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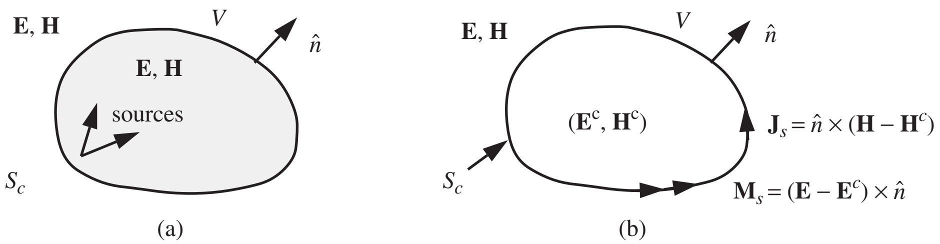

FIGURE 1-6 ■ General form of the equivalence theorem: (a) original problem and (b) equivalence problem.

These will now generate the field ($\mathbf{E}$, $\mathbf{H}$) exterior to S_c and the field ($\mathbf{E}^c$, $\mathbf{H}^c$) interior to S_c.

1.11 | RECIPROCITY AND REACTION THEOREMS

The reciprocity theorem for electromagnetics parallels the familiar theorem in circuit theory. It simply states that the fields and sources can be interchanged in a given problem or set-up without affecting the system's response. This implies that the transmitting and receiving antenna patterns are the same, even though in the first case the source was at the feed whereas for the receiving antenna the source is at infinity. Another example refers to the case of plane wave scattering illustrated in Figure 1-7. Let us assume that the far-zone scattered field $\mathbf{E}^s$ is measured along $\hat{r}$ and is caused by a plane wave excitation $\mathbf{E}^i$ incident along $\hat{r}^i$. Based on the reciprocity theorem, one can then state that the scattered field is unchanged when we let $\hat{r}^i \to -\hat{r}$ and $\hat{r} \to -\hat{r}^i$.

To derive a mathematical statement of the reciprocity theorem, we assume the existence of two sets of fields caused by two different sets of sources radiating in the same environment. In particular, suppose that the field ($\mathbf{E}_1$, $\mathbf{H}_1$) are associated with the sources ($\mathbf{J}_1$, $\mathbf{M}_1$), whereas the fields ($\mathbf{E}_2$, $\mathbf{H}_2$) are due to the sources ($\mathbf{J}_2$, $\mathbf{M}_2$). Each set of these fields and sources will then satisfy the equations

$$\nabla \times \mathbf{E}_{\substack{1\\2}} = -j\omega\mu\mathbf{H}_{\substack{1\\2}} - \mathbf{M}_{\substack{1\\2}} \tag{1.118a}$$

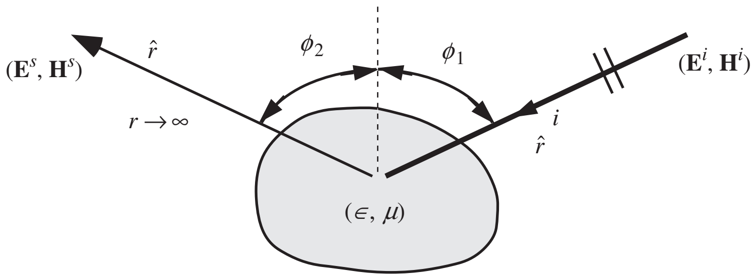

FIGURE 1-7 ■ Illustration of reciprocity for plane wave incidence and far-zone observation.

and

$$\nabla \times \mathbf{H}_{\frac{1}{2}} = j\omega\epsilon\mathbf{E}_{\frac{1}{2}} - \mathbf{J}_{\frac{1}{2}}. \tag{1.118b}$$

By invoking the identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.119}$$

we then have

$$\begin{aligned} \nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 \\ - \mathbf{H}_1 \cdot \nabla \times \mathbf{E}_2 + \mathbf{E}_2 \cdot \nabla \times \mathbf{H}_1. \end{aligned} \tag{1.120}$$

