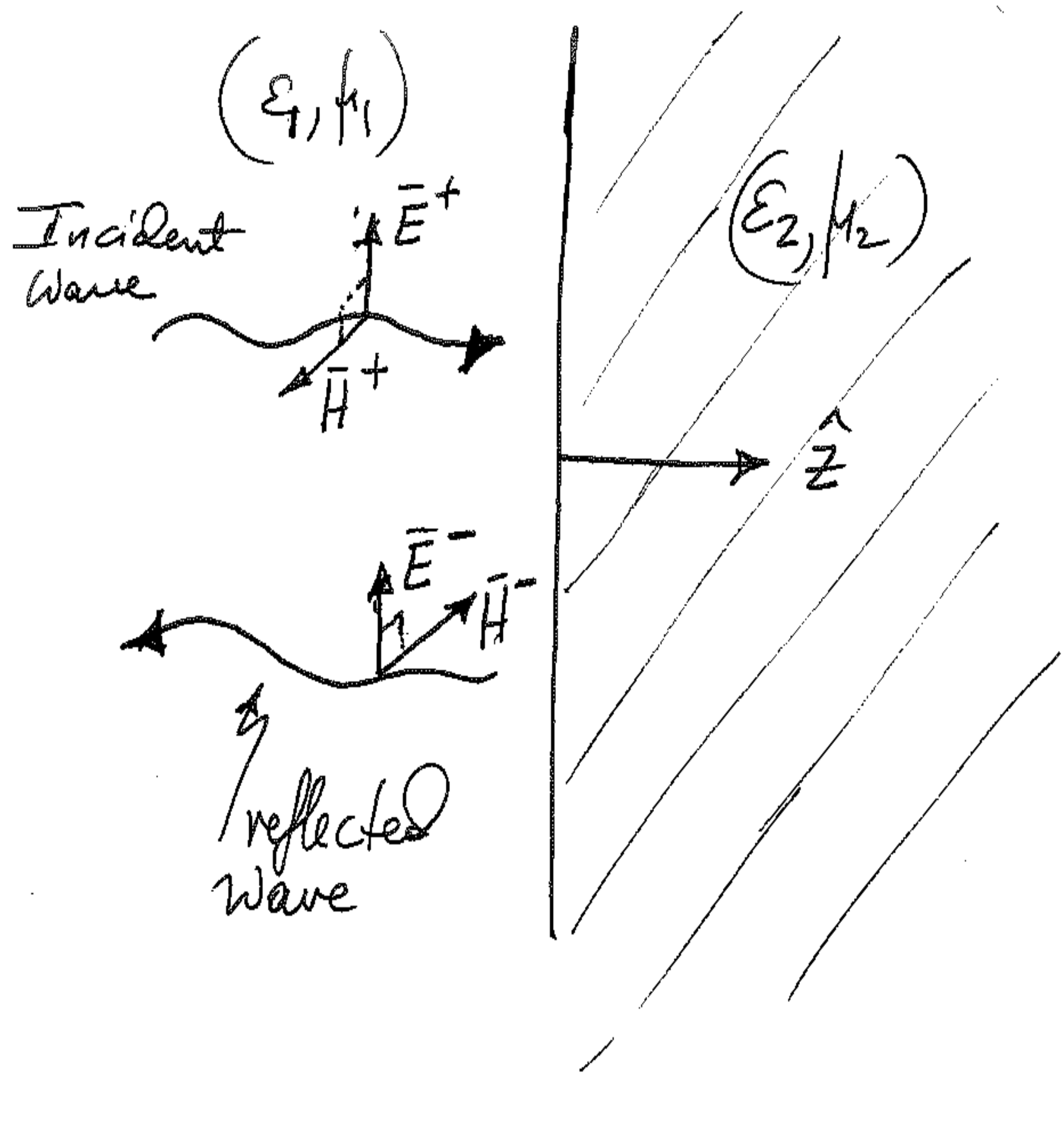
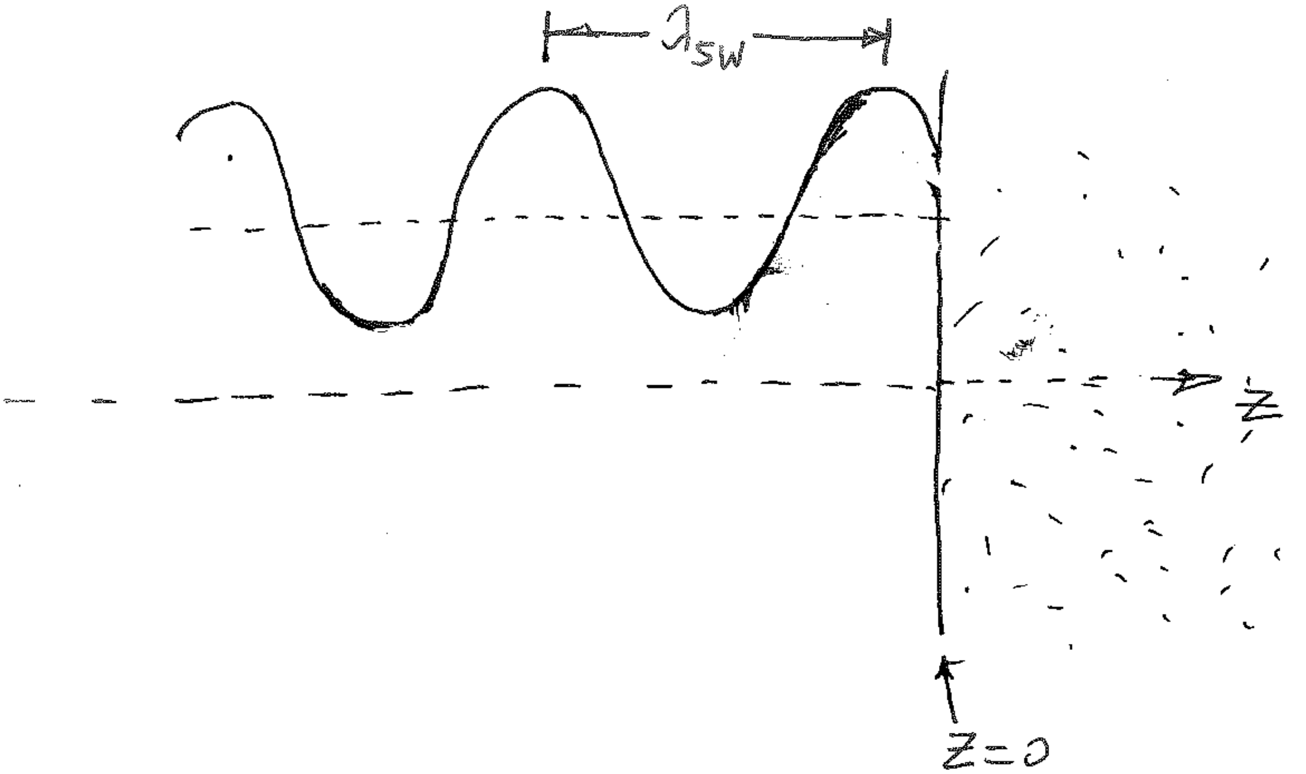
Standing waves

