Solution of Maxwell's equations

Having concluded that 3 equations are sufficient to solve for $\mathbf{E}$, $\mathbf{H}$ and ρ, we next proceed to solve these equations for specific situations. Let us begin by stating the derived equations. We have (in time domain)

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{i}} + \left(\sigma + \epsilon \frac{\partial}{\partial t}\right) \mathbf{E}$$
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

(We cannot use complex ϵ in time domain.) The last equation can actually be combined with the divergence condition

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}$$

a by-product of the second Maxwell's equation, and the constitutive relation $\mathbf{J} = \sigma \mathbf{E}$ to get

$$\nabla \cdot (\sigma \mathbf{E}) = -\frac{\partial}{\partial t}(\epsilon \nabla \cdot \mathbf{E})$$

Rearranging yields

$$\left(\sigma + \frac{\partial}{\partial t}\epsilon\right) \nabla \cdot \mathbf{E} = 0$$

or

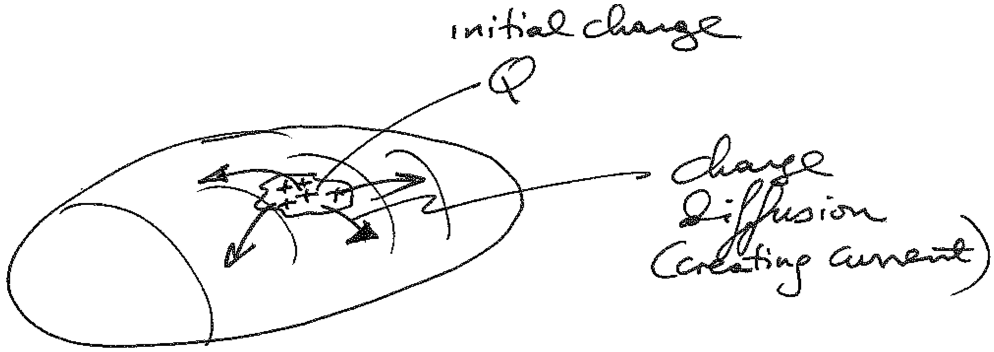
$$\left(\sigma + \frac{\partial}{\partial t}\epsilon\right) \frac{\rho}{\epsilon} = 0$$

Thus, $\rho(t)$ satisfies the first order D.E.

$$\boxed{\frac{\partial \rho(t)}{\partial t} + \frac{\sigma}{\epsilon}\,\rho(t) = 0}$$

Solution of this is

$$\rho(t) = \rho(0)\, e^{-(\sigma/\epsilon)t} = \rho(0)\, e^{-t/\tau}$$
$$\tau = \text{relation time constant in seconds} = \epsilon/\sigma$$
$$= 8 \times 10^{-21} \text{ sec for } \sigma = 10^{7} \text{ s/m and } \epsilon = \epsilon_0$$



That is, ρ decays/diffuses rapidly when some initial charge $\rho(0) = \rho_0$ is placed on the conductor. In most cases, we will assume $\sigma = 0$ or $\sigma = \infty$. Clearly for $\sigma = 0$, then $\nabla \cdot \mathbf{E} = 0$ and this is the relation most often quoted or used.

Maxwell's equations in terms of E and H only

Solution of Maxwell's equations

Using the constitutive relations we can rewrite Maxwell's equations using **E** and **H** only. Let us assume a simple medium (isotropic, homogeneous (constant ϵ and μ) and linear—all the good stuff).

