HW Assignment Solution for EML 4806 CH 6

Problem 2:

${}^{C_i}I$ given by (6.27). $\boxed{{}^{C_i}I_{ZZi} = \frac{Mi}{12}(L_i^2 + W_i^2)}$

Other moments will not matter.

$${}^1P_{C_1} = r_1\hat{X}_1, \quad {}^2P_{C_2} = r_2\hat{X}_2 \quad \left(r_i = \frac{L_i}{2}\right)$$

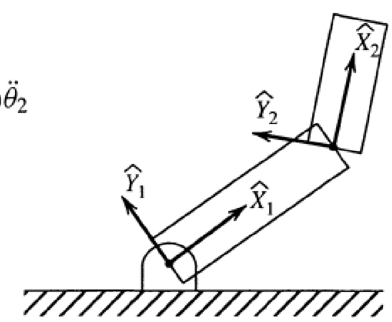
Derivation follows that of section 6.7 quite closely.
The answer is:

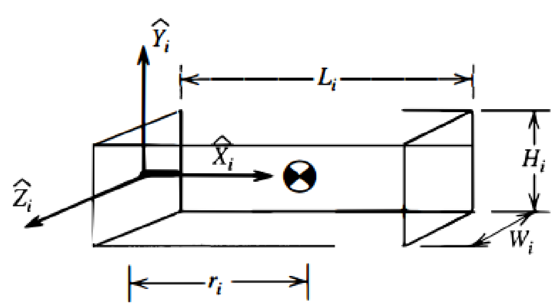
$$\tau_1 = (I_{ZZ1} + I_{ZZ2} + 2M_2r_2L_1C_2 + M_2L_1^2 + M_1r_1^2 + M_2r_2L_2)\ddot{\theta}_1$$
$$+ (M_2r_2L_2 + I_{ZZ2} + M_2L_1r_2C_2)\ddot{\theta}_2 - M_2L_1r_2S_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$$
$$+ M_2r_2gC_{12} + M_1r_1gC_1 + M_2L_1gC_1$$

(*Continued*)

$$\tau_2 = (I_{ZZ2} + M_2L_1r_2C_2 + M_2L_2r_2)\ddot{\theta}_1 + (I_{ZZ2} + M_2L_2r_2)\ddot{\theta}_2$$
$$+ M_2L_1r_2S_2\dot{\theta}_1^2 + M_2r_2gC_{12}$$

Further compaction could be done since $r_i = \frac{L_i}{2}$.





Problem 5:

α_{i-1}	a_{i-1}	d_i	θ_i
0	0	0	θ_1
90°	L_1	0	θ_2
0	L_2	0	0

$${}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} {}^1P_2 &= L_1\hat{X}_1 \\ {}^2P_3 &= L_2\hat{X}_2 \end{aligned}$$

$${}^1P_{C_1} = L_1\hat{X}_1 \; {}^2P_{C_2} = L_2\hat{X}_2 \; {}^{C_1}I_1 = 0 \; {}^{C_2}I_2 = 0$$

$${}^0\dot{V}_0 = g\hat{Z}_0. \quad \text{(since gravity points in } -\hat{Z}_0 \text{ Dir.)}$$

$$W_0 = \dot{W}_0 = 0 \quad \text{(base stationary)}$$

$$F_3 = x_3 = 0 \quad \text{(no forces on hand)}$$

Forward Velocity & Acceleration Iterations:

Link 1

$${}^1W_1 = {}^1_0R\,{}^0W_0 + \dot{\theta}_1\,{}^1\hat{Z}_1 = \dot{\theta}_1\,{}^1\hat{Z}_1 = [0 \quad 0 \quad \dot{\theta}_1\,]^T$$

$${}^1\dot{W}_1 = {}^1_0R\,{}^0\dot{W}_0 + {}^1_0R\,{}^0W_0 \otimes \dot{\theta}_1\,{}^1\hat{Z}_1 + \ddot{\theta}_1\,{}^1\hat{Z}_1$$

$${}^1\dot{W}_1 = \ddot{\theta}_1\,{}^1\hat{Z}_1 = [0 \quad 0 \quad \ddot{\theta}_1\,]^T$$

$${}^1\dot{V}_1 = {}^1_0R({}^0\dot{W}_0 \otimes {}^0P_1 + {}^0W_0 \otimes ({}^0W_0 \otimes {}^1P_2) + {}^0\dot{V}_0)$$

$${}^1\dot{V}_1 = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$${}^1\dot{V}_{C_1} = {}^1\dot{W}_1 \otimes {}^1P_{C_1} + {}^1W_1 \otimes ({}^1W_1 \otimes {}^1P_{C_1}) + {}^1\dot{V}_1$$