$$\mathbf{E} = \overbrace{\mathbf{E}_0^+ e^{-j\beta z}}^{E^+} + \overbrace{\mathbf{E}_0^- e^{+j\beta z}}^{E^-} = \mathbf{E}^+ + \mathbf{E}^-$$

$$\mathbf{H}^+ = Y\hat{z} \times \mathbf{E}_0^+ e^{-j\beta z}$$

$$\mathbf{H}^- = -Y\hat{z} \times \mathbf{E}_0^- e^{+j\beta z}$$





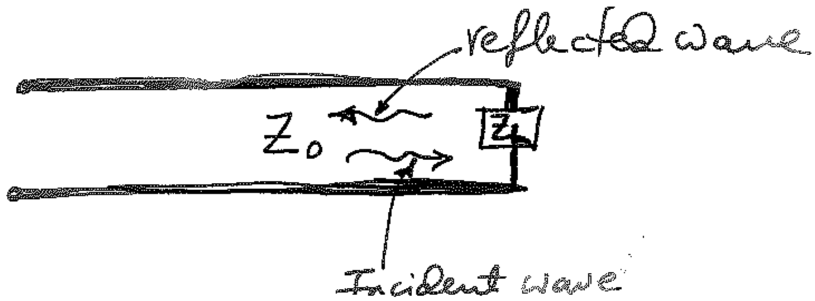
$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \hat{x}[(E_0^+ + E_0^-)\cos\beta z - j(E_0^+ + E_0^-)\sin\beta z]$$

$$\text{VSWR} = \left|\frac{E_{\text{max}}}{E_{\text{min}}}\right| = \left|\frac{E_0^+ + E_0^-}{E_0^+ - E_0^-}\right| = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$\Gamma = \left|\frac{E_0^-}{E_0^+}\right| = \text{reflection coefficients}$$

