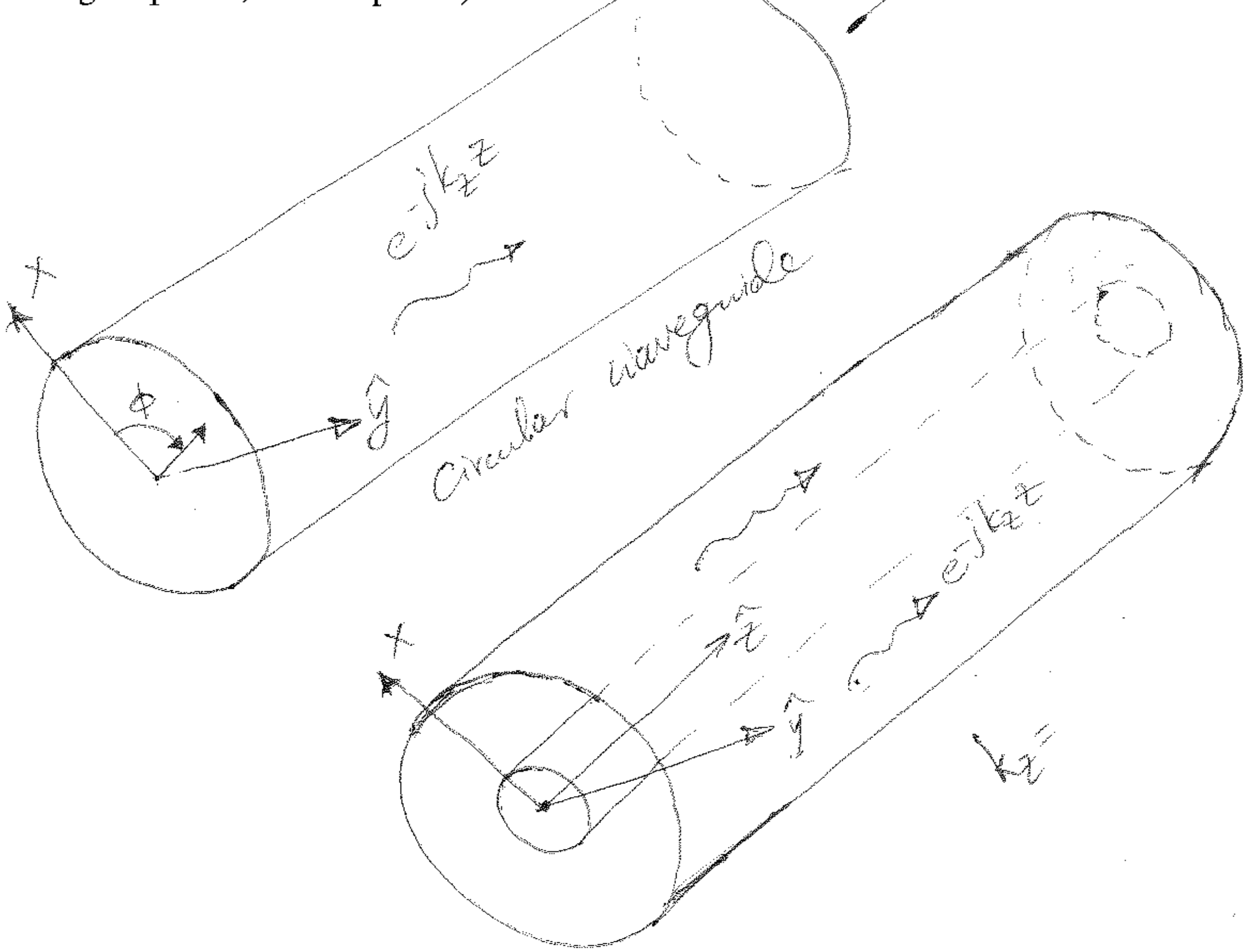
Cylindrical waveguide

(Harrington p. 198; Balanis p. 471)



TE modes

$$\nabla^2 F_z + \beta^2 F_z = 0 \qquad \Rightarrow$$

$$F_z = f(\rho)\; g(\phi)\; h(z) \qquad \text{separation of variables}$$

$$\nabla^2 F_z = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F_z}{\partial \rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 F_z}{\partial\phi^2} + \frac{\partial^2 F_z}{\partial z^2}$$

