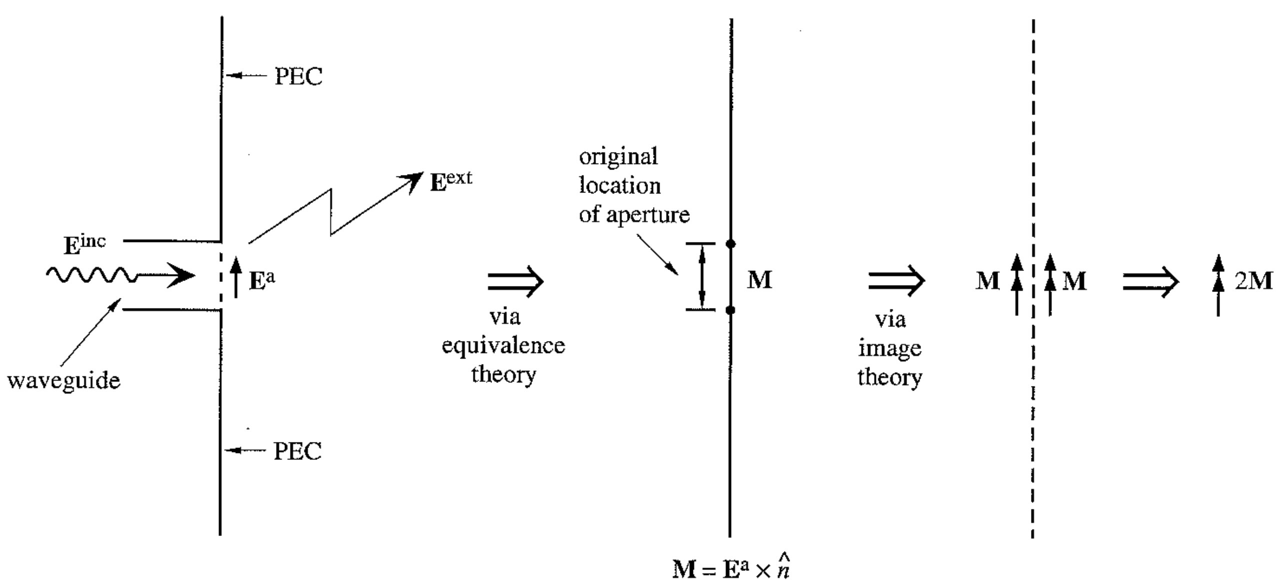
Second equivalence

Based on the uniqueness theorem, only **M** or **J** currents (viz., **E** or **H**) fields are needed on any portion of the surface for a unique determination of the fields outside S. A way to exploit this statement is to specialize the first equivalence as follows:

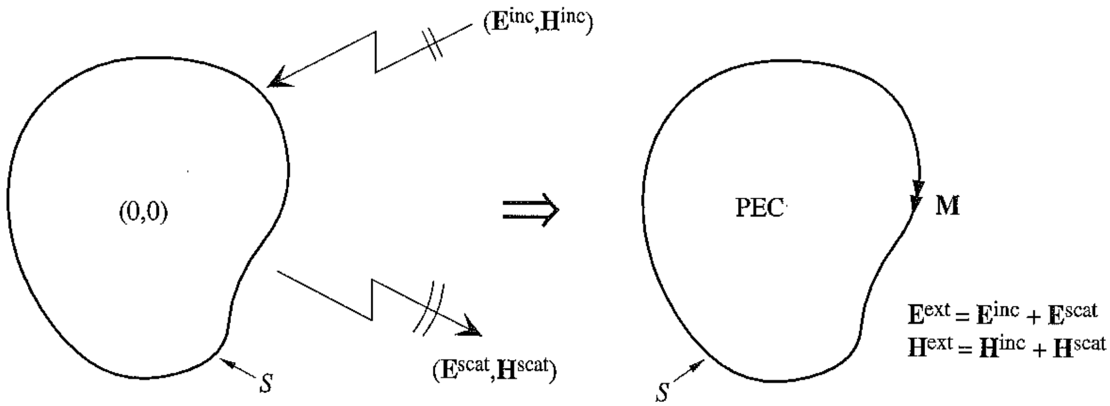
Since we choose the fields inside S to be zero, we can proceed one step further and associate these fields with a metal medium. As a result, we get the following set-up:



A typical application of this equivalence considered earlier was that of radiation through an aperture. Let us again review the waveguide radiation problem.