Compare this to

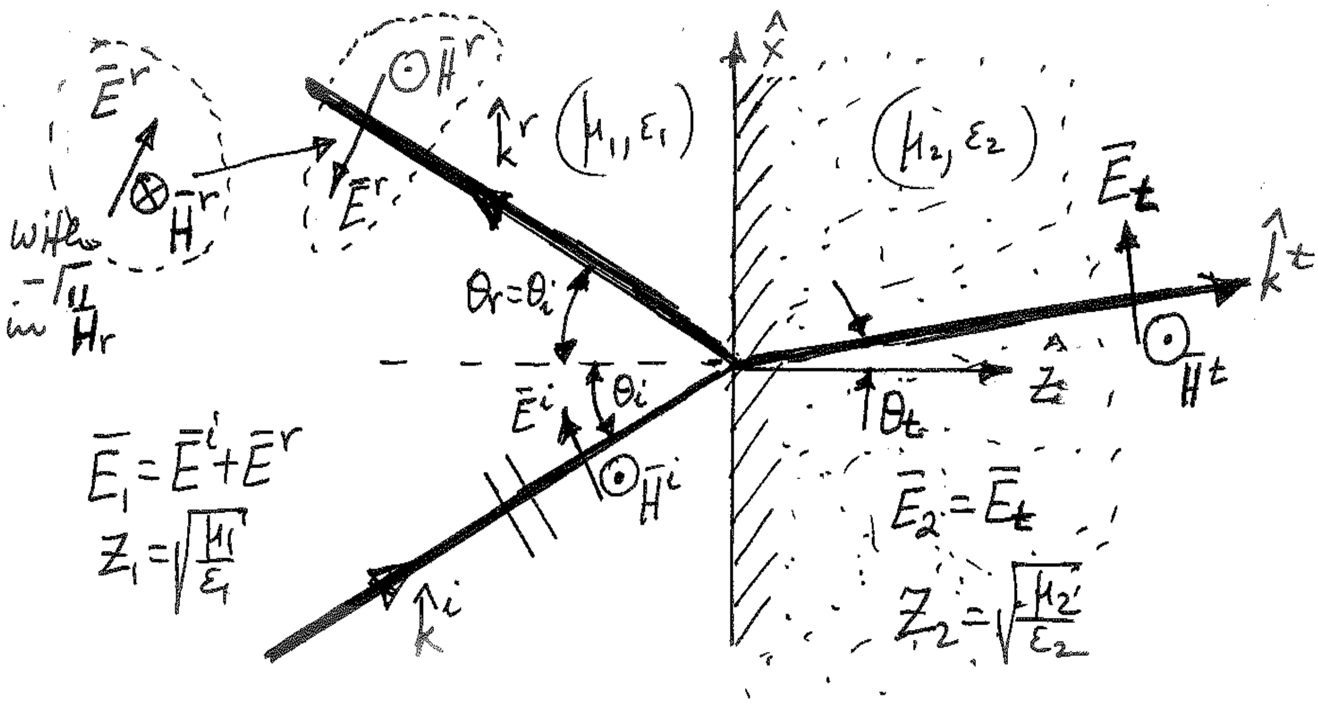
$$\Gamma = \frac{Z_{\text{L}} - Z_0}{Z_{\text{L}} + Z_0}$$



Reflection from planar dielectric interfaces (parallel polarization)

(We will later use duality to analyze the $\perp$ polarization.)

Duality Principle: $E^{\mathrm{i}} \to H^{\mathrm{i}} \Rightarrow$ (by duality) $E^{\mathrm{r}} \to H^{\mathrm{r}}$ and $E^{\mathrm{t}} \to H^{\mathrm{t}}$ if $\mu_{\mathrm{r}} \to \varepsilon_{\mathrm{r}}$ and $\varepsilon_{\mathrm{r}} \to \mu_{\mathrm{r}}$.



$$\mathbf{H}_{\|}^{\mathrm{i}} = \hat{y}\left(\frac{E_0}{Z_1}\right) e^{-j\mathbf{k}^{\mathrm{i}}\cdot\mathbf{r}} = \hat{y}\left(\frac{E_0}{Z_1}\right) e^{-jk_1(x\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}})}$$

$$\mathbf{E}_{\|}^{\mathrm{i}} = -Z_1\hat{k}^{\mathrm{i}} \times \mathbf{H}_{\|}^{\mathrm{i}} = Z_1\mathbf{H}_{\|}^{\mathrm{i}} \times \hat{k}^{\mathrm{i}} = E_0(+\hat{x}\cos\theta_{\mathrm{i}} - \hat{z}\sin\theta_{\mathrm{i}})e^{-j\mathbf{k}^{\mathrm{i}}\cdot\mathbf{r}}$$

$$\mathbf{H}_{\|}^{\mathrm{r}} = \hat{y}\frac{E_0}{Z_1}(-\Gamma_{\|})e^{-j\mathbf{k}^{\mathrm{r}}\cdot\mathbf{r}}$$ (minus sign in front of $\Gamma_{\|}$ serves so that $\Gamma_{\|} \to -1$ as $\varepsilon_2 \to 1 - j\infty$)