This can be simplified by introducing (1.118a), giving, for example,

$$\begin{aligned} \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 &= \mathbf{H}_2 \cdot (-j\omega\mu\mathbf{H}_1 - \mathbf{M}_1) = -j\omega\mu\mathbf{H}_1 \cdot \mathbf{H}_2 - \mathbf{M}_1 \cdot \mathbf{H}_2 \\ -\mathbf{H}_1 \cdot \nabla \times \mathbf{E}_2 &= -\mathbf{H}_1 \cdot (-j\omega\mu\mathbf{H}_2 - \mathbf{M}_2) = j\omega\mu\mathbf{H}_1 \cdot \mathbf{H}_2 + \mathbf{M}_2 \cdot \mathbf{H}_1 \end{aligned}$$

which upon substitution into (1.120) yield

$$-\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = \mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2. \tag{1.121}$$

Integrating both sides of this equation over a volume V enclosed by the surface S_c, and applying the divergence theorem, it is further deduced that

$$\begin{aligned} -\oiint_{S_c} (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) \cdot \hat{n}\, ds = \iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_2 \cdot \mathbf{M}_1 \\ - \mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv. \end{aligned} \tag{1.122}$$

It will be shown in the next chapter that in the far field (let S_c become an infinite circle),

$$\mathbf{E} = -Z_o\hat{r} \times \mathbf{H} \quad \text{and} \quad \mathbf{H} = \frac{1}{Z_o}\hat{r} \times \mathbf{E}$$

where $Z_o = \sqrt{\mu_o/\epsilon_o}$ is the free-space intrinsic impedance, and thus

$$\begin{aligned} \mathbf{E}_1 \times \mathbf{H}_2 = \mathbf{E}_1 \times \frac{(\hat{r} \times \mathbf{E}_2)}{Z_o} &= \frac{1}{Z_o}[(\mathbf{E}_1 \cdot \mathbf{E}_2)\hat{r} - (\mathbf{E}_1 \cdot \hat{r})\mathbf{E}_2] = \frac{1}{Z_o}(\mathbf{E}_1 \cdot \mathbf{E}_2)\hat{r} \\ \mathbf{E}_2 \times \mathbf{H}_1 = \frac{\mathbf{E}_2 \times (\hat{r} \times \mathbf{E}_1)}{Z_o} &= \frac{1}{Z_o}[(\mathbf{E}_2 \cdot \mathbf{E}_1)\hat{r} - (\mathbf{E}_1 \cdot \hat{r})\mathbf{E}_1] = \frac{1}{Z_o}(\mathbf{E}_2 \cdot \mathbf{E}_1)\hat{r} \end{aligned}$$

implying that the surface integral in (1.122) vanishes when S_c is a sphere of infinite radius. Consequently, we conclude that

$$\iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv = \iiint_V (\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1)\, dv \tag{1.123}$$

which is a mathematical statement of the reciprocity theorem (special case of the Lorentz reciprocity theorem given by (1.122)). It states that the fields and sources

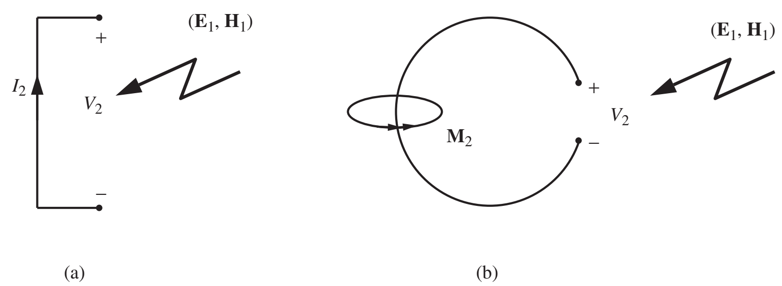

FIGURE 1-8 Illustration of circuit source: (a) current source and (b) voltage source.

can be interchanged without altering the outcome of (1.123). Integrals of the type in (1.123) are also referred to as reactions of one set of sources with the fields caused by another set of sources. Based on this reasoning, (1.123) is often written as

$$\langle 1, 2\rangle = \langle 2, 1\rangle \tag{1.124}$$

where

$$\langle 1, 2\rangle = \iiint_V (\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2)\, dv. \tag{1.125}$$

