Differential form of Maxwell's equations

Direct solution of Maxwell's equations is difficult since all available math tools are for differential equations. So, we shall seek to rewrite Maxwell's equations in *differential or point form*:

Two integral identities play a crucial role in achieving this.

Stokes's Theorem

(George G. Stokes, 1819–1903, Cambridge, England)

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = \iint_S \underbrace{\nabla \times \mathbf{F}}_{\text{curl of } \mathbf{F}} \cdot \hat{n}\, ds$$

The proof of this identity was given in 1854.

Note that $\nabla \times \mathbf{F} = \text{curl } \mathbf{F}$ is an operation on $\mathbf{F}$ and not necessarily the cross product of ∇ and $\mathbf{F}$. The latter is true for rectangular coordinates only, in which case we can write

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

For expressions of $\nabla \times \mathbf{F}$ in all three popular coordinate systems see the handout "Vector Differential Operators."

Divergence Theorem

$$\oiint_S \mathbf{F} \cdot \hat{n}\, ds = \iiint_V \underbrace{\nabla \times \mathbf{F}}_{\text{Div } \mathbf{F}}\, d\upsilon$$

S encloses V. This theorem is particularly connected with Gauss' Law, to be stated later.

Again, $\nabla \cdot \mathbf{F} = \text{Div. } \mathbf{F}$ is not necessarily the dot product of ∇ and $\mathbf{F}$. The latter is true for rectangular coordinate systems, viz.,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

For the definitions of $\nabla \cdot \mathbf{F}$ refer to the handout "Vector Differential Operators." From now on, we will assume that

$$\begin{array}{ccc} \nabla f(x,y,z), & \nabla \cdot \mathbf{F}, & \nabla \times \mathbf{F} \\ \text{grad } f & \text{Div. } \mathbf{F} & \text{curl } \mathbf{F} \end{array}$$

are math quantities that can be looked up and used appropriately as math expressions. See the handouts or the text for list of formulas/identities associated with these operators.

Differential form of Faraday's Law

Starting from Stokes's theorem, we have

$$\oint_C \mathbf{E} \cdot \mathbf{dl} = \iint_S \nabla \times \mathbf{E} \cdot \hat{n}\, ds$$

where $\mathbf{E}$ is the electric field. Next, on introducing the first of Maxwell's equations (in integral form), we get

$$\iint_S \nabla \times \mathbf{E} \cdot \hat{n}\, ds = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{n}\, ds$$

Moving everything to the left-hand side gives

$$\iint_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot \hat{n}\, ds = 0 \quad \Rightarrow \quad \text{Integral is zero, implying}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad \textbf{Faraday's Law}$$

Differential form of Ampère-Maxwell's Equations

Again, based on Stokes's theorem

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = \iint_S \nabla \times \mathbf{H} \cdot \mathbf{ds}$$

and from Ampère's Law we get

$$\Rightarrow \quad \iint_S \nabla \times \mathbf{H} \cdot \mathbf{ds} = \iint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \hat{n}\, ds$$

$$\Rightarrow \quad \boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \qquad \textbf{Ampère-Maxwell's Law}$$

Differential form of Continuity Equation

Here we invoke the Div. Theorem, giving

$$\oint\!\!\!\!\oint_S \mathbf{J} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \mathbf{J}\, d\upsilon$$

$$\Rightarrow \quad \iiint_V \nabla \cdot \mathbf{J}\, dv = -\frac{\partial}{\partial t} \iiint_V \rho\, d\upsilon$$

$$\Rightarrow \quad \boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}} \qquad \textbf{Continuity Equation}$$

Gauss' Law

We can derive some more useful and commonly used relations between fields by using the above three Maxwell's equation which are considered as mutually independent.

To begin, note the identity

$$\boxed{\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0}$$

On taking the div of the first M.E. we get

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t}$$

But since $\nabla \cdot (\nabla \times \mathbf{E}) \equiv 0$,

$$0 = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{B}$$

$$\Rightarrow \quad \boxed{\nabla \cdot \mathbf{B} = 0}$$

The corresponding integral form for $\nabla \cdot \mathbf{B} = 0$ is

$$\boxed{\oiint_S \mathbf{B} \cdot \mathbf{ds} = 0} \qquad \textbf{Gauss' Magnetic Law}$$

Similarly, on taking the div of the second Maxwell's equation, we have

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \quad \Rightarrow$$

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{D}$$

But since $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

$$-\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} \quad \Rightarrow \quad \boxed{\nabla \cdot \mathbf{D} = \rho}$$

which in integral form is

$$\boxed{\oiint_S \mathbf{D}\, \mathbf{ds} = \iiint \rho \, dv = Q} \qquad \textbf{Gauss' Electric Law}$$

Summary of independent equations

Differential form	**Integral form**
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \iint \mathbf{B} \cdot \mathbf{ds}$
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot \mathbf{dl} = \iint \mathbf{J} \cdot \mathbf{ds} + \frac{\partial}{\partial t} \iint \mathbf{D} \cdot \mathbf{ds}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$	$\iint_S \mathbf{J} \cdot \hat{n}\, ds = -\frac{\partial}{\partial t} \iiint_V \rho \, dv$

Dependent Equations

(but necessary for DC fields/statics)

$$\nabla \cdot \mathbf{B} = 0 \qquad \oiint_S \mathbf{B} \cdot ds = 0$$

$$\nabla \cdot \mathbf{D} = \rho \qquad \oiint_S \mathbf{D} \cdot ds = \iiint_V \rho \, dv = Q_{\text{encl.}}$$

All are needed for statics in which case **E** and **B** become independent of one another.

Considering a solution of these in the rectangular coordinate system where $\mathbf{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$ etc., a quick look reveals that the 3 independent equations have a total of

$$\begin{aligned} 3(\mathbf{E}) + 3(\mathbf{H}) &= 6 \\ 3(\mathbf{D}) + 3(\mathbf{B}) + 3(\mathbf{J}) &= 9 \\ 1(\rho) &= 1 \\ \text{Total} &= 16 \text{ unknowns} \end{aligned}$$

for a total of $3+3+1=7$ equations. Thus, we need an additional 9 equations that relate **E**, **B**, **H**, **J**, **D**, and ρ for a unique solution of the system. These are the well-known constitutive relations:

$\mathbf{D} = \varepsilon \mathbf{E}$ (3 eqns.) $\qquad \varepsilon =$ permittivity, in free space $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

$\mathbf{B} = \mu \mathbf{H}$ (3 eqns.) $\qquad \mu =$ permeability, in free space $\mu_0 = 4\pi \times 10^{-7}$ F/m

$\mathbf{J} = \sigma \mathbf{E}$ (3 more eqns.)

and (Ohm's Law)

(3 eqns.) $\quad \mathbf{J}_c = \sigma \mathbf{E}, \qquad \sigma =$ conductivity, $\qquad \sigma_{\text{metals}} \approx 10^7$ S/m; $\sigma_{\text{free space}} = 0$ (no current)

These additional 9 equations provide us with a complete set for the field solution. Often, Maxwell's equations are written as

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \end{aligned}}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}$$

involving only $(2 \times 3 = 6)$ unknowns **E** and **H**. Here, **J** or ρ are assumed to be given and refer to the known excitation.

Note that

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_c = J_i + \sigma \mathbf{E}$$

where "i" means "impressed."