$${}^1\dot{V}_{C_1} = \ddot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1 + \dot{\theta}_1\hat{Z}_1 \otimes (\dot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1) + g\hat{Z}_1$$

$${}^1\dot{V}_{C_1} = L_1\ddot{\theta}_1\hat{Y}_1 + \dot{\theta}_1\hat{Z}_1 \otimes L_1\dot{\theta}_1\hat{Y}_1 + g\hat{Z}_1$$

$${}^1\dot{V}_{C_1} = L_1\ddot{\theta}_1\hat{Y}_1 + (-L_1\dot{\theta}_1^2)\hat{X}_1 + g\hat{Z}_1 = \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ L_1\ddot{\theta}_1 \\ g \end{bmatrix}$$

Link 2

$${}^2W_2 = {}^2_1R\,{}^1W_1 + \dot{\theta}_2\,{}^2\hat{Z}_2$$

$${}^2W_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^2\dot{W}_2 = {}^2_1R\,{}^1\dot{W}_1 + {}^2_1R\,{}^1W_1 \otimes \dot{\theta}_2\hat{Z}_2 + \ddot{\theta}_2\hat{Z}_2$$

$$^2\dot{W}_2 = \begin{bmatrix} S_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$^2\dot{W}_2 = \begin{bmatrix} S_2\ddot{\theta}_1 \\ C_2\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} C_2\dot{\theta}_1\dot{\theta}_2 \\ -S_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} S_2\ddot{\theta}_1 + C_2\dot{\theta}_1\dot{\theta}_2 \\ C_2\ddot{\theta}_1 - S_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$^2\dot{V}_2 = {}^2_1R({}^1\dot{W}_1 \times {}^1P_2 + {}^1W_1 \otimes ({}^1W_1 \otimes {}^1P_2) + {}^1\dot{V}_1)$$

$$^2\dot{V}_2 = {}^2_1R(\ddot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1 + \dot{\theta}_1\hat{Z}_1 \otimes (\dot{\theta}_1\hat{Z}_1 \otimes L_1\hat{X}_1) + g\hat{Z}_1)$$

$$^2\dot{V}_2 = {}^2_1R(L_1\ddot{\theta}_1\hat{Y}_1 - L_1\dot{\theta}_1^2\hat{X}_1 + g\hat{Z}_1)$$

$$= \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ L_1\ddot{\theta}_1 \\ g \end{bmatrix}$$

$$^2\dot{V}_2 = [\,-L_1C_2\dot{\theta}_1^2 + S_2g,\; L_1S_2\dot{\theta}_1^2 + C_2g,\; -L_1\ddot{\theta}_1\,]^T$$

$$^2\dot{V}_{C_2} = {}^2\dot{W}_2 \otimes {}^2P_{C_2} + {}^2W_2 \otimes ({}^2W_2 \otimes {}^2P_{C_2}) + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} S_2\ddot{\theta}_1 + C_2\dot{\theta}_1\dot{\theta}_2 \\ C_2\ddot{\theta}_1 - S_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\otimes \left(\begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \right) + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} 0 \\ L_2\ddot{\theta}_2 \\ -L_2C_2\ddot{\theta}_1 + L_2S_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ L_2\dot{\theta}_2 \\ -L_2C_2\dot{\theta}_1 \end{bmatrix} + {}^2\dot{V}_2$$

$$^2\dot{V}_{C_2} = \begin{bmatrix} -(L_1C_2 + L_2C_2^2)\dot{\theta}_1^2 - L_2\dot{\theta}_2^2 + S_2g \\ (L_1S_2 + L_2S_2C_2)\dot{\theta}_1^2 + L_2\ddot{\theta}_2 + C_2g \\ 2L_2S_2\dot{\theta}_1\dot{\theta}_2 - L_1\ddot{\theta}_1 - L_2C_2\ddot{\theta}_1 \end{bmatrix}$$

Inertial Equations:

Link 1

$$^1F_1 = M_1\,{}^1\dot{V}_{C_1}$$

$$^1F_1 = \begin{bmatrix} -M_1L_1\dot{\theta}_1^2 \\ M_1L_1\ddot{\theta}_1 \\ M_1g \end{bmatrix}$$

$$^1N_1 = {}^{C_1}I_1\,{}^1\dot{W}_1 + {}^1W_1 \otimes {}^{C_1}I_1\,{}^1W_1$$

$$^1N_1 = [0 \quad 0 \quad 0]^T$$

Link 2

$$^2F_2 = M_2\,{}^2\dot{V}_{C_2}$$

$$
{}^2F_2 = \begin{bmatrix} -M_2(L_1 + L_2C_2)C_2\dot{\theta}_1^2 - M_2L_2\dot{\theta}_2^2 + M_2S_2g \\ M_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 + M_2L_2\ddot{\theta}_2 + M_2C_2g \\ 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 - M_2L_1\ddot{\theta}_1 - M_2L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^2N_2 = {}^{C_2}I_2{}^2\dot{W}_2 + {}^2W_2 \otimes {}^{C_2}I_2{}^2W_2 = [0 \quad 0 \quad 0]^T
$$