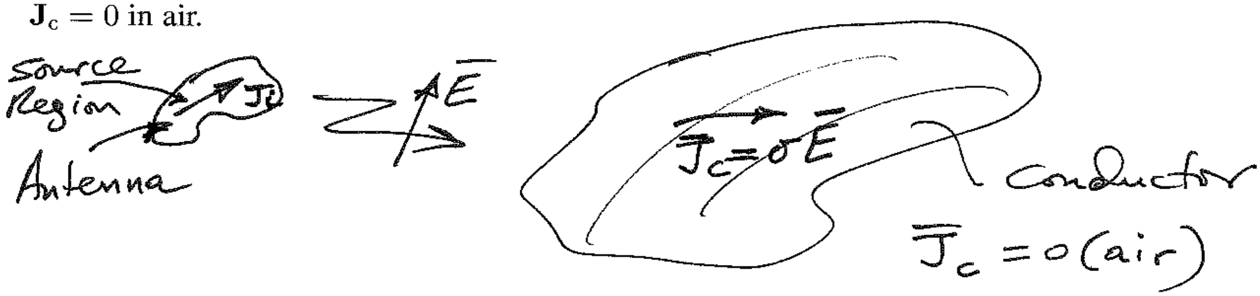
$$\begin{aligned}
&\text{Faraday's Law:} \qquad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad (\mathbf{B} = \mu \mathbf{H}) \\
&\text{Ampère's Law:} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \qquad (\mathbf{D} = \epsilon \mathbf{E})
\end{aligned}$$

Before using $\mathbf{J} = \sigma \mathbf{E}$, we need to classify $\mathbf{J}$ in two parts:

$$\mathbf{J}_\mathrm{i} = \text{source current} \tag{1}$$

$$\mathbf{J}_\mathrm{c} = \text{conduction current on nearby conductors due to } \mathbf{J}_\mathrm{i} \tag{2}$$

$\mathbf{J}_\mathrm{c} = 0$ in air.



Thus we can write

$$\mathbf{J} = \mathbf{J}_\mathrm{i} + \mathbf{J}_\mathrm{c} = \mathbf{J}_\mathrm{i} + \sigma \mathbf{E}$$

and the two independent Maxwell's equations take the form

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{i} + \left(\sigma + \epsilon \frac{\partial}{\partial t} \right) \mathbf{E}$$

($\mathbf{J}_\mathrm{i} = 0$ outside the source region.)

The divergence equations also take the form

$$\nabla \cdot \mathbf{D} = \rho \qquad \begin{array}{c} \nabla \cdot \mathbf{H} = 0 \\ \nabla \cdot (\epsilon \mathbf{E}) = \rho \end{array} \qquad \text{or} \qquad \nabla \times \mathbf{E} = \frac{\rho}{\epsilon} \quad \text{for } \epsilon = \text{constant}$$

without a need to invoke the continuity equation.

Solution of Maxwell's equations in time domain

To simplify the solution of Maxwell's pair of coupled equations, a customary approach is to combine them to obtain a single (partial) differential equation. Let's begin this process by taking the curl of the first equation and substituting it into the second.

Taking the curl of first Maxwell's equation (Faraday's Law) gives

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}$$

Substituting the second Maxwell's equation into this yields

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} \left(\sigma + \epsilon \frac{\partial}{\partial t} \mathbf{E} \right)$$

Rearranging yields

$$\nabla \times \nabla \times \mathbf{E} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2}{\partial t} \mathbf{E} = 0$$

In fluid mechanics $\mu\sigma$ is called the *viscous* or *drag term* producing decay.
We will assume $\sigma = 0$. Thus, for lossless media

$$\nabla \times \nabla \times \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t} = 0$$

Further, since

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 E = -\nabla^2 E$$

(since $\sigma = 0$) we obtain the equation

$$\boxed{\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0} \qquad \nabla \cdot \mathbf{E} = 0$$

What is meant by this vector equation?
It is actually a set of 3 equations in Cartesian coordinates. They are

$$\left.\begin{matrix} \nabla^2 E_x \\ \nabla^2 E_y \\ \nabla^2 E_z \end{matrix}\right\} + \mu\epsilon \left\{\begin{matrix} \dfrac{\partial^2 E_x}{\partial t^2} \\ \dfrac{\partial^2 E_y}{\partial t^2} \\ \dfrac{\partial^2 E_z}{\partial t^2} \end{matrix}\right\} = 0$$

Each of them is called the *Helmholtz* or *scalar wave equation.*
We note that by following a similar procedure, we can also derive the corresponding wave equation for the magnetic field $\mathbf{H}$ given by

$$\nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

where $1/\mu\epsilon = v^2$ in which v is referred to as the *wave phase velocity*. This is the dual of the corresponding $\mathbf{E}$ wave equation. In general, two equations are called *dual* if one is obtained from the other via the replacements

$$\begin{aligned} \mathbf{E} &\rightarrow \mathbf{H} \\ \mathbf{H} &\rightarrow -\mathbf{E} \\ \mu &\rightarrow \epsilon \\ \epsilon &\rightarrow \mu \\ \mathbf{J} &\rightarrow \mathbf{M} \\ \mathbf{M} &\rightarrow -\mathbf{J} \end{aligned}$$

Example: Examine the field expression

$$\mathbf{E} = \hat{x} e^{-\alpha y} \cos(\omega t - k_z z)$$

where α and k_z are constants. Determine whether this is a Maxwellian solution or not.