Thus, we have

$$\frac{\partial^2 F_z}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial F_z}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2 F_z}{\partial\phi^2} + \frac{\partial^2 F_z}{\partial z^2} + \beta^2 F_z = 0$$

or

$$ghf'' + gh\frac{1}{\rho}f' + fh\frac{1}{\rho^2}g'' + fgh'' = -\beta^2 fhg$$

Diving by "fgh" gives

$$\frac{1}{f}f'' + \frac{1}{f}\frac{1}{\rho}f' + \frac{1}{g}\frac{1}{\rho^2}g'' + \frac{1}{h}h'' = -\beta^2$$

The last term is only a function of z and thus

$$\frac{h''(z)}{h(z)} = -\beta_z^2 \qquad \Rightarrow$$
$$h(z) = Ae^{-j\beta_z z} + Be^{+j\beta_z z}$$

Then

$$\frac{f''}{f} + \frac{1}{\rho}\frac{f'}{f} + \frac{1}{\rho^2}\frac{g''}{g} = (\beta_z^2 - \beta^2)$$

or

$$\rho^2\frac{f''}{f} + \rho\frac{f'}{f} + \underbrace{(\beta_0^2 - \beta_z^2)}_{\beta_\rho^2}\rho^2 + \frac{g''}{g} = 0$$

$$\Rightarrow \qquad \frac{g''}{g} = -\beta_\phi^2$$

$$\Rightarrow \qquad g(\phi) = A'e^{j\beta_\phi\phi} + B'e^{-j\beta_\phi\phi}$$

Because $g(\phi)$ must be periodic with respect to ϕ we must have $\beta_\phi = m,\ \ m = 0, 1, 2, 3, \ldots$

$$g(\phi) = A'e^{jm\phi} + B'e^{-jm\phi}$$
$$g(\phi) = A\cos m\phi + B\sin m\phi$$

and

$$\rho^2\frac{f''}{f} + \rho\frac{f'}{f} + [(\beta_\rho\rho)^2 - m^2] = 0$$

or

$$\rho^2 f'' + \rho f' + [(\beta_\rho\rho)^2 - m^2]f = 0$$

This is the general Bessel's equation and has the solution

$$f(\rho) = A\ J_m(\beta_\rho\rho) + B\ Y_m(\beta_\rho\rho)$$
$$\text{or } A\ H_m^{(1)}(\beta_\rho\rho) + B\ H_m^{(2)}(\beta_\rho\rho)$$
$$\text{or } A\ J_m(\beta_\rho\rho) + B\ H_m^{(1)}(\beta_\rho\rho)$$

Thus, the complete solution for F_z is

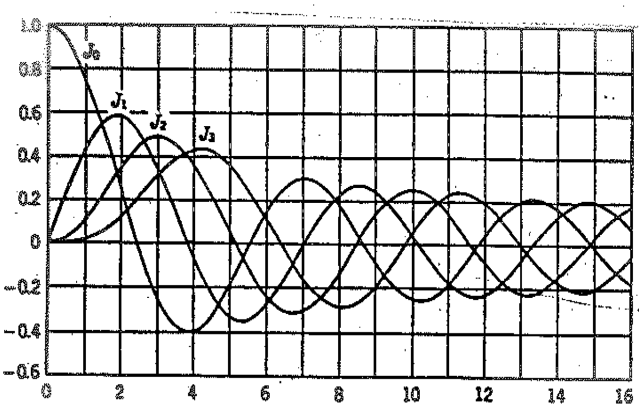
$$F_{z_m}(\rho, \phi, z) = [A_1 J_m(\beta_\rho\rho) + B_1 H_m^{(2)}(\beta_\rho\rho)][A_2\cos m\phi + B_2\sin m\phi][A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}]$$