A very powerful integral equation for the determination of **M** in scattering occurs in conjunction with this equivalence. Let us consider for example the case of an incidence on a metal structure.



$\mathbf{E}^{\text{scat}}$ is due to the magnetic current generated on the target, viz.

$$\mathbf{E}^{\text{scat}} = \iint_S \mathbf{M} \times \nabla G \, ds'$$

But on S

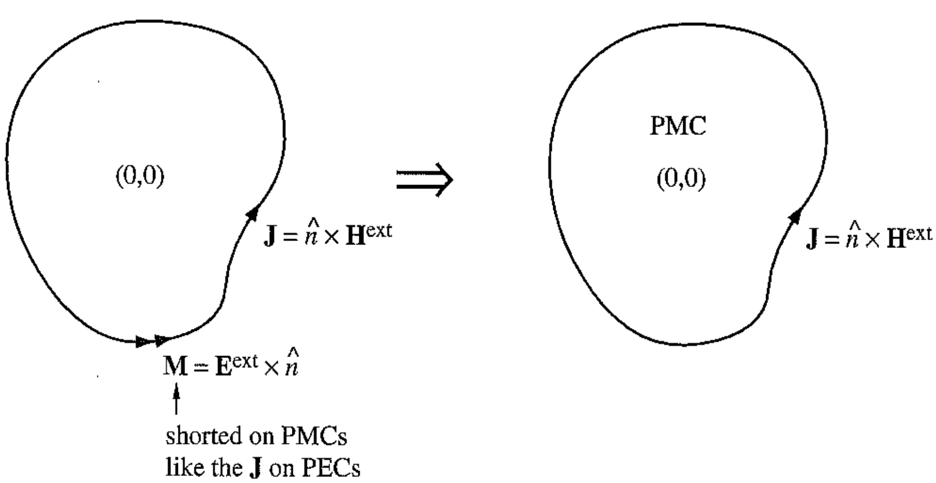
$$\hat{n} \times (\mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{scat}}) = 0 \qquad \Longrightarrow$$

$$\boxed{-\hat{n} \times \mathbf{E}^{\text{inc}} = \iint \mathbf{M} \times \nabla G \, ds'}$$

This is the so-called MFIE used for the determination of the magnetic current M. Note that in this evaluation **M** is simply a quantity which generates the desired field so that the boundary condition is satisfied. One does not have to relate it to $\mathbf{E}^{\text{ext}} \times \hat{n}$. Another way to think of **M** is as the required surface current to generat the "jump" in the **E** field from zero in the PEC body to $\mathbf{E}^{\text{ext}} \times \hat{n}$ just outside S.

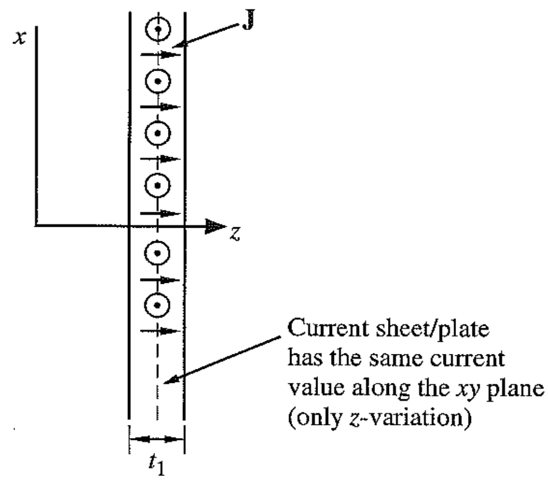
Third equivalence

We can develop a parallel equivalence to that of #2 by instead making the interior to S a PMC. This is a choice of ours since PMCs also yield $(0, 0)$ internal fields. Thus, we have