1) Must satisfy $\nabla^2 E_x - \mu\epsilon(\partial^2 E_x/\partial t^2) = 0$.

   Substituting in the given E_x we get

$$\begin{aligned} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} &= 0 \\ +\alpha^2 E_x - k_z^2 E_x + \omega^2 \mu\epsilon E_x &= 0 \end{aligned}$$

   or

$$\boxed{\omega^2 \mu\epsilon + \alpha^2 - k_z^2 = 0}$$

   This is called the *characteristic* or *consistency equation* and must be satisfied if the given $\mathbf{E}$ is to be Maxwellian.

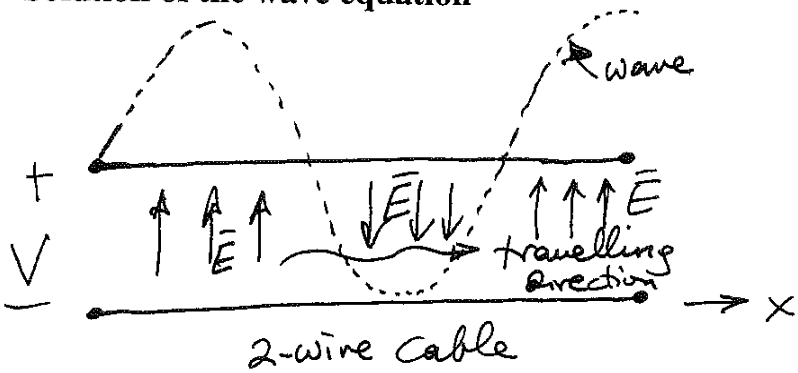
2) Must satisfy $\nabla \times \mathbf{E} = 0$

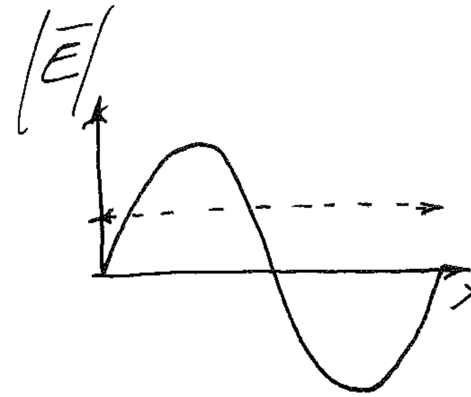
   Indeed,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

We remark that in general, two components (one from $\mathbf{E}$ and one from $\mathbf{H}$) are required for a unique determination of all other field components. Specifically we just need (E_x and H_x) or (E_z and H_z) to obtain every other field component via Maxwell's equations (see handout notes on the web).

Solution of the wave equation





Let us consider the case where $\mathbf{E} = \mathbf{E}(x,t)$, i.e., $\mathbf{E}$ is a function of x only and time. Such a functionality may correspond to the so-called transmission line problem. This is basically the wave inside your VHF cable that arrives from the antenna to your TV. Assume

$$\mathbf{E} = \mathbf{E}_0\, f(x,t)$$

$$\mathbf{E}_0 = \begin{cases} \hat{y} E_{y0} & \text{etc.} \\ \hat{z} E_{z0} & \text{etc.} \end{cases}$$

We seek to find $f(x,t))$ so that it satisfies the wave equation and $\nabla \cdot \mathbf{E} = 0$. Let us try the form (from experience)

$$\mathbf{E} = \mathbf{E}_0\, f(x - vt)$$

or

$$\mathbf{E} = \mathbf{E}_0\, f\left(t - \frac{x}{v}\right)$$

where $v =$ velocity of the wave. Substituting into the wave equation

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial^2 \mathbf{E}}{\partial x^2} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t} = 0$$

yields

$$\frac{\partial^2}{\partial x^2}\, f(x - vt) - \mu\epsilon \frac{\partial^2}{\partial t^2}\, f(x - vt) = 0$$