$$\mathbf{E}_{\|}^{\mathrm{r}} = E_0\Gamma_{\|}(\hat{x}\cos\theta_{\mathrm{r}} + \hat{z}\sin\theta_{\mathrm{r}})e^{-jk_1(x\sin\theta_{\mathrm{r}} - z\cos\theta_{\mathrm{r}})}$$

$$\mathbf{H}^{\mathrm{t}} = \hat{y}\left(\frac{E_0}{Z_2}\right) T_{\|}e^{-j\mathbf{k}^{\mathrm{t}}\cdot\mathbf{r}} = \hat{y}\left(\frac{E_0}{Z_2}\right) T_{\|}e^{-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}})}$$

$$\mathbf{E}^{\mathrm{t}} = E_0T_{\|}(\hat{x}\cos\theta_{\mathrm{t}} - \hat{z}\sin\theta_{\mathrm{t}})e^{-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}})}$$

The above are the proposed reflected and transmitted fields. If they satisfy all boundary conditions using these forms, they are the unique solutions to the problem. The boundary conditions are

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = \hat{n} \times (\mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{r}} - \mathbf{E}^{\mathrm{t}}) = 0$$
$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \hat{n} \times (\mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{r}} - \mathbf{H}^{\mathrm{t}}) = 0$$

and these enforce continuity of $E_{\tan}$ and $H_{\tan}$ across the dielectric. We have two unknowns ($\Gamma_\parallel$ and $T_\parallel$) in the above expressions and we will proceed to find these constants by enforcing the above boundary conditions. Also, θ_t must be found. We begin by first noting that since the boundary conditions must be valid for all x, the exponents (the only portion of the field expressions that depend on x) must be equal, viz.

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

Thus,

$$\boxed{\theta_i = \theta_r} \qquad \text{and} \qquad \boxed{\sin\theta_t = \frac{k_1}{k_2}\sin\theta_i}$$

This is the well-known Snell's law.

After eliminating the x-dependence in this manner, enforcement of the boundary conditions gives $\Gamma_\parallel$ and $T_\parallel$. Specifically (dual quantities for $\perp$ polarization are also noted)

$$\begin{aligned} \Gamma_\parallel &\to -\Gamma_\perp(Z_1 \to Y_1,\ Z_2 \to Y_2) \\ &= \boxed{\frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} = \Gamma_\parallel} \end{aligned}$$

and

$$\begin{aligned} T_\parallel &\to T_\perp(Z_1 \to Y_1,\ Z_2 \to Y_2) \\ &= \frac{2Y_2\cos\theta_i}{Y_2\cos\theta_i + Y_1\cos\theta_t} = \frac{2Z_1\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t} \end{aligned}$$

or

$$\boxed{T_\parallel = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}}$$

We conclude that

$$\boxed{T_\parallel \frac{\cos\theta_t}{\cos\theta_i} = 1 + \Gamma_\parallel}$$

and $\Gamma_\parallel \to -1$ when $Z_2 = 0$, i.e., Medium 2 is a perfect electric conductor (PEC).

Brewster angle

Also, $\Gamma_\parallel = 0$ (Brewster angle) when

$$Z_2\cos\theta_t = Z_1\cos\theta_i$$

Since

$$\cos\theta_t = \sqrt{1 - \left(\frac{k_1}{k_2}\sin\theta_i\right)^2}$$

$$Z_{1,2} = \sqrt{\frac{\mu_{1,2}}{\varepsilon_{1,2}}}$$

$$k_{1,2} = \omega\sqrt{\varepsilon_{1,2}\mu_{1,2}}$$

it follows that

$$\sin\theta_i = \sin\theta_{iB_\parallel} = \sqrt{\frac{(\varepsilon_2/\varepsilon_1) - (\mu_2/\mu_1)}{(\varepsilon_2/\varepsilon_1) - (\varepsilon_1/\varepsilon_2)}}\Bigg|_{\mu_1=\mu_2} = \sqrt{\frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_1}}$$

Also,

$$\begin{aligned}\sin\theta_{iB\perp} &= \sin\theta_{iB\parallel} \quad (\varepsilon \to \mu, \quad \mu \to \varepsilon)\\ &= \sqrt{\frac{(\mu_2/\mu_1) - (\varepsilon_2/\varepsilon_1)}{(\mu_2/\mu_1) - (\mu_1/\mu_2)}}\end{aligned}$$