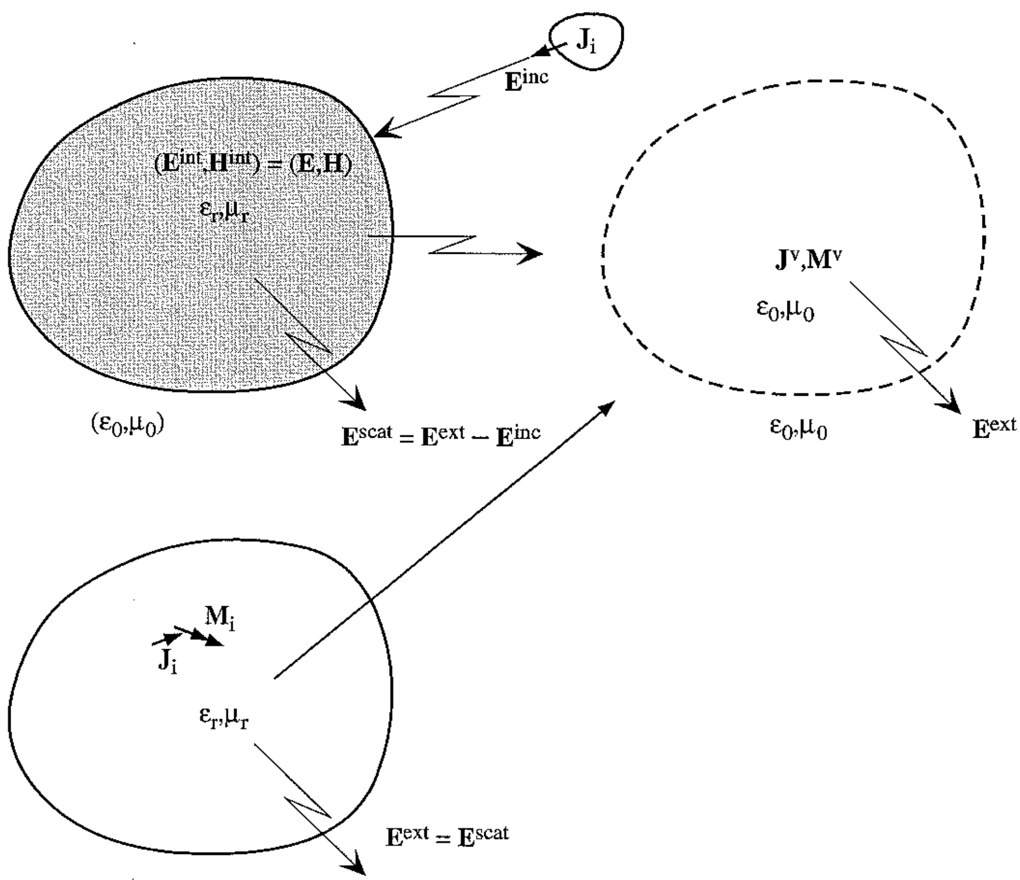
It is important to note that the introduction of PEC and PMC interior is not necessary at all. Of importance throughout the surface equivalences is the uniqueness theorem itself. The latter states that knowledge of either tangential $\mathbf{E}$ or $\mathbf{H}$ on the surface S is sufficient to uniquely determine the fields exterior to the closed surface. We can translate this to mean that knowledge (or determination) of some $\mathbf{J}$ or $\mathbf{M}$ on S is sufficient for a unique solution of the exterior fields. If both interior and exterior fields need to be determined uniquely, then knowledge of the pair of current $(\mathbf{J}, \mathbf{M})$ on S is required.

1D sources



$$\mathbf{A} = \mu \int_{t_1}^{t} J(z') \, \frac{e^{-jk_0|z-z'|}}{2jk_0} \, dz'$$

Volume equivalence



The above $(\mathbf{J}_\mathrm{v}, \mathbf{M}_\mathrm{v})$ are equivalent volume currents which generate the same fields $(\mathbf{E}^{\mathrm{ext}}, \mathbf{H}^{\mathrm{ext}})$ as if the dielectric was there. The derivation in terms of $(\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}})$ is straightforward.

