Sample Problems in Coordinate Systems

#1) The electric and magnetic fields generated by a small wire antenna is given by:

$$\bar{E} = \nabla(\nabla \cdot \bar{\pi}_e) + k^2 \bar{\pi}_e \quad , k^2 = \omega^2 \mu \epsilon$$

$$\bar{H} = j\omega\epsilon \, \nabla \times \bar{\pi}_e$$

ω = frequency, ϵ = permittivity
μ = permeability
k = propagation constant

where $\bar{\pi}_e = A \dfrac{e^{-jkr}}{4\pi r} \hat{z}$, $r = \sqrt{x^2 + y^2 + z^2}$

A = Constant

- Carry out the required operations to find an explicit expression for $\bar{H}$ in Cartesian Coordinates:

- Find the spherical components of $\bar{H}$.

2)

Solution)

* Cartesian Coordinates:

Write $\bar{H}$ in terms of x, y, and z:

$$\bar{H} = j\omega\epsilon\, \nabla\times\left(A\frac{e^{-jkr}}{4\pi r}\hat{z}\right) = j\omega\epsilon\, \nabla\times\left(A\frac{e^{-jk\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}\hat{z}\right)$$

$\nabla\times H$ in Cartesian Coordinates is:

$$\nabla\times\bar{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cancel{H_x} & \cancel{H_y} & H_z \end{vmatrix}$$

$$0 \qquad 0$$

Noting that
$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{e^{-jkr}}{r}\right) = \frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right)\frac{\partial r}{\partial x} \\ \qquad = -\frac{e^{-jkr}}{r^3}(1+jkr)x \;, r = \sqrt{x^2+y^2+z^2} \\ \frac{\partial}{\partial y}\left(\frac{e^{-jkr}}{r}\right) = \cdots \\ \frac{\partial}{\partial z}\left(\frac{e^{-jkr}}{r}\right) = \cdots \end{cases}$$

Then:

3)

$$\bar{H} = \hat{x}\,\frac{\partial H_z}{\partial y} - \hat{y}\,\frac{\partial H_z}{\partial x}$$

$$- \frac{e^{-jkr}}{r^3}(1+jkr)\,y\,\hat{x} + \frac{e^{-jkr}}{r^3}(1+jkr)\,x\,\hat{y}$$

$$\boxed{\bar{H} = \frac{j\omega\epsilon A}{4\pi}\,\frac{e^{-jkr}}{r^3}(1+jkr)(-y\,\hat{x} + x\,\hat{y})}$$

* <u>spherical coordinates :</u>

For finding $\bar{H}$ in spherical coordinates we can do either of the followings:

1 - Write the above expression in spherical coordinates:

we already know that: $y = r\sin\theta\sin\varphi$

$x = r\sin\theta\cos\varphi$

Note:

$\hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta$

(see table)

and

$$\hat{x} = \hat{r}\,\boxed{\overset{\hat{r}\cdot\hat{x}}{\sin\theta\cos\varphi}} + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi$$

$$\hat{y} = \hat{r}\sin\theta\sin\varphi + \hat{\theta}\underbrace{\cos\theta\sin\varphi}_{\hat{y}\cdot\hat{\theta}} + \hat{\varphi}\cos\varphi$$

(refer to the table)

4)

$\Rightarrow$ after some algebraic manipulation :

$$\boxed{\bar{H} = j\omega\epsilon \frac{A}{4\pi r^2} \sin\theta \, e^{-jkr} (1 + jkr)\, \hat{\varphi}}$$

2 - perform the curl in spherical coordinate :

$$\bar{H} = j\omega\epsilon \nabla \times \bar{\Pi}_e = j\omega\epsilon \, \nabla \times \left(\frac{A}{4\pi r} e^{-jkr} \hat{z} \right)$$

refer to the table $\longleftarrow$

$$= j\omega\epsilon \, \nabla \times \left[\left(\frac{A \, e^{-jkr}}{4\pi r} \right) \left(\hat{r}\cos\theta - \hat{\theta}\sin\theta \right) \right]$$

$$\nabla \times \bar{H} = \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ H_r & rH_\theta & r\sin\theta H_\varphi \end{vmatrix} \times \frac{1}{r^2 \sin\theta}$$

which leads to the same result.

#2)

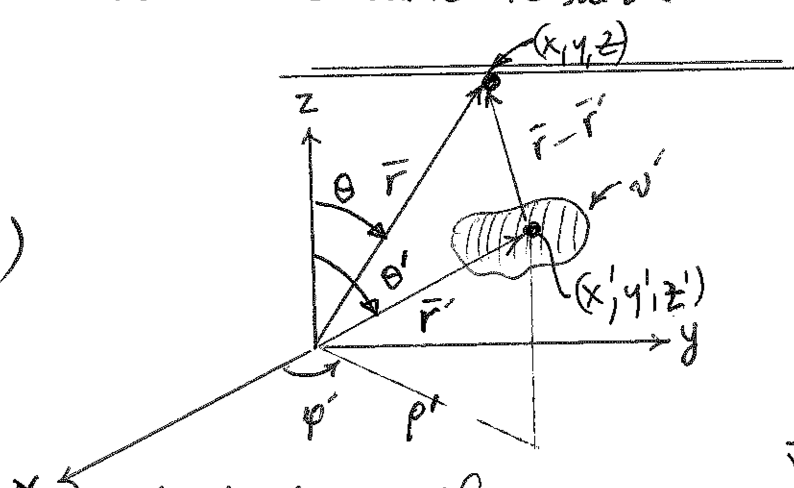

Illustration of the primed and unprimed coordinate system

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta$$

$$\bar{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$\hat{r}' = \hat{x}\cos\phi'\sin\theta' + \hat{y}\sin\phi'\sin\theta' + \hat{z}\cos\theta'$$