The symbolism $\langle 1, 2\rangle$ denotes the reaction of fields $(\mathbf{E}_1, \mathbf{H}_1)$ with the sources $(\mathbf{J}_2, \mathbf{M}_2)$ and (1.125) is often referred to as the reaction theorem. If $\mathbf{J}_2$ represents a linear source of strength I_2 (i.e., $\mathbf{J}_2\, dv = I_2 \hat{\ell}\, d\ell$) and $\mathbf{M}_2 = 0$, (1.125) reduces to

$$\langle 1, 2\rangle = \iiint_V \mathbf{E}_1 \cdot \mathbf{J}_2\, dv = I_2 \int \mathbf{E}_1 \cdot \hat{\ell}\, d\ell = -I_2 V_2^{(1)} \tag{1.126}$$

where $V_2^{(1)}$ is the voltage across the terminals of source 2 due to some unspecified source 1. Similarly, across the terminals of a magnetic source $\mathbf{M} = K\hat{\ell}$ (current loop), shown in Figure 1-8, $V = -K$, and if we set $\mathbf{M}_2\, dv = K_2 \hat{\ell}\, d\ell$ and $\mathbf{J}_2 = 0$, (1.125) gives

$$\langle 1, 2\rangle = -\iiint \mathbf{H}_1 \cdot \mathbf{M}_2\, dv = -K_2 \int \mathbf{H}_1 \cdot \hat{\ell}\, d\ell = +V_2 I_2^{(1)} \tag{1.127}$$

where $I_2^{(1)}$ is now the current flowing to the terminal of source 2 due to the field excitation $\mathbf{H}_1$ from some unspecified source 1.

To illustrate the application of the reciprocity theorem in electromagnetics, we consider the radiation of two antenna elements in free space as illustrated in Figure 1-9. Each of these radiates the fields $(\mathbf{E}_1, \mathbf{H}_1)$ and $(\mathbf{E}_2, \mathbf{H}_2)$, respectively, and their equivalent circuit parameters can be characterized by the usual system

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1.128}$$

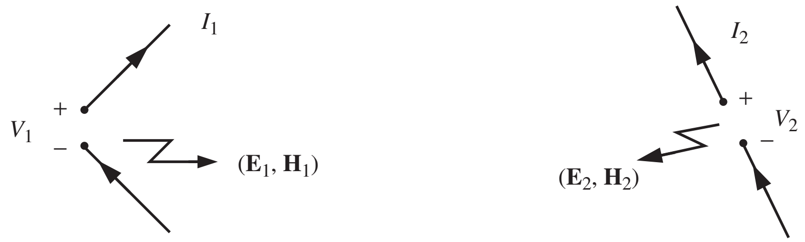

FIGURE 1-9 Reaction between two antennas.

which is identical to that for a two-port network in circuit theory. Reciprocity and the reaction theorem will now prove useful in determining the elements Z_{ij} of the impedance matrix. These elements can be easily determined by shorting or open-circuiting the antennas one at a time. Setting $I_2 = 0$, gives

$$Z_{21} = \frac{V_2^{(1)}}{I_1}$$

and by referring to (1.126) we may express Z_{21} as

$$Z_{21} = -\frac{\langle 1, 2\rangle}{I_1 I_2}. \tag{1.129}$$

By invoking the reciprocity theorem (1.123), we also have $Z_{12} = Z_{21}$ and in general

$$Z_{ij} = -\frac{\langle j, i\rangle}{I_i I_j}. \tag{1.130}$$

This expression is valid for computing the self-impedance elements Z_{ii} as well and is useful in numerical simulations of antenna and scattering problems.

1.12 APPROXIMATE BOUNDARY CONDITIONS

In Section 1.4, we discussed the boundary conditions that must be imposed on material interfaces. These are the usual natural or exact boundary conditions. However, in many cases, it is possible to employ approximate boundary conditions that effectively account for the presence of some inhomogeneous interface, a material coating on a conductor, or a dielectric layer without actually having to include their geometry explicitly in the analysis.

1.12.1 Impedance Boundary Conditions

The most common approximate boundary condition (ABC) is the impedance boundary condition attributed to Leontovich (1948), which often carries his name in the literature. It can be derived by considering the simple problem of a plane wave incidence on a material half space. Choosing the interface to be the plane $y = 0$ with the y axis directed out of the half space, the Leontovich impedance boundary condition takes the form

$$E_z = -\eta Z_o H_x, \qquad E_x = \eta Z_o H_z \tag{1.131}$$