Note that

$$\frac{\partial}{\partial x}\, f(x - vt) = f'(x - vt)\, \frac{\partial}{\partial x}(x - vt) = f'(x - vt)$$

$$\frac{\partial^2}{\partial x^2}\, f(\alpha) = f''(\alpha), \quad \alpha = (x - vt)$$

$$\frac{\partial}{\partial t}\, f(\alpha) = f'(\alpha)\, \frac{\partial \alpha}{\partial t} = f'(\alpha)\,(-v) = -v\, f'(\alpha)$$

$$\frac{\partial^2}{\partial t^2}\, f(\alpha) = v^2\, f''(\alpha)$$

Thus, we get

$$\boxed{f''(x - vt) - \mu\epsilon v^2\, f''(x - vt) = 0}$$

and since $v^2 = 1/\mu\epsilon$, it is apparent that $\mathbf{E} = \mathbf{E}_0\ f(x - vt)$ is a solution to the wave equation (note that $f(t)$ is still undefined!) and can be a solution to Maxwell's equations subject to $\nabla \cdot \mathbf{E} = 0$. Generally, since we are having a second order partial differential equation, two linearly independent solutions exist. The second solution is $f(x + vt)$ and thus a general solution is

$$\mathbf{E} = \mathbf{E}_0^+\ f(x - vt) + \mathbf{E}_0^-\ f(x + vt)$$

or

$$\mathbf{E} = \mathbf{E}_0^+\ f\left(t + \frac{x}{v}\right) + \mathbf{E}_0^-\ f\left(t - \frac{x}{v}\right)$$

For sinusoidal fields, i.e., for $f(t) = \cos\omega t = \mathrm{Re}\{e^{j\omega t}\}$, $\omega =$ frequency of the wave, we have

$$f(x \pm vt) = \cos[\omega(x \pm vt)]$$

or

$$f\left(t + \frac{x}{v}\right) = \cos\left[\omega\left(t \pm \frac{x}{v}\right)\right] = \cos\left(\omega t \pm \frac{\omega}{v}x\right)$$

Thus, the solution to the wave equation can be written as

$$\mathbf{E} = E_0^+ \cos\left(\omega t - \frac{\omega}{v}x\right) + E_0^- \cos\left(\omega t + \frac{\omega}{v}x\right)$$

where we identify the parameter

$$k = \frac{\omega}{v} = \omega\sqrt{\mu\epsilon} = \text{propagation constant}$$

and v as the wave velocity. We will also use the quantity/variable

$$\beta = \frac{\omega}{v} = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}, \qquad \lambda = \text{wavelength}$$

to define the propagation constant. Typically, for general material/media, we will use the relation

$$k = \beta - j\alpha \qquad (\alpha = \text{loss factor})$$

Let us now see how this solution implies propagation or movement of information with a velocity v. To do so, let us first rewrite $\mathbf{E}$ as

$$\mathbf{E} = E_0^+ \cos(\omega t - \beta x)\big|_{t=t_0} = E_0^+ \cos\left[\beta\left(x - \frac{\omega}{\beta}t_0\right)\right]$$