5)

The far field electric field due to an electric current source is given by:

$$E_\theta = \frac{-j\omega\mu}{4\pi r} e^{-jkr} \iiint_{V'} (J_x \cos\theta\cos\varphi + J_y \cos\theta\sin\varphi - J_z \sin\theta) e^{jkg} dv'$$

$$E_\varphi = \frac{-j\omega\mu}{4\pi r} e^{-jkr} \iiint_{v'} (-J_x \sin\varphi + J_y \cos\varphi) e^{jkg} dv'$$

where $dv' = dx'dy'dz'$, $g = x'\cos\varphi\sin\theta + y'\sin\varphi\sin\theta + z'\cos\theta$

- set $x' = \rho'\cos\varphi'$, $y' = \rho'\sin\varphi'$

$$J_x = J_{\rho'}\cos\varphi' - J_{\varphi'}\sin\varphi', \quad J_y = J_{\rho'}\sin\varphi' + J_{\varphi'}\cos\varphi'$$

(refer to the table)

Note that (ρ', φ', z') and (ρ, φ, z) are associated with $\bar{r}'$ and $\bar{r}$ respectively.

Show that E_φ can be rewritten as:

$$E_\varphi = \frac{j\omega\mu}{4\pi r} e^{-jkr} \iiint_{v'} \left(J_{\rho'} \sin(\varphi - \varphi') - J_{\varphi'} \cos(\varphi - \varphi') \right) e^{jkg} dv'$$

where $dv' = \rho' d\rho' d\varphi' dz'$ and $g = \rho'\cos(\varphi' - \varphi)\sin\theta + z'\cos\theta$

6)

– Similarly show that E_θ can be rewritten as :

$$E_\theta = \frac{j\omega\mu}{4\pi r} e^{-jkr} \iiint_{V'} \left(-J_{\rho'} \cos(\varphi-\varphi')\cos\theta - J_{\varphi'} \sin(\varphi-\varphi')\cos\theta + J_z \sin\theta\right) e^{jkg} dv'$$

Solution)

Do not get confused by primed and unprimed coordinates.

These coordinates represent source and observation point respectively.

So since source should be represented by primed coordinates, we

wrote: $J_x = J_{\rho'} \cos\varphi' - J_{\varphi'} \sin\varphi'$, $J_y = J_{\rho'} \sin\varphi' + J_{\varphi'} \cos\varphi'$

$$\Rightarrow \left(J_x \cos\theta\cos\varphi + J_y \cos\theta\sin\varphi - J_z \sin\theta\right) e^{jkg} \overset{\text{algebraic manipulation}}{=}$$

$$\left(J_{\rho'} \sin(\varphi-\varphi') - J_{\varphi'} \cos(\varphi-\varphi')\right) e^{jkg}$$

where
$$g = x' \cos\varphi \sin\theta + y' \sin\varphi \sin\theta + z' \cos\theta$$

$$= \overbrace{\rho' \cos\varphi'}^{x'} \cos\varphi \sin\theta + \overbrace{\rho' \sin\varphi'}^{y'} \sin\varphi \sin\theta + z' \cos\theta$$

$$= \rho' \sin\theta \cos(\varphi-\varphi') + z' \cos\theta$$

$$\Rightarrow E_\varphi = \frac{-j\omega\mu}{4\pi r} e^{-jkr} \iiint_{V'} \left(J_{\rho'} \sin(\varphi-\varphi') - J_{\varphi'} \cos(\varphi-\varphi')\right) e^{jkg} dv'$$

Similarly E_θ can be written as above.

·		Rectangular $\hat{x}$	$\hat{y}$	$\hat{z}$	Cylindrical $\hat{\rho}$	$\hat{\phi}$	$\hat{z}$	Spherical $\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
Rectangular	$\hat{x}$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
	$\hat{y}$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
	$\hat{z}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Cylindrical	$\hat{\rho}$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
	$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
	$\hat{z}$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
Spherical	$\hat{r}$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
	$\hat{\theta}$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
	$\hat{\phi}$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

Note that the unit vectors $\hat{r}$ in the cylindrical and spherical systems are *not* the same. Example:

$\hat{r}\cdot\hat{x} = \sin\theta\cos\phi$

$\hat{r}\cdot\hat{y} = \sin\theta\sin\phi$

$\hat{r}\cdot\hat{z} = \cos\theta$

Examples

$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$

$\hat{\rho} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta.$

Example uses of the table