Solving Maxwell's Equations Using Potentials: Given J find (E,H)

- Start with:

$$\nabla\times\bar{\mathrm{E}}_e = -j\omega\mu\bar{H}_{\mathrm{e}} \qquad (1)$$

$$\nabla\times\bar{\mathrm{H}}_e = +j\omega\varepsilon\bar{E}_{\mathrm{e}} + \bar{J} \qquad (2)$$

- Introduce a simple differential equation:

$$\bar{\mathrm{B}}_e = \mu\bar{H}_e = \nabla\times\bar{A} \qquad (3)$$

$$\bar{\mathrm{H}}_{\mathrm{e}} = \frac{1}{\mu}\nabla\times\bar{A} \qquad (4)$$

$$\because\ \nabla\bullet\nabla\times\bar{A}\equiv 0 \therefore\ \nabla\times\bar{\mathrm{B}}=\nabla\bullet(\mu\bar{H})=0$$

Substituting (4) into (1): $\nabla\times\bar{\mathrm{E}}_e = -j\omega\frac{1}{\mu}\nabla\times\bar{A} \Rightarrow \nabla\times(\bar{\mathrm{E}}_{\mathrm{e}} + j\omega\bar{A}) = 0$

- Recall the old static equations: $\nabla\times\nabla\Phi = 0, \quad \bar{\mathrm{E}}_{\mathrm{static}} = -\nabla\Phi$

Thus, we set $\bar{\mathrm{E}}_{\mathrm{A}} + j\omega\bar{A} = -\nabla\Phi_e \Rightarrow \bar{\mathrm{E}}_{\mathrm{A}} = -\nabla\Phi_e - j\omega\bar{A}$

From (2):

$$\nabla\times\left(\frac{\nabla\times\bar{A}}{\mu}\right) = j\omega\varepsilon(-\nabla\Phi_e - j\omega\bar{A}) + \bar{\mathrm{J}} = -j\omega\varepsilon\nabla\Phi_e + \underbrace{\omega^2\mu\varepsilon}_{k^2}\bar{A} + \bar{\mathrm{J}} \quad (\mathrm{k}=\frac{\omega}{\mathrm{c}}) \quad (5)$$

- To simplify the double curl, note

$$\nabla\times\nabla\times\bar{A}=-\nabla^2\bar{A}+\nabla(\nabla\bullet\bar{A}) \qquad (6)$$

Substitute (6) into (5): $\nabla^2\bar{A}+k^2\bar{A}=\nabla\left(j\omega\mu\varepsilon\Phi_e+\nabla\bullet\bar{A}\right)+\mu\bar{\mathrm{J}}$

$\because$ $\nabla\times\bar{A}$ has derivatives different from $\nabla\bullet\bar{A}$ and a function is not completely defined until all derivatives (recall Taylor series) are known,

$\therefore$ we can arbitrarily set $\nabla\bullet\bar{A}=-j\omega\mu\varepsilon\Phi_e \Rightarrow \Phi_e=\dfrac{\nabla\bullet\bar{A}}{-j\omega\mu\varepsilon}$

$$\nabla^2\bar{A}+k^2\bar{A}=-\mu\bar{\mathrm{J}} \quad (\text{component form:}\nabla^2 A_a+k^2 A_a=-\mu \mathrm{J}_\mathrm{a})$$

- In spherical coordinates, for one component, this becomes

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2\frac{\partial A_a}{\partial \mathrm{r}}\right)+k^2A_a=0 \qquad (7)$$

- To solve (7), set $A_a(r) = \dfrac{f(r)}{r} \quad \Rightarrow \quad \dfrac{\partial^2}{\partial \mathrm{r}^2} f(r) + k^2 f(r) = 0$

  whose solution is $\quad \mathrm{f(r)} = \mathrm{C}_1 e^{-jkr} + \mathrm{C}_2 e^{+jkr}$

  From the Sommerfeld condition, keep only the outgoing wave:

$$\mathrm{A_a} = \mathrm{C}_1 \frac{e^{-jkr}}{r}$$

  To account for sources, recall that from statics the solution of

$$\frac{1}{\mathrm{r}^2}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}^2 \frac{\partial V}{\partial \mathrm{r}}\right) = 0 \quad is \quad \mathrm{V} = \frac{\mathrm{C}_1}{\mathrm{r}}$$