Consider Maxwell's equations for the incident field in free space

$$\nabla \times \mathbf{H}^{\text{inc}} = \mathbf{J}_{\text{i}} + j\omega\epsilon_0\mathbf{E}^{\text{inc}}, \qquad \nabla \times \mathbf{E}^{\text{inc}} = -\mathbf{M}_{\text{i}} - j\omega\mu_0\mathbf{H}^{\text{inc}}$$

Also for total fields inside the dielectric, satisfies the equations

$$\nabla \times \mathbf{H}^{\text{int}} = j\omega\epsilon\mathbf{E}^{\text{int}}, \qquad \nabla \times \mathbf{E}^{\text{int}} = -j\omega\mu\mathbf{H}^{\text{int}},$$

where (μ, ϵ) refer to the constitutive parameters in the dielectric volume. Subtracting the two sets of equations gives

$$\nabla \times (\mathbf{H}^{\text{int}} - \mathbf{H}^{\text{inc}}) = j\omega(\epsilon - \epsilon_0)\mathbf{E}^{\text{int}} + j\omega\epsilon_0\mathbf{E}^{\text{scat}}$$
$$\nabla \times (\mathbf{E}^{\text{int}} - \mathbf{E}^{\text{inc}}) = j\omega(\mu - \mu_0)\mathbf{H}^{\text{int}} - j\omega\mu_0\mathbf{H}^{\text{scat}}$$

in which $(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}})$ was replaced by $\mathbf{H}^{\text{int}} = \mathbf{H}^{\text{scat}} + \mathbf{H}^{\text{inc}}$ and $\mathbf{E}^{\text{int}} = \mathbf{E}^{\text{scat}} + \mathbf{E}^{\text{inc}}$ to obtain the right hand sides. Doing

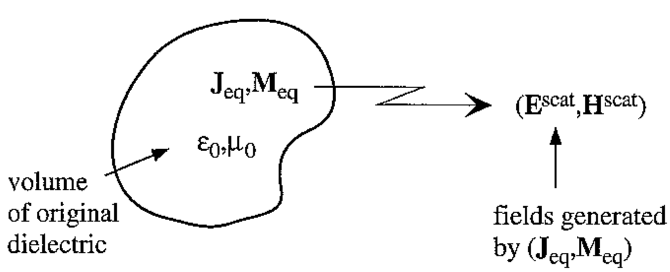
$$\nabla \times \mathbf{H}^{\text{scat}} = \underbrace{j\omega(\epsilon - \epsilon_0)\mathbf{E}^{\text{int}}}_{\text{like } \mathbf{J}_{\text{i}}} + j\omega\epsilon_0\mathbf{E}^{\text{scat}}$$
$$\nabla \times \mathbf{E}^{\text{scat}} = \underbrace{-j\omega(\mu - \mu_0)\mathbf{H}^{\text{int}}}_{\text{like } -\mathbf{M}_{\text{i}}} - j\omega\mu_0\mathbf{H}^{\text{scat}}$$

We now note that the first term of the left hand side is analogous to electric and magnetic current densities $\mathbf{J}_{\text{i}}$ and $\mathbf{M}_{\text{i}}$, respectively.

With this in mind, the scattered fields $(\mathbf{E}^{\text{scat}}, \mathbf{H}^{\text{scat}})$ are due to some equivalent sources:

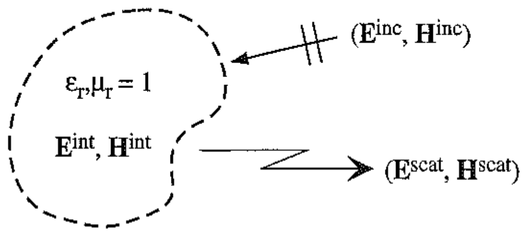
$$\mathbf{J}_{\text{eq}} = j\omega(\epsilon - \epsilon_0)\mathbf{E}^{\text{int}} = j\omega\epsilon_0(\epsilon_{\text{r}} - 1)\mathbf{E}^{\text{int}}$$
$$\mathbf{M}_{\text{eq}} = j\omega(\mu - \mu_0)\mathbf{H}^{\text{int}} = j\omega\mu_0(\mu_{\text{r}} - 1)\mathbf{H}^{\text{int}}$$

Pictorial explanation:



In practice, to find $\mathbf{E}^{\text{scat}}$, it is first necessary to determine $\mathbf{J}_{\text{eq}}/\mathbf{M}_{\text{eq}}$. However, these depend on $(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}})$, which are the actual fields inside the dielectric and are not available. To obtain them, we need to solve integral equations. However, certain approximations to $\mathbf{J}_{\text{eq}}/\mathbf{M}_{\text{eq}}$ can give us a "quick answer." One such type of approximation (very popular) is called the *Born approximation* or *Rayleigh-Debye approximation*. Let us describe it:

Consider a dielectric scatterer as shown below (with $\mu_{\text{r}} = 1$ for simplicity).