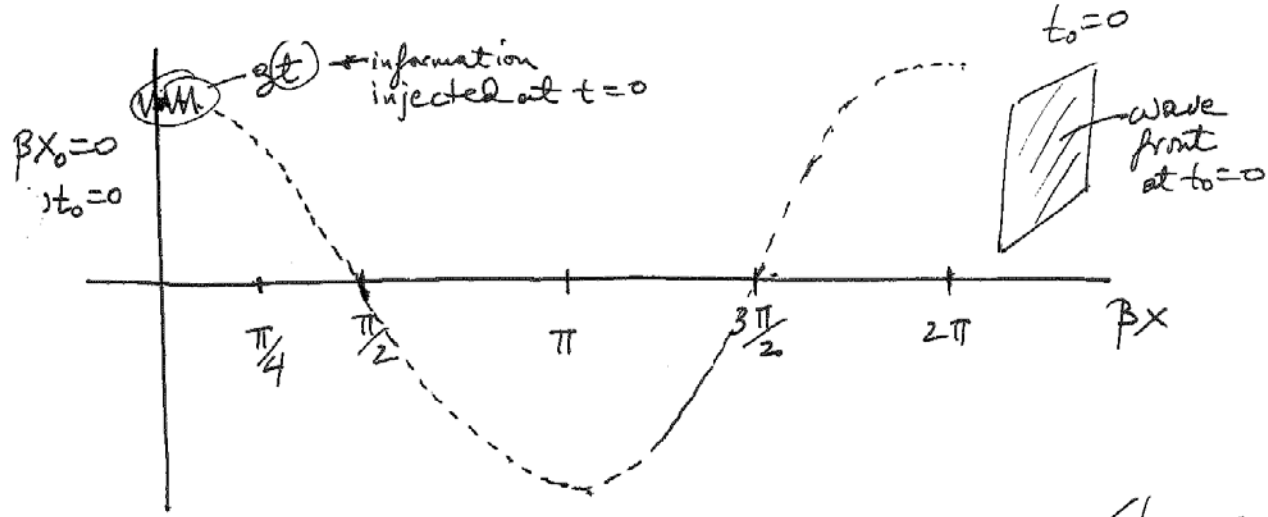
or

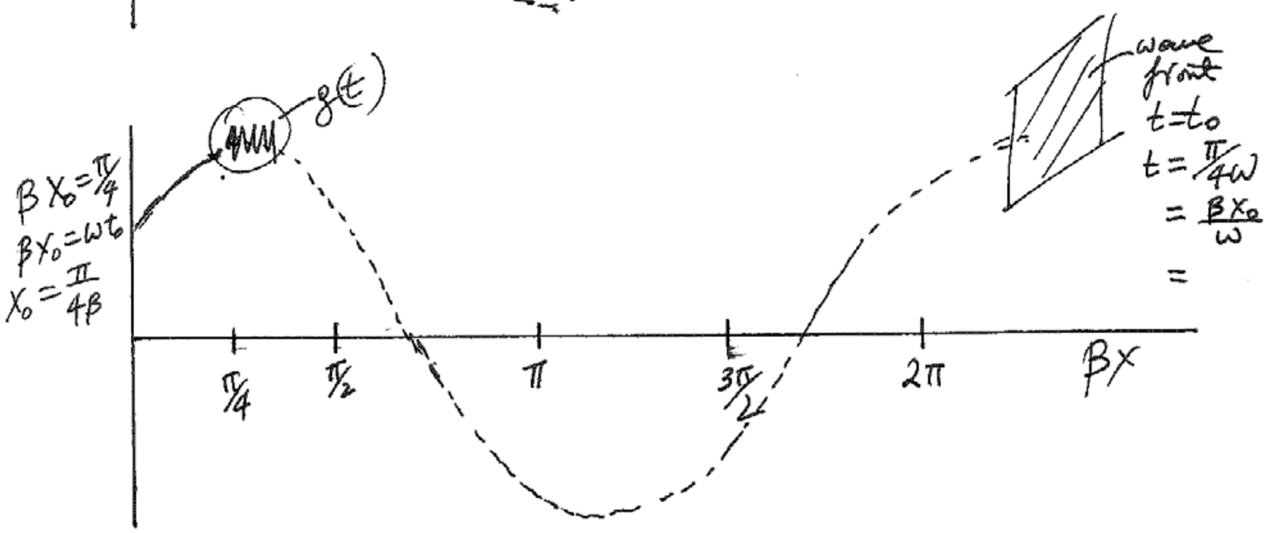
$$\mathbf{E}|_{t=t_0} = E_0^+ cos[\beta(x - x_0)], \qquad x_0 = \frac{\omega}{\beta}t_0 = vt_0$$