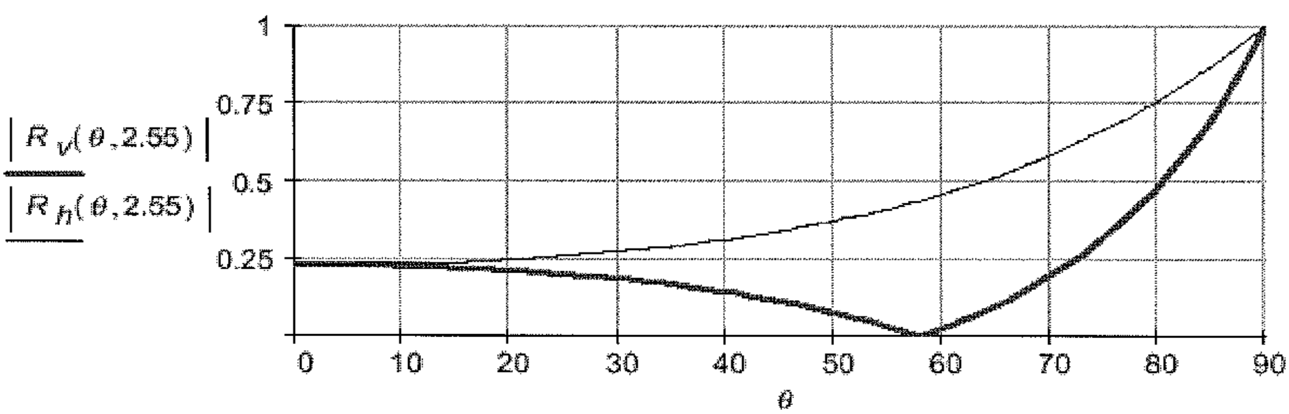
The Brewster angle for parallel polarization can be simplified further, viz.

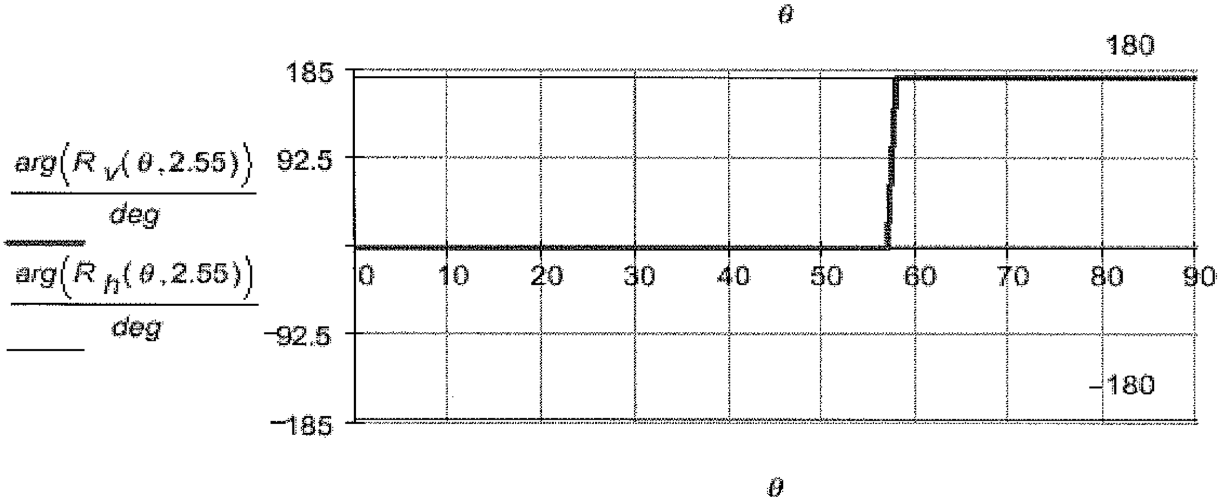
$$\begin{aligned}\theta_{iB_\parallel} &= \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_2 + \varepsilon_1}}\right)\\ &= \cos^{-1}\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}}\right)\\ &= \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \tan^{-1}\left(\sqrt{\varepsilon_r}\right)\end{aligned}$$

if medium 1 is air and $\varepsilon_2 = \varepsilon_0\varepsilon_r$, where ε_r = relative permittivity.