Then, we can approximate the interior fields as follows:

$$\mathbf{E}^{\text{int}} \approx \mathbf{E}^{\text{inc}}, \qquad \mathbf{H}^{\text{int}} \approx \mathbf{H}^{\text{inc}}$$

Thus

$$\mathbf{J}_{\text{eq}} = j\omega\epsilon_0(\epsilon_{\text{r}} - 1)\mathbf{E}^{\text{inc}}$$
$$\mathbf{M}_{\text{eq}} = 0$$

The scattered field is that generated by $\mathbf{J}_{\text{eq}}$. More explicitly

$$\mathbf{E}^{\text{scat}} = -j\omega\mathbf{A} + \frac{1}{j\omega\mu\epsilon}\nabla\nabla\cdot\mathbf{A}, \qquad \mathbf{A} = \mu\iiint \mathbf{J}G\,dV$$

In the far zone, the above reduces to

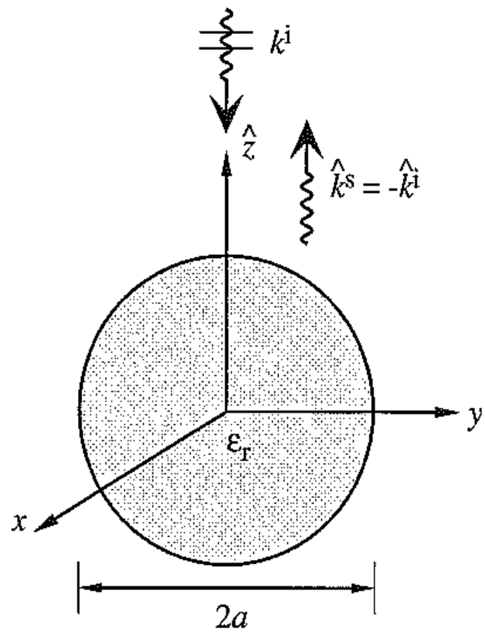
$$\mathbf{E}^{\text{scat}}(\mathbf{r}) = jkZ\frac{e^{-jkr}}{4\pi r}\iiint_V \hat{r}\times(\hat{r}\times\mathbf{J}(\mathbf{r}'))e^{+jk\mathbf{r}'\cdot\hat{r}}\,dv'$$

Since $\mathbf{J}(\mathbf{r}') = \hat{e}^{\text{i}}e^{jk\cdot\mathbf{r}}$ and $e^{jk\mathbf{r}'\cdot\hat{r}} = e^{+jk\mathbf{r}'\cdot\hat{k}_{\text{s}}}$ here $\hat{k}_{\text{s}} = \hat{r}$ is the direction of scattering, we obtain

$$\begin{aligned}
\mathbf{E}^{\text{scat}} &= jkZ\frac{e^{-jkr}}{4\pi r}\iiint_V \hat{r}\times(\hat{r}\times\hat{e}^{\text{i}})\underbrace{[j\omega\epsilon_0(\epsilon_{\text{r}}-1)]}_{jk_0Y_0}e^{-jk_0(\hat{k}^{\text{i}}-\hat{k}^{\text{s}})\cdot\mathbf{r}'}\,dv' \\
&= -k^2\frac{e^{-jkr}}{4\pi r}\iiint_V \hat{r}\times(\hat{r}\times\hat{e}^{\text{i}})(\epsilon_{\text{r}}-1)e^{-jk_0(\hat{k}^{\text{i}}-\hat{k}^{\text{s}})\cdot\mathbf{r}'}\,dv' \\
&= -k^2\frac{e^{-jkr}}{4\pi r}[\hat{k}^{\text{s}}\times(\hat{k}^{\text{s}}\times\hat{e}^{\text{i}})](\epsilon_{\text{r}}-1)\iiint_V e^{-jk_0(\hat{k}^{\text{i}}-\hat{k}^{\text{s}})\cdot\mathbf{r}'}\,dv'
\end{aligned}$$