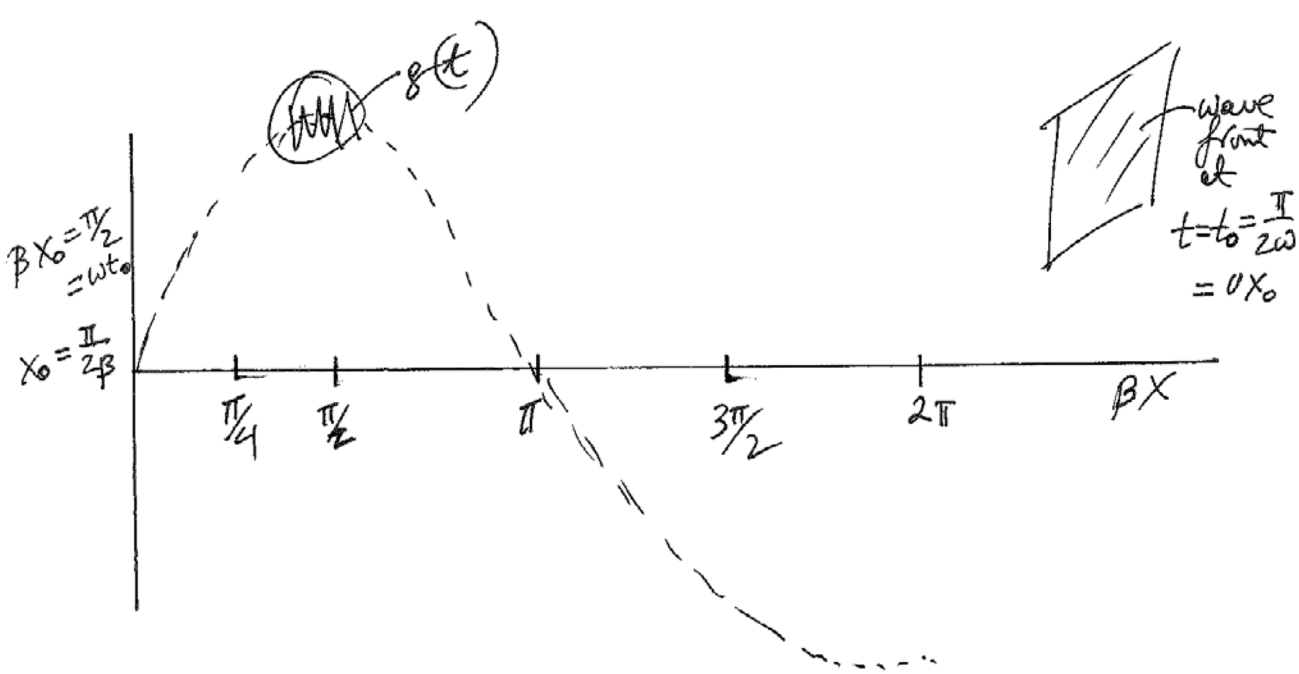
For relevance, let us actually refer to the AM radio signal which has the form

$$\mathbf{E} = \hat{z}\ g(t)\ \cos[\beta(x - x_0)] = \hat{z}\ g(t)\ \cos[\beta x - \omega t_0)]$$

in which $\hat{z}\ g(t)$ is the voice signal (very small in amplitude) and ω is the AM station frequency.


$t_0=0$
$g(t)$
information injected at $t=0$
$\beta X_0=0$
$\omega t_0=0$
wave front at $t_0=0$
$\pi/4$
$\pi/2$
π
$3\pi/2$
2π
βX


$g(t)$
$\beta X_0=\pi/4$
$\beta X_0=\omega t_0$
$X_0=\frac{\pi}{4\beta}$
wave front
$t=t_0$
$t=\pi/4\omega$
$=\frac{\beta X_0}{\omega}$
$=$
$\pi/4$
$\pi/2$
π
$3\pi/2$
2π
βX


$g(t)$
$\beta X_0=\pi/2$
$=\omega t_0$
$X_0=\frac{\pi}{2\beta}$
wave front at
$t=t_0=\frac{\pi}{2\omega}$
$=vX_0$
$\pi/4$
$\pi/2$
π
$3\pi/2$
2π
βX

H field from E field

Since $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$ we can easily get $\mathbf{H}$ from $\mathbf{E}$. Note that for

$$\mathbf{E} = \mathbf{E}_0\, f(x - vt)$$

we can use the identity

$$\begin{aligned}
\nabla \times \mathbf{E} &= \nabla f \times \mathbf{E}_0 + f \nabla \times \mathbf{E}_0 \\
&= \left(\hat{x} \frac{\partial f}{\partial x} \right) \times \mathbf{E}_0 \\
&= +(\hat{x} \times \mathbf{E}_0)\, \frac{\partial}{\partial x}\, f(x - vt) \\
&= -\frac{1}{v} (\hat{x} \times \mathbf{E}_0)\, \frac{\partial}{\partial t}\, f(x - vt)
\end{aligned}$$