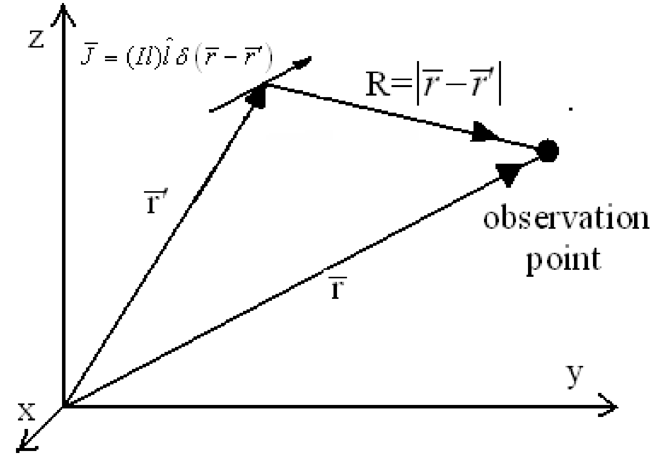
- The solution of $\quad \nabla^2 V = -\dfrac{\rho(r)}{\varepsilon}$

$$is \quad \mathrm{V(r,\phi,\theta)} = \frac{\mathrm{Q}}{4\pi\varepsilon \mathrm{r}} \rightarrow \iiint \frac{\rho\,(\bar{r}')}{4\pi\varepsilon_0 \left|\bar{r} - \bar{r}'\right|} dv' = \Phi_e(\bar{r}')$$

- Therefore $\quad \mathrm{C}_1 = \dfrac{Q}{4\pi\varepsilon_0}$

• Similarly, $\nabla^2 A_a = -\mu J_a$ implies $\quad A_a(r) = \iiint \frac{\mu J_a(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} dV$

The $|\bar{r}-\bar{r}'|$ term provides the shift due to the off center location of the source.



• Because the fundamental solution function

is now $\frac{e^{-jkr}}{r}$ and not $\frac{1}{r}$,

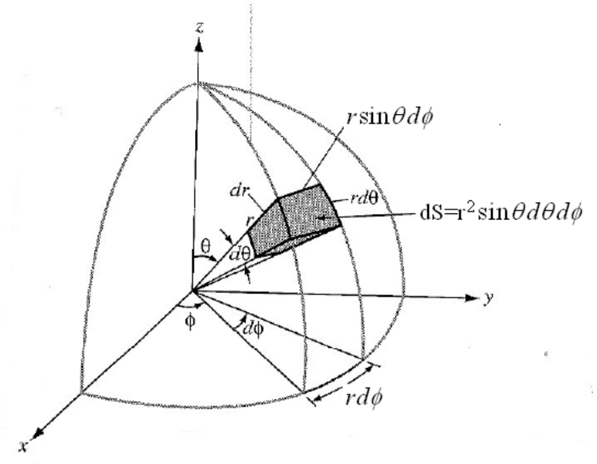
the integral for A_a is modified to read:

(Compare $A_a = C_1 \frac{e^{-jkr}}{r}$ *with* $V = \frac{C_1}{r}$)

$$A_a = \iiint \frac{\mu J_a(\bar{r}') e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} d\upsilon' \rightarrow \iiint \frac{\mu J_a(r) e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} d\upsilon' = \bar{A}(\bar{r}')$$

This can be a volume, surface or line integral, depending on the form of $\bar{J}_a$.

- Generally $\bar{A}=\int_{\ell}\frac{\mu\bar{J}(\bar{r}')e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}dl'$ with $dl'=\begin{cases}dx\\dy\\dz\end{cases}$



- Volume (not often used for antenna analysis)

$$\bar{A}=\iiint\frac{\mu\bar{J}(\bar{r}')e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}dV'\left(\begin{array}{c}dV'=dx'dy'dz'\\dV'=r^2\sin\theta dr d\phi d\theta\end{array}\right)$$

Note: $G(\bar{r},\bar{r}')=\frac{e^{-jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}$ or $G(R)=\frac{e^{-jkR}}{4\pi R}$

Then $\bar{A}=\int_{\text{line, surface or volume}}\mu\bar{J}(\bar{r}')G(\bar{r},\bar{r}')dV'$

- *For* $\iint_S\mu\bar{J}(\bar{r}')G(\bar{r},\bar{r}')ds$ $\quad ds=\begin{cases}dxdy\\dydz\\dxdz\end{cases}$ or $ds=r^2\underbrace{\sin\theta d\theta d\phi}_{d\Omega}$ $\left(\iint d\Omega=4\pi\ \textit{steradians}\right)$

- *Field* Solution $\bar{E}_e=-\nabla\Phi_e-j\omega\bar{A}=-j\omega\bar{A}-\nabla\left[\frac{\nabla\bullet\bar{A}}{-j\omega\mu\varepsilon}\right]=-j\omega\bar{A}-j\frac{1}{\omega\mu\varepsilon}\nabla\nabla\bullet\bar{A}$