For the case of backscattering by a sphere (of radius a) as shown, viz. $\hat{k}^{\text{i}} = -\hat{k}^{\text{s}} = -\hat{z}$, the volume integral simplifies to

$$\begin{aligned}
\iiint_V e^{-jk_0(\hat{k}^{\text{i}}-\hat{k}^{\text{s}})\cdot\mathbf{r}'}\,dv' &= \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta'\,d\theta' \int_0^{a} r'^2\,dr'\,e^{-j2k_0\hat{z}\cdot\mathbf{r}'} \\
&= \frac{3V}{(2k_0)^3a^3}\underbrace{[\sin(2k_0a) - 2k_0a\cos(2k_0a)]}_{\text{II}}
\end{aligned}$$



Setting $\mathrm{II} = [\sin(2k_0a) - 2k_0a\cos(2k_0a)]$, we get

$$\mathbf{E}^{\mathrm{scat}} = -k^2\frac{e^{-jkr}}{4\pi r}(\mathrm{II})[\hat{k}^{\mathrm{s}} \times (\hat{k}^{\mathrm{s}} \times \hat{e}^{\mathrm{i}})]$$

Using this for computing the radar cross section of the dielectric sphere we get

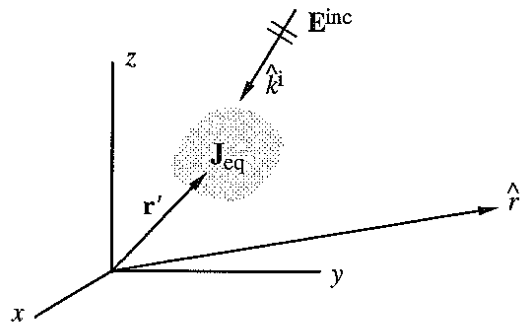
$$\boxed{\sigma = k^2(\mathrm{II})^2\left|\hat{k}^{\mathrm{s}} \times (\hat{k}^{\mathrm{s}} \times \hat{e}^{\mathrm{i}})\right|^2}$$

Note that in this $\hat{e}^{\mathrm{i}}$ denotes the incident wave polarization.

Scattered field definition

Source in free space	Same source in presence of dielectric
$\nabla \times \mathbf{E}_0 = -\mathbf{M}_\mathrm{i} - j\omega\mu_0\mathbf{H}_0$	$\nabla \times \mathbf{E} = -\mathbf{M}_\mathrm{i} - j\omega\mu\mathbf{H}$
$\nabla \times \mathbf{H}_0 = \mathbf{J}_\mathrm{i} + j\omega\epsilon_0\mathbf{E}_0$	$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + j\omega\epsilon\mathbf{E}$

We want to compute the new **E**, **H**. Usually the difference between the two is called the *scattered field:*



$$\mathbf{E}^{\mathrm{scat}} = \mathbf{E} - \mathbf{E}_0$$

to get $\mathbf{E}^{\mathrm{scat}}$ subtract the two equations, and this gives

$$\begin{aligned}
\nabla \times \mathbf{E}^{\mathrm{scat}} &= -j\omega\mu_0\mu_\mathrm{r}\mathbf{H} + j\omega\mu_0\mathbf{H}_0 \\
\nabla \times \mathbf{H}^{\mathrm{scat}} &= j\omega\epsilon\epsilon_0\mu_\mathrm{r}\mathbf{E} - j\omega\epsilon_0\mathbf{E}_0 \\
\nabla \times \mathbf{E}^{\mathrm{scat}} &= -j\omega\mu_0[\mu_\mathrm{r}\mathbf{H} - \mathbf{H}_0] \\
\nabla \times \mathbf{E}^{\mathrm{scat}} &= -j\omega\mu_0[\mu_\mathrm{r}\mathbf{H} - \mathbf{H} + \mathbf{H}^{\mathrm{scat}}] \\
&= -j\omega\mu_0(\mu_\mathrm{r} - 1)\mathbf{H} - j\omega\mu_0\mathbf{H}^{\mathrm{scat}}
\end{aligned}$$

Comparing the above to the usual Maxwell's equation of $\nabla \times \mathbf{E} = -\mathbf{M} - j\omega\mu_0\mathbf{H}$, it is concluded that $(\mathbf{E}^{\mathrm{scat}}, \mathbf{H}^{\mathrm{scat}})$ can be thought of as generated from a magnetic source $\mathbf{M} = j\omega\mu_0(\mu_\mathrm{r} - 1)\mathbf{H}$.