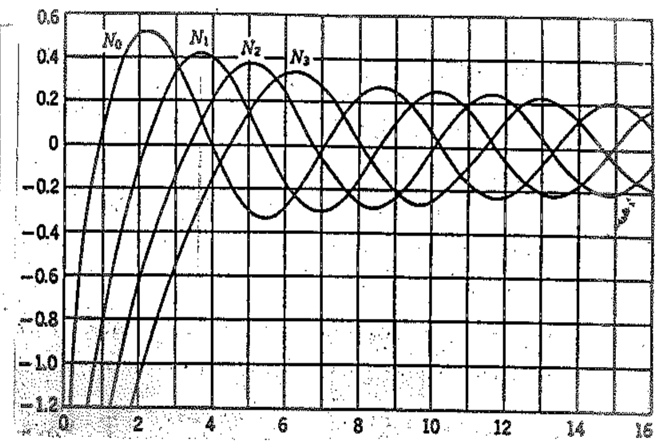
with

$$\beta^2 = \beta_\rho^2 + \beta_z^2$$

Bessel functions

(see p. 934 of Balanis)





	Bessel function	$x \to 0$ Small arg. approximation	$x \to \infty$ Large arg. approximation
Bessel function of the 1st kind	$J_v(x)$ (compare to $\cos x$)	$\frac{1}{\Gamma(v+1)}\left(\frac{x}{2}\right)^v$ $J_0(0) = 1$	$\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$
Bessel function of the 2nd kind	Neumann function $N_v(x) = Y_v(x)$ compare to $\sin x$	$v = 0$ $N_0(x) = \frac{2}{\pi}\ln\left(\frac{\gamma x}{2}\right)$ $\gamma = 1.781$ $v \neq 0$ $N_v(x) = \frac{-(v-1)!}{\pi}\left(\frac{2}{x}\right)^v$	$\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$
Hankel function of the 1st kind	$H_v^{(1)}(x) =$ $J_v(x) + jN_v(x)$ compare to e^{+jkx}	$H_1^{(1)}(x) = \frac{-j2}{\pi x}$	$\sqrt{\frac{2}{j\pi x}}\, e^{-jv\pi/2} e^{jx}$
Hankel function of the 2nd kind	$H_v^{(2)}(x) =$ $J_v(x) - jN_v(x)$ compare to e^{-jkx}	$H_1^{(2)}(x) = \frac{+j2}{\pi x}$	$\sqrt{\frac{2j}{\pi x}}\, e^{+jv\pi/2} e^{-jx}$
Modified Bessel functions $I_v(x) = j^v\, J_v(-jx)$ $K_v(x) = \frac{\pi}{2}(-j)^{v+1}\, H_v^{(2)}(-jx)$		Identity ($B(x) \leftarrow$ any Bessel) $B1_v(x) = B_{v-1} - \frac{v}{x}B_v$ $B_v'(x) = -B_{v+1} + \frac{v}{x}B_v$	Identity $B_0'(x) = -B(x)$

Let us now apply the required boundary condition and rad. conditions (circular waveguide).

1) Assume propagation along $+z$.

2) $E_\phi(\rho = a, \phi, z) = 0$.

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho} = \beta_\rho \frac{A_{mn}}{\epsilon} J'_m(\beta_\rho \rho)\,[A_2 \cos m\phi + B_2 \sin m\phi] e^{-j\beta_z z}$$
$$= \beta_\rho \frac{A_{mn}}{\epsilon} \frac{\partial}{\partial(\beta_\rho \rho)} J(\beta_\rho \rho)\,[A_2 \cos m\phi + B_2 \sin m\phi] e^{-j\beta_z z}$$
$$k_0^2 = k_\rho^2 + k_z^2$$

since

$$J'_m(\beta_\rho \rho) = \frac{\partial}{\partial(\beta_\rho \rho)} J(\beta_\rho \rho)$$

Note also

$$J'_m(\beta\rho) = \beta\, J_{m-1}(\beta\rho) - \frac{m}{\rho} J_m(\beta\rho)$$

From

$$E_\phi(\rho = 0) = 0 \qquad \Rightarrow \qquad \beta_\rho a = \chi'_{mn} \qquad \Rightarrow \qquad \beta_\rho = \frac{\chi'_{mn}}{a}, \qquad n = 1, 2, 3, \ldots$$

χ'_{mn}

n \ m	0	1	2
1	3.832	1.841	3.054
2	7.016	5.331	6.706
3	10.173	8.536	9.969

For

$$m = 1 = \chi'_{11} = 1.8412$$
$$\Rightarrow \qquad f_c = \frac{\chi_{mn}}{2\pi a\sqrt{\mu\epsilon}}$$