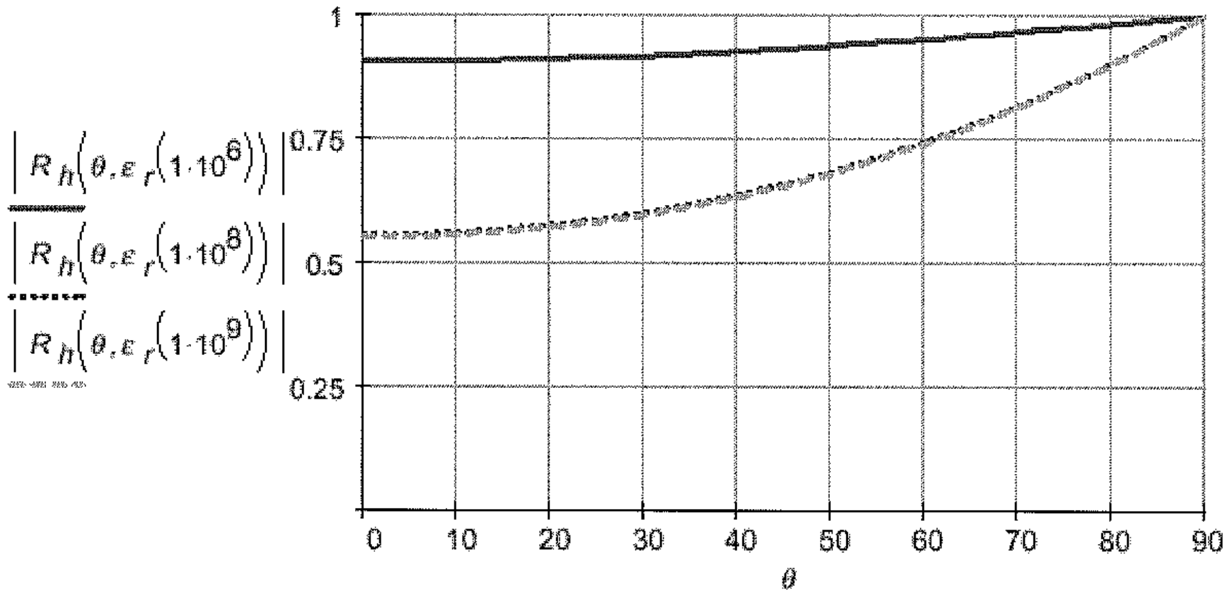
Actual plots of reflection coefficients for dielectric interfaces:

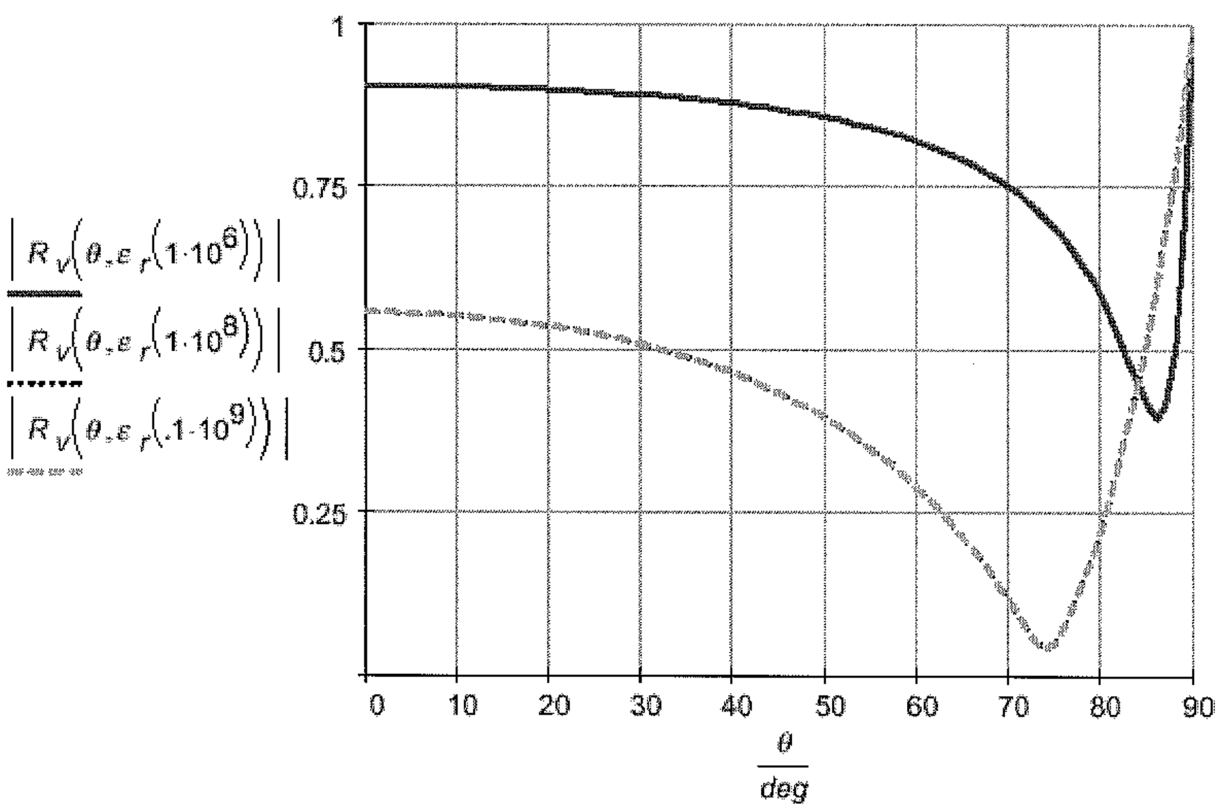
1) $\varepsilon_r = 2.55$





2) $\varepsilon_r = 12 - j\dfrac{0.012}{2\pi f \varepsilon_0}$

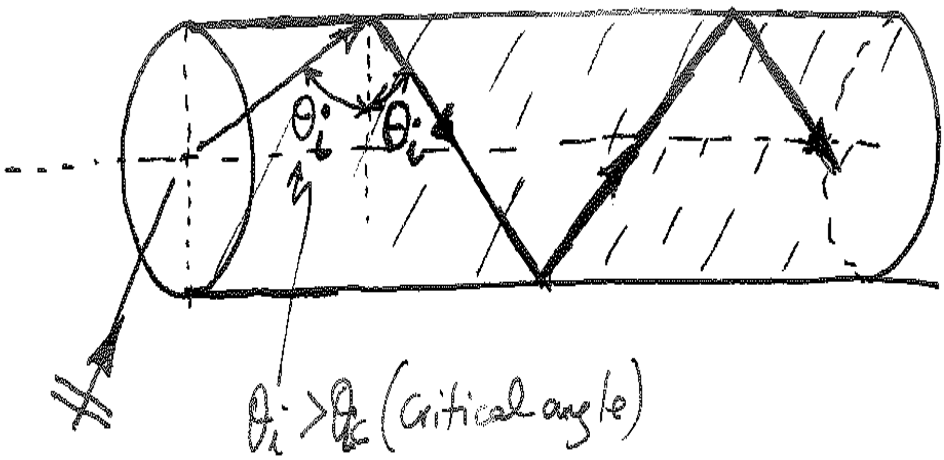




Note the condition:

$$\boxed{\theta_{iB} + \theta_t = \frac{\pi}{2}}$$

Critical angle



optical Fiber

The critical angle occurs when $|\Gamma_{\|}| = 1$ or $\Gamma = 1e^{j\delta}$. For this to be satisfied, we must have $\varepsilon_1 > \varepsilon_2$ or $N_1 > N_2$. For $|\Gamma_{\|}|$ to be unity, we must have

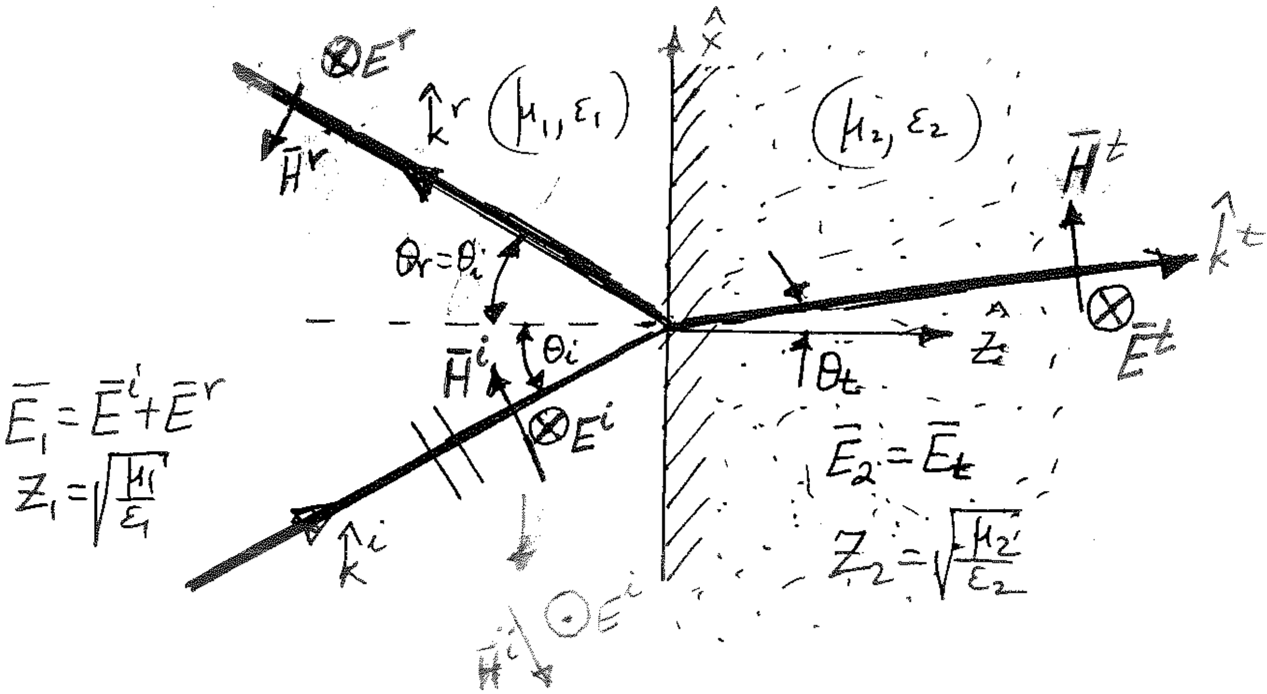
$$\cos\theta_t = 0 \quad \Longrightarrow$$