Thus,

$$\begin{aligned}
-\mu \frac{\partial \mathbf{H}}{\partial t} &= -\frac{1}{v} (\hat{x} \times \mathbf{E}_0)\, \frac{\partial}{\partial t}\, f(x - vt) \\
\mathbf{H} &= \frac{1}{\mu v}\, (\hat{x} \times \mathbf{E}_0)\, f(x - vt) + \mathbf{H}_0
\end{aligned}$$

and $\mathbf{H}_0 = 0$ since $\mathbf{H} = 0$ at $t = 0$. The quantity

$$\frac{1}{\mu v} = \frac{\sqrt{\mu \epsilon}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} = \frac{1}{Z} = Y, \quad Z = \sqrt{\frac{\mu}{\epsilon}}$$

In general (for plane waves)

$$\begin{aligned}
\mathbf{H} &= Y \hat{p} \times \mathbf{E} \\
\mathbf{E} &= Z \mathbf{H} \times \hat{p}
\end{aligned}$$

Remember this special relation for plane waves in lieu of Maxwell's equations.

$$\begin{aligned}
\hat{p} &= \hat{x} \text{ (in above example)} = \text{direction of wave} \\
\mathbf{E}_{\text{phasor}} &= \mathbf{E}_0 e^{-jk_x x}
\end{aligned}$$

Alternatively, if

$$\mathbf{E} = \mathbf{E}_0 e^{-jk_y y}$$

then

$$\hat{p} = \hat{y}$$

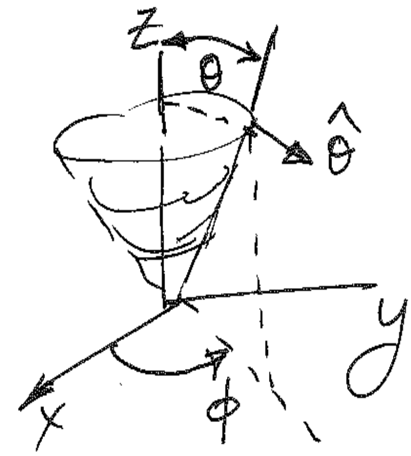
In general

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}}$$

where

$$\mathbf{k} = \hat{p} \frac{\mathbf{k}}{|\mathbf{k}|}$$

and $k = |\mathbf{k}|$ = propagation constant.



Spherical coordinates

$\mathbf{E} = \hat{\theta} E_0(r,t)$

$$\nabla^2 E_\theta - \mu\epsilon \frac{\partial^2 E_\theta}{\partial t^2} = 0$$
$$\frac{\partial^2 E_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial E_\theta}{\partial r} - \mu\epsilon \frac{\partial^2 E_\theta}{\partial t^2} = 0$$

To solve, set $E_\theta = \frac{U}{r} \quad \Rightarrow$

$$\frac{1}{r}\frac{\partial U^2}{\partial r^2} - \frac{\mu\epsilon}{r}\frac{\partial U^2}{\partial t^2} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial U^2}{\partial r^2} - \mu\epsilon \frac{\partial U^2}{\partial t^2} = 0}$$

which is the same as the equation just solved for rectangular coordinates.

Thus,

$$U = f\left(t - \frac{r}{v}\right) \qquad \text{(instead of } f(t - \tfrac{x}{v})\text{)}$$

or

$$U = f(vt - r)$$

Thus, since $E_\theta = \frac{U}{r} \quad \Rightarrow$

$$E_\theta = C\frac{f\left(t - \frac{r}{v}\right)}{r} = C'\frac{f\left(vt - r\right)}{r}$$

An example solution would be

$$E_\theta = \hat{\theta}\,\frac{\cos(\omega t - kr)}{r}$$
$$E_\theta = \hat{\theta}\,\frac{e^{-jkr}}{r}$$

$1/r$ is the decay of the field.