TE_{11} is the lowest order mode.
$\Rightarrow$ Cutoff frequency occurs when

$$\beta_z = 0 = \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\frac{\chi'_{mn}}{a}\right)} = 0 \qquad \Rightarrow$$
$$\beta = \beta_c = \frac{\chi'_{mn}}{a} = \omega\sqrt{\mu\epsilon} = 2\pi f_c \sqrt{\mu\epsilon} \qquad \Rightarrow$$
$$\boxed{f_c = \frac{\chi'_{mn}}{2\pi a\sqrt{\mu\epsilon}}}$$

where f_c is the cutoff frequency of the mn mode. Note that

$$\beta_c = \frac{2\pi}{\lambda_c}, \qquad \lambda_c = \frac{2\pi}{\beta_c}, \qquad \lambda_c = \frac{2\pi a}{\chi'_{mn}}$$

Also,

$$\Rightarrow \qquad Z_{mn}^{\mathrm{TE}} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\omega\mu}{\beta_z} = \frac{j\omega\mu}{\gamma}$$

$$\boxed{Z_{mn}^{\mathrm{TE}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1-(f_c/f)^2}} = \frac{\omega\mu}{\beta\sqrt{1-(f_c/f)}}}$$

with

$$\beta_z = k_z - j\gamma \qquad \text{or} \quad \gamma = jk_z$$

We can rewrite β_z as

$$\beta_z = \sqrt{\beta^2 - \beta_\rho^2} = \beta\sqrt{1-\left(\frac{\beta_\rho}{\beta}\right)^2} = \beta\sqrt{1-\left(\frac{\chi'_{mn}}{a/(2\pi f/c)}\right)^2} = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\boxed{\lambda_g = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta}\frac{1}{\sqrt{1-(f_c/f)^2}}}$$

Explicit TE^z fields

$$\begin{aligned}
E_\rho &= -\frac{1}{\epsilon\rho}\frac{\partial F_z}{\partial \phi} = -A_{mn}\frac{m}{\epsilon\rho}\, J_m(\beta_\rho\rho)[-A_2\sin m\phi + B_2\cos m\phi]e^{-j\beta_z z} \\
E_\phi &= -\frac{1}{\epsilon}\frac{\partial F_z}{\partial \rho} = A_{mn}\frac{\beta\rho}{\epsilon}\, J'_m(\beta_\rho\rho)[A_2\cos m\phi + B_2\sin m\phi]e^{-j\beta_z z} \\
E_z &= 0 \\
H_\rho &= -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_z}{\partial\rho\,\partial z} \\
H_\phi &= -j\frac{1}{\omega\mu}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} \\
H_z &= \frac{-j}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)F_z
\end{aligned}$$

As another example, if $E_\rho = 0$ at $\phi = 0$ and $\phi = \pi = 2$, then $E_\rho(\phi = 0) = 0 \Rightarrow B_2 = 0$. Consequently

$$\begin{aligned}
E_\rho &= -\frac{1}{\epsilon\rho}\frac{\partial F_z}{\partial\phi} \\
&= -\widetilde{A}_{mn}\frac{m}{\epsilon\rho}\, J_m(\beta_\rho\rho)\sin(m\phi)e^{-j\beta_z z}
\end{aligned}$$

TM^z modes

$$A = \hat{z}\, A_z(\rho, \phi, z) \qquad \rightarrow \text{ gives } E_z \text{ components for } H_z = 0$$

$$A_z = [A_1\, J_m(\beta_\rho \rho) + B_1\, Y_m(\beta_\rho \rho)][A_2 \cos m\phi + B_2 \sin m\phi] e^{-j\beta_z z} \qquad \text{for } +z\text{-propagation}$$

$\beta_\rho^2 + \beta_z^2 = \beta^2$ from wave equation. Must have $E_\phi(\rho = 0, \phi, z) = 0$ *or* $E_z(\rho = a, \phi, z) = 0$ and E must be finite at $\rho = 0$.

$$E_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial}{\partial z^2} + \beta^2\right) A_z \sim A_z \qquad H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi}$$

$$E_\phi = -j\frac{1}{\omega\mu\epsilon}\frac{1}{\rho}\frac{\partial^2}{\partial \phi\, \partial z} A_z \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho}$$

$$E_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 A_z}{\partial \rho\, \partial z}$$