Backward Force Iterations:

Link 2

$$
{}^2F_2 = {}^2_3R^3F_3 + {}^2F_2 = {}^2F_2 \text{ (see above)}
$$

$$
{}^2n_2 = {}^2N_2 + {}^2_3R^3n_3 + {}^2P_{C_2} \otimes {}^2F_2 + {}^2P_3 \otimes {}^2_3R^3F_3
$$

$$
{}^2n_2 = {}^2P_{C_2} \otimes {}^2F_2 = L_2\hat{X}_2 \otimes {}^2F_2
$$

$$
{}^2n_2 = \begin{bmatrix} 0 \\ M_2L_1L_2\ddot{\theta}_1 - 2M_2L_2^2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2^2C_2\ddot{\theta}_1 \\ M_2L_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 - M_2gL_2C_2 + M_2L_2^2\ddot{\theta}_2 \end{bmatrix}
$$

Link 1

$$
{}^1F_1 = {}^1_2R^2F_2 + {}^1F_1 \text{ (not needed, so skip)}
$$

$$
{}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{C_1} \otimes {}^1F_1 + {}^1P_2 \otimes {}^1_2R\,{}^2F_2
$$

$$
{}^1_2R\,{}^2n_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} {}^2n_2
$$

$$
{}^1_2R\,{}^2n_2 = \begin{bmatrix} * \\ * \\ M_2L_1L_2C_2\ddot{\theta}_1 - 2M_2L_2^2S_2C_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2^2C_2^2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^1P_{C_1} \otimes {}^1F_1 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} -M_1L_1\dot{\theta}_1^2 \\ M_1L_1\ddot{\theta}_1 \\ M_1g \end{bmatrix} = \begin{bmatrix} 0 \\ -M_1gL_1 \\ M_1L_1^2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^1_2R\,{}^2F_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ 0 & 0 & -1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} -M_2(L_1 + L_2C_2)C_2\dot{\theta}_1^2 - M_2L_2\dot{\theta}_2^2 + M_2S_2g \\ M_2(L_1 + L_2C_2)S_2\dot{\theta}_1^2 + M_2gC_2 \\ 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 - M_2L_1\ddot{\theta}_1 - M_2L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
{}^1_2R\,{}^2F_2 = \begin{bmatrix} * \\ M_2L_2C_2\ddot{\theta}_1 + M_2L_1\ddot{\theta}_1 - 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 \\ * \end{bmatrix}
$$

$$
{}^1P_2 \otimes {}^1_2R\,{}^2F_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} * \\ M_2L_1\ddot{\theta}_1 - 2M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_2C_2\ddot{\theta}_1 \\ * \end{bmatrix}
$$

$$
{}^1P_2 \otimes {}^1_2R\,{}^2F_2 = \begin{bmatrix} * \\ * \\ M_2L_1^2\ddot{\theta}_1 - 2M_2L_1L_2S_2\dot{\theta}_1\dot{\theta}_2 + M_2L_1L_2C_2\ddot{\theta}_1 \end{bmatrix}
$$

$$
\therefore {}^1n_1 = \begin{bmatrix} * \\ * \\ (M_1L_1^2 + M_2L_1^2 + M_2L_2^2C_2 + 2M_2L_1L_2C_2)\ddot{\theta}_1 - 2(L_1 + L_2C_2)M_2L_2S_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}
$$

Joint Torque Equations

$\tau_i = {}^i n_i \cdot {}^i \hat{Z}_i + V_i \dot{\theta}_i$ Added viscous friction term.

$$\tau_1 = (M_1 L_1^2 + M_2 (L_1 + L_2 C_2)^2) \ddot{\theta}_1$$

$$- 2(L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 + V_1 \dot{\theta}_1$$

$$\tau_2 = M_2 L_2^2 \ddot{\theta}_2 + (L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1^2 + M_2 g L_2 C_2 + V_2 \dot{\theta}_2$$

or

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

where:

$$M(\theta) = \begin{bmatrix} (M_1 L_1^2 + M_2 (L_1 + L_2 C_2)^2) & 0 \\ 0 & M_2 L_2^2 \end{bmatrix}$$

$$V(\theta_1 \theta) = \begin{bmatrix} -2(L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 \\ (L_1 + L_2 C_2) M_2 L_2 S_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ M_2 g L_2 C_2 \end{bmatrix}$$