One-Dimensional Green's Function

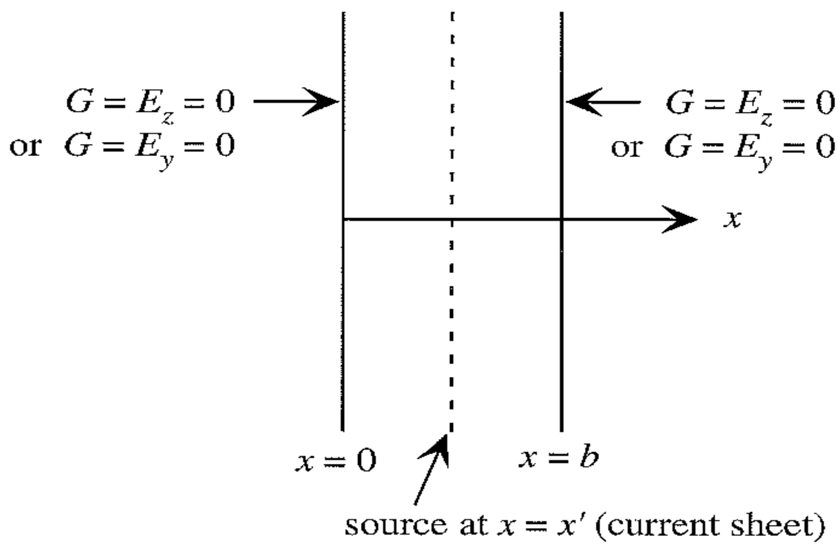
(see also Balanis, Ch. 14)

Consider the solution of the D.E.

$$\frac{d^2G}{dx^2} + \beta_0^2 G = -\delta(x - x')$$

with

$$G(x=0) = 0 = G(x=b)$$



1. Closed-Form Solution

Propose the solution

$$G(x,x') = \begin{cases} \dfrac{U(x)\,T(x')}{C} & x > x' \\ \dfrac{U(x')\,T(x)}{C} & x < x' \end{cases}$$

where $U(x)$ and $T(x)$ satisfy

$$U'' + \beta_0^2 U = 0 \qquad U(b) = 0$$
$$T'' + \beta_0^2 T = 0 \qquad T(0) = 0$$

leading to the solutions

1)

$$T(x) = A\sin\beta_0 x + B\cos\beta_0 x$$

but since $T(0) = 0 \quad \Longrightarrow$

$$\boxed{T(x) = A\sin\beta_0 x}$$

2)

$$U(x) = A' \sin\beta_0 x + B' \cos\beta_0 x$$

with $U(b) = 0$

$$U(b) = A' \sin\beta_0 b + B' \cos\beta_0 b = 0 \quad \Longrightarrow$$
$$\frac{A'}{B'} = -\frac{\cos\beta_0 b}{\sin\beta_0 b}$$
$$U(x) = B'\left(-\frac{\cos\beta_0 b}{\sin\beta_0 b}\sin\beta + \cos\beta_0 x\right)$$
$$\boxed{U(x) = B'' \sin\beta_0 (x-b)}$$

in which $B'' = -B'/\sin(\beta_0 b)$.

The expression for the constant C is

$$\begin{aligned}
C &= -\left[U(x')\,T'(x') - U'(x')\,T(x')\right] \\
&= +(AB'')\left[\beta_0 \sin\beta_0 x' \cos\beta_0(x'-b) - \beta_0 \cos\beta_0 x' \sin\beta_0(x'-b)\right] \\
&= +(AB'')\beta_0\left[\sin\beta_0[x' - (x'-b)]\right] \\
&= +(AB'')\beta_0\left[\sin(\beta_0 b)\right]
\end{aligned}$$

Thus, the final expression for $G(x,x')$ is

$$G(x,x') = \begin{cases} \dfrac{\sin\beta_0 x \sin\beta_0(x'-b)}{\beta_0 \sin\beta_0 b} & x < x' \\[2ex] \dfrac{\sin\beta_0 x' \sin\beta_0(x-b)}{\beta_0 \sin\beta_0 b} & x > x' \end{cases}$$

2. Eigenfunction Representation of $G(x,x')$

Here we construct $G(x,x')$ as a superposition of functions which satisfy the wave equation and the associated boundary conditions, viz.