Since E_z must be finite at $\rho = 0$,

$$A_z = [A_1\, J_m(\beta_\rho \rho)][A_2 \cos \phi + B_2 \sin m\phi] e^{-j\beta_z z}$$

and from $E_z = E_\phi = 0$ on $\rho = a \Rightarrow$

$$J_m(\beta_\rho a) = 0 \qquad \Rightarrow \qquad \beta_\rho a = \chi_{mn}$$

since E_z is finite at $\rho = 0$ we have

$$A_z = J_m(\beta_\rho \rho)\, [A_2 \cos m\phi + B_2 \sin m\phi] e^{-j\beta_z z}$$

and from $E_z = E_\phi = 0$ on $\rho = a$ we get

$$J_m(\beta_\rho a) = 0 \qquad \Rightarrow \qquad \beta_\rho a = \chi_{mn}, \qquad n = 1, 2, 3, \ldots$$

χ_{mn} have been tabulated.

$$\beta_z \sqrt{\beta^2 - \beta_\rho^2} = \sqrt{\beta^2 - \left(\frac{\chi_{mn}}{a}\right)^2}$$

The cutoff frequency of each mnth mode is

$$\beta^2 = \omega_c^2 \mu\epsilon = \left(\frac{\chi_{mn}}{a}\right)^2 \qquad \Rightarrow$$

$$\boxed{(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\left(\frac{\chi_{mn}}{a}\right)}$$

χ_{mn}

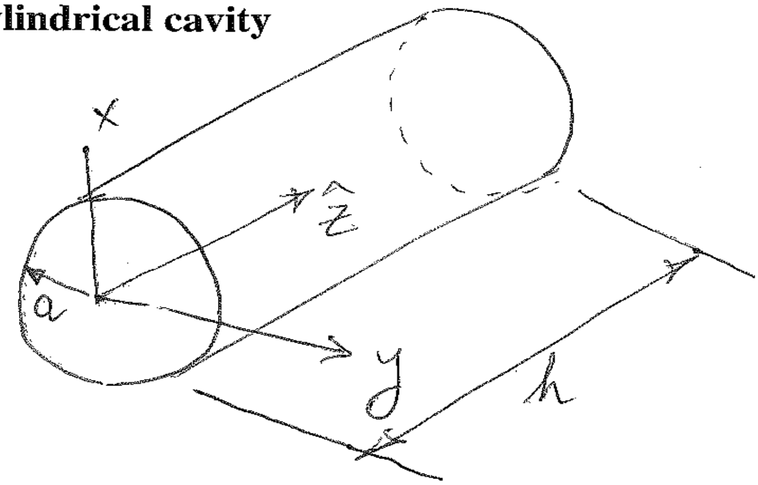
	m			
n	0	1	2	
1	2.405	3.832	5.136	6.38
2	5.52	7.016	8.417	9.761
3	8.654	10.173	11.620	13.015

TM_{11} is degenerate with TE_{01}.

$$\mathrm{TM}_{13} \leftrightarrow \mathrm{TE}_{03}$$

Formulas for λ_c, λ_g, etc., are the same as for the TE case with $\chi'_{mn} \rightarrow \chi_{mn}$.

Example of cylindrical cavity



$E_\rho = 0$ at $z = 0$ and $z = h \Rightarrow$

$$
\begin{aligned}
&F_z = A_{mn}\ J_m(\beta_\rho \rho)\ [A_2 \cos m\phi + B_2 \sin m\phi][A_3 \sin \beta_z z + B_3 \cos \beta_z z] \\
&\beta^2 = \beta_\rho^2 + \beta_z^2 \\
&E_\rho = -\frac{1}{\epsilon\rho}\frac{\partial F_z}{\partial \phi} = 0 \qquad \text{at } z = 0 \text{ and } z = h \\
&\quad \Rightarrow \quad B_3 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\sin \beta_z h = 0 \quad \Rightarrow \quad \beta_z h = p\pi \quad \Rightarrow \quad \beta_z = \frac{p\pi}{h} \\
&F_z = \widetilde{A}_{mn}\ J_m(\beta_\rho \rho)\ [A_2 \cos m\phi + B_2 \sin m\phi] \sin \frac{p\pi}{h} z
\end{aligned}
$$