$$1 - \frac{k_1}{k_2}\sin\theta_{ic} = 0 \quad \Longrightarrow$$

$$\sin\theta_{ic} = \frac{k_2}{k_1} = \frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}}$$

$$\theta_{ic} = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right), \qquad \mu_1 = \mu_2$$

Summary for perpendicular polarization



$$\Gamma_\perp = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}, \qquad T_\perp = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

$$T_\perp = 1 + \Gamma_\perp$$

(a) PEC medium 2

$$T_\perp = 0, \qquad T_\perp = 1 + \Gamma_\perp \quad \Longrightarrow \quad \Gamma_\perp = -1$$

(b) Brewster angle

$$\Gamma_\perp = 0 \quad \longrightarrow \quad \sin\theta_{iB_\perp} = \sqrt{\frac{(\varepsilon_2/\varepsilon_1) - (\mu_2/\mu_1)}{(\mu_1/\mu_2) - (\mu_2/\mu_1)}} \rightarrow \infty \text{ when } \mu_1 = \mu_2$$

i.e., it does not exist for $\perp$ polarization.

(c) Critical Angle (important for propagation in optical fibers)

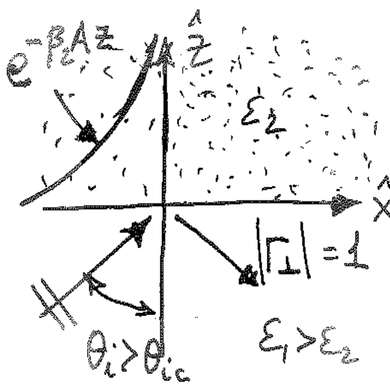
$$|\Gamma_\perp| = 1 \quad \Longrightarrow \quad \Gamma_\perp = 1e^{j\delta} = \frac{B + jA}{B - jA} = 1e^{j\delta}$$

$$\theta_{ic} = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

$$= \text{real, only for } \varepsilon_1 > \varepsilon_2$$

That is, the critical angle does not exist when the wave impinges from a less dense medium.

When $\theta_i > \theta_{ic}$, then $\sin\theta_t > 1$ and therefore $\cos\theta_t$ is complex. For this case, we have



$$\Gamma_\perp = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

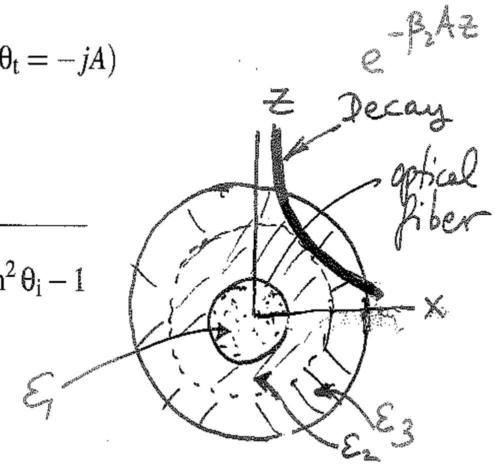
$$= \frac{Z_2\cos\theta_i + jZ_1 A}{Z_2\cos\theta_i - jZ_1 A}, \qquad (\cos\theta_t = -jA)$$

$$\cos\theta_t = \sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta_i}$$

$$= -jA = -j\sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta_i - 1}$$

$$\mathbf{E}_2 = \mathbf{E}_1^t = \hat{y} T_\perp e^{-j\beta_2 x \sin\theta_t} e^{-j\beta_2 z \cos\theta_t}$$

$$= \hat{y} T_\perp e^{-j\beta_2 x \cos\theta_t} e^{-\beta_2 A z}$$



This means that transmission does occur but the transmitted field is highly attenuated when incident at $\theta_i > \theta_c$. In this manner, the field is contained within the fiber. A dielectric layer around the fiber is typically included ("sheath") to suppress interference from nearby fibers that form the bundles of fibers.