$$G(x,x') = \sum_{n=-\infty}^{\infty} a_n \psi_n(x)$$

where $\psi_n(x)$ are the solutions to

$$\frac{d^2\psi_n}{dx^2} + \beta_n^2 \psi_n = 0 \qquad \psi_n(0) = \psi(b) = 0$$

and $\psi_n(x)$ are referred to as the natural (source-free) solutions/modes or simply the eigenfunctions of the problem. To find $\psi_n(x)$ we proceed to solve the wave equation in the usual manner. We set

$$\psi_n(x) = A \sin\beta_n x + B \cos\beta_n x$$

and since

$$\psi_n(0) = 0 \quad \Longrightarrow \quad B = 0$$

Also,

$$\psi_n(b) = 0 \quad \Longrightarrow \quad \psi_n(b) = A \sin\beta_n b = 0 \quad \Longrightarrow$$
$$\beta_n b = n\pi \qquad n = 1, 2, \ldots$$

Thus, $\beta_n = \frac{n\pi}{b}$ and are referred to as the eigenvalues of the problem. Finally,

$$\boxed{\psi_n(x) = A \sin \frac{n\pi}{b} x}$$

For normalization we will set $A = \sqrt{2/b}$ so that $\int_0^b [\psi_n(x)]^2\, dx = 1$.

To find a_n, let's consider the D.E. satisfied by G and ψ_n:

$$\left.\begin{aligned} \psi_n'' + \beta_n^2 \psi_n &= 0 \\ G'' + \beta_0^2 G &= -\delta(x-x') \end{aligned}\right\} \Longrightarrow \begin{aligned} G\psi_n'' + \beta_n^2 \psi_n G &= 0 \\ \psi_n G'' + \beta_0^2 G \psi_n &= -\delta(x-x')\psi_n \end{aligned}$$

Subtracting the latter gives

$$\begin{aligned} [G\psi_n'' + \beta_n^2 \psi_n G - \psi_n G'' - \beta_0^2 G \psi_n] &= \delta(x-x')\, \psi_n(x) \\ [G\psi_n'' - \psi_n G''] &= (\beta_0^2 - \beta_n^2)\psi_n G + \delta(x-x')\, \psi_n(x) \end{aligned}$$

Next, we integrate both sides to eliminate the delta function. We have

$$\int_a^b [G\psi_n'' - \psi_n G'']\, dx = (\beta_0^2 - \beta_n^2) \int_a^b \psi_n G\, dx + \int_a^b \psi_n(x)\, \delta(x-x')\, dx$$

and upon introducing the eigenfunction representing for $G(x,x')$ we get

$$(\beta_0^2 - \beta_n^2) \sum_{m=0} a_m \int_a^b \psi_n \psi_m\, dx \equiv - \int_a^b \psi_n(x)\, \delta(x-x')\, dx$$

Finally, noting that

$$\int_0^b \sin \frac{n\pi}{b} x \sin \frac{m\pi}{b} x\, dx = \begin{cases} 0, & m \neq n \\ \displaystyle\int_0^b \sin^2 \frac{n\pi}{b} x\, dx = \int_0^b \frac{1 - \cos \frac{2n\pi}{b} x}{2}\, dx = \left(\frac{b}{2}\right), & m = n \end{cases}$$

(i.e., making use of the eigenfunction orthogonality) yields

$$(\beta_0^2 - \beta_n^2) a_n = - \int_0^b \psi_n(x)\, \delta(x-x')\, dx \Longrightarrow$$

or

$$\boxed{a_n = -\frac{\psi_n(x')}{\beta_0^2 - \beta_n^2}}$$

If the excitation is a function $f(x)$ instead of the impulsive source $-\delta(x-x')$, then

$$\boxed{a_n = +\frac{1}{\beta^2 - \beta_n^2} \int_a^b \psi_n(x)\, f(x)\, dx}$$

The complete Green's function then takes the form

$$\boxed{G = -\sum_{n=0}^{\infty} \frac{2}{b} \frac{\sin\left(\frac{n\pi}{b} x'\right) \sin\left(\frac{n\pi}{b} x\right)}{\beta_0^2 - \beta_n^2}, \qquad \beta_n = \frac{n\pi}{b}}$$

Corollary:

Since $G'' + \beta_0^2 G = -\delta(x-x')$, it follows that

$$-\left[-\left(\frac{2}{b}\right)\sum_{n=-\infty}^{\infty}\frac{\sin\frac{n\pi}{b}x'\sin\frac{n\pi}{b}x}{\beta-\beta_n^2}+\frac{2}{b}\sum\frac{\sin\frac{n\pi}{b}x'\sin\frac{n\pi}{b}x}{\beta-\beta_n^2}\right]=-\delta(x-x')$$

$$\sum\frac{2}{b}\sin\frac{n\pi}{b}x'\sin\frac{n\pi}{b}x=+\delta(x-x')$$

$$\Longrightarrow\sum\psi_n(x)\,\psi_n^*(x')=\delta(x-x')$$

This latter result is referred to as the *completeness identity* or *completeness theorem.*

Summary

In general, to solve the D.E.

$$\frac{d^2 g}{dx^2} + \beta_0^2 g(x) = f(x) \qquad + \text{B.C.}$$

write

$$g(x) = \sum_{n=-\infty}^{\infty} a_n \psi_n(x) \qquad \begin{aligned} \alpha_a g'(a) + \beta_a g(a) &= 0 \\ \alpha_b g'(b) + \beta_b g(b) &= 0 \end{aligned}$$

Next consider the solution of

$$\frac{d^2 \psi_n}{dx^2} + \beta_n^2 \psi_n = 0$$

with B.C.

$$\alpha_a \psi_n'(a) + \beta_a \psi_n(a) = 0$$
$$\alpha_b \psi_n'(b) + \beta_b \psi_n(b) = 0$$

Note: Choose

$$\int_a^b \psi_n(x)\, \psi_n(x) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

to normalize the orthogonal functions $\psi_n(x)$. Upon finding the eigenfunctions (ψ_n) and eigenvalues (β_n), the coefficients are found from

$$a_n = \frac{1}{\beta_0^2 - \beta_n^2} \int_a^b f(x)\, \psi_n(x)\, dx$$

Sturm-Liouville Problem

(See pp. 867–869 Balanis.) Consider

$$[L + \beta_0^2 r(x)]\, g(x) = f(x) \qquad + \text{B.C.}$$

where

$$L = \frac{d}{dx}\left(p(x) \frac{d}{dx} \right) - q(x)$$

is a general differential operator. To obtain the solution for $g(x)$, write

$$g(x) = \sum_{m=-\infty}^{\infty} a_m \psi_m(x)$$

where

$$[L + \beta_n^2 r(x)]\, \psi_n(x) = 0$$

and we normalize $\psi_n(x)$ so that

$$\int_a^b r(x)\, \psi_n \psi_m \, dx = \delta_{mn} = \delta(m-n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

The functions $\psi_n(x)$ and $g(x)$ satisfy the D.E.'s

$$g(x)\,[L+\beta_n^2\,r(x)]\,\psi_n(x)=0$$
$$\psi_n(x)\,[L+\beta_0^2\,r(x)]\,g(x)=f(x)\,\psi_n(x)$$

Subtracting we get

$$[gL\psi_n-\psi_nLg]+(\beta_n^2-\beta_0^2)\,r(x)\,g\psi_n=-f(x)\,\psi_n(x)$$

Next we integrate both sides to get

$$\int_a^b[gL\psi_n-\psi_nLg]\,dx+\int_a^b(\beta_n^2-\beta_0^2)\,r(x)\,g(x)\,\psi_n(x)\,dx=-\int_a^b f(x)\,\psi_n(x)\,dx$$

Next, upon making use of the property

$$\int g(x)\,\{L\psi_n(x)\}\,dx=\int\psi_n(x)\,\{Lg(x)\}\,dx \qquad \text{or} \qquad \langle g,L\psi_n\rangle=\langle\psi_n,Lg\rangle$$

(a property of Hermitian operators), we get

$$(\beta_n^2-\beta_0^2)\sum_{m=-\infty}^{\infty}\int_a^b a_m\,r(x)\,\psi_m\psi_n\,dx=-\int_a^b f(x)\,\psi_n(x)\,dx$$

Finally, on invoking the orthogonality of ψ_n, we obtain

$$\boxed{a_n=-\frac{1}{(\beta_n^2-\beta_0^2)}\int_a^b f(x)\,\psi_n(x)\,dx}$$

implying that

$$\boxed{g(x)=\int_a^b f(x')\,G(x,x')\,dx'}$$