Note that

$$
\beta_\rho = \frac{\chi'_{mn}}{a} \quad \Rightarrow \quad \beta^2 = \left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2
$$

must be satisfied. This implies that only certain frequencies will excite the guide, i.e., when

$$
\beta' = \beta_\mathrm{r} = \omega_\mathrm{r}\sqrt{\mu\epsilon} = 2\pi f_\mathrm{r}\sqrt{\mu\epsilon}
$$

$$
(f_\mathrm{r})_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\left[\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2\right], \qquad
\begin{aligned}
m &= 0, 1, 2, 3, \ldots \\
n &= 0, 1, 2, 3, \ldots \\
p &= 0, 1, 2, 3, \ldots
\end{aligned}
$$

The lowest resonance is T.

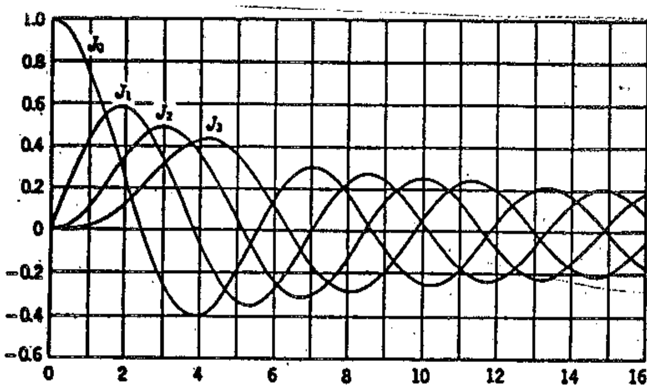
For $h/a > 2$ the $(f_\epsilon)_{111}^{\mathrm{TE}}$ is the lowest resonant frequency, i.e., TE_{111} is dominant.

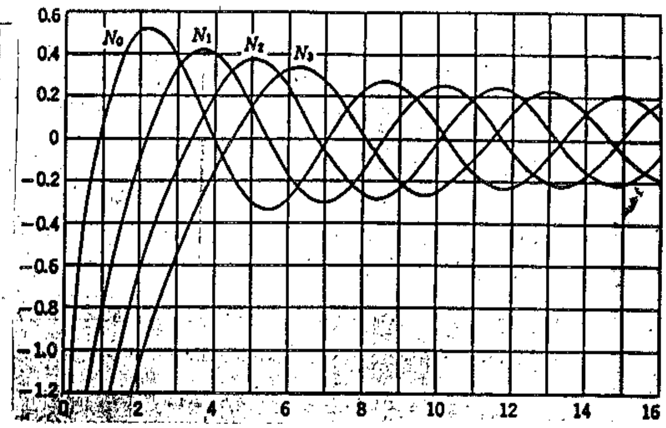
For $h/a < 2$ the TM_{010} mode is the dominant mode.

$$
\begin{aligned}
(f_\mathrm{r})_{mnp} &= \frac{1}{2\pi\sqrt{\mu\epsilon}}\left[\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2\right] \\
(f_\mathrm{r})_{111} &= \frac{1}{2\pi\sqrt{\mu\epsilon}}\left[\left(\frac{1.8}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2\right], \qquad p = 1
\end{aligned}
$$

Bessel functions

(see p. 934 of Balanis)





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Hankel function of the 2nd kind	$H_v^{(2)}(x) =$ $J_v(x) - jN_v(x)$ compare to e^{-jkx}	$H_1^{(2)}(x) = \frac{+j2}{\pi x}$	$\sqrt{\frac{2j}{\pi x}}\, e^{+jv\pi/2}e^{-jx}$
Modified Bessel functions $I_v(x) = j^v\, J_v(-jx)$ $K_v(x) = \frac{\pi}{2}(-j)^{v+1}\, H_v^{(2)}(-jx)$		Identity ($B(x) \leftarrow$ any Bessel) $B1_v(x) = B_{v-1} - \frac{v}{x}B_v$ $B'_v(x) = -B_{v+1} + \frac{v}{x}B_v$	Identity $B'_0(x) = -B(x)$