Overview

Signal Representation

- Signals carry overwhelming amounts of data in which relevant information is often more difficult to find than a needle in a haystack.

- Processing is faster and simpler if signals are represented in such a way that few coefficients reveal the information we are looking for.
Such representations can be constructed by decomposing signals signals over elementary waveforms chosen in a family called dictionary.
When the needed information is concentrated in few coefficients, we have <u>sparse representation</u>

Orthogonal basis is a dictionary of minimum size that can yield a sparse representation if designed to concentrate the signal energy over a set of few vectors

Nice but what ~~co~~ could be the problem with this orthogonal representation:
If one or more coefficients are missing it will create a bigger loss of information

<u>Fourier and Wavelet representations</u>

They decompose signals over oscillatory waveforms that reveal many signal properties and provide a path to sparse representations.

<u>Fourier Representations</u>

Why? If the system is linear and time-Invariant.

linear.

$$x(t)\ \boxed{\quad H \quad} \longrightarrow y(t) = H(x(t))$$

H: operator.

$$H(\alpha_1 x_1(t) + \alpha_2 x_2(t)) = \alpha_1 H(x_1(t)) + \alpha_2 H(x_2(t$$

Time - Invariant

$$x(t) \rightarrow \boxed{H} \rightarrow y(t) = H(x(t))$$

$$x(t-t_0) \rightarrow \boxed{H} \rightarrow H(x(t-t_0)) = y(t-t_0)$$

then :

$$e^{j\omega t} \; - \boxed{H} - \; y(t) = H(e^{j\omega t})$$

$$= \lambda e^{j\omega t}$$

- $h(t)$: impulse response of the system :

$$y(t) = h(t) * x(t)$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

if $x(t) = e^{j\omega t}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$y(t) = e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \lambda\, e^{j\omega t}$$

$$\lambda = H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

The motivation is there to write signals as superpositions of sinewaves ($e^{j\omega t}$)

1) <u>Periodic Signals</u>

$f(t)$ periodic period T_0

$\omega_0 = \dfrac{2\pi}{T_0}$ Then:

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k\, e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t)\, e^{-jk\omega_0 t}\, dt$$

Fourier Series . c_k : Fourier Coefficients

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} \longrightarrow \boxed{} \longrightarrow y(t)$$

$$= \sum_{k=-\infty}^{+\infty} \underline{\underline{H(k\omega_0) c_k}} e^{jk\omega_0 t}$$

$\cancel{H(k\omega_0) =}$

$$H(k\omega_0) = \int_{-\infty}^{+\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau$$

$h(t)$: impulse response

$$H(k\omega_0) = |H(k\omega_0)| e^{j \angle H(k\omega_0)}$$

2) <u>non-periodic signal</u>

If $f(t)$ is a finite energy signal.

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt < +\infty$$

$f(t)$ can be written as a continuous sum of weighted sinewaves

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega) e^{j\omega t} d\omega$$

where $$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$\hat{f}(\omega)$ measures how much oscillation at frequency $2\pi\omega$ does $f(t)$ have?

Fourier Transform ~~of~~ of $f(t)$

$|\hat{f}(\omega)|$ decays rapidly when ω increases if the signal is regular and smooth

Back to

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\langle f, g \rangle_T = \int_{-T/2}^{T/2} f(t)\, g^*(t)\, dt$$

Inner Product of f, g

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i^*$$

dot product

Fourier analysis is great for:

Linear. Time. Invariant systems

stationary signals steady state . .

Problems rise when:

one wants to represent a transient phenomenon, a word pronounced or. music. or a singularity in an image.

The Fourier Transform becomes a cumbersome tool that requires many coefficients to represent a localized event.

$\hat{f}(\omega)$ depends on the values of $f(t)$ for all times

This global mix of information makes it difficult to analyze or represent any local property of $f(t)$ from $\hat{f}(\omega)$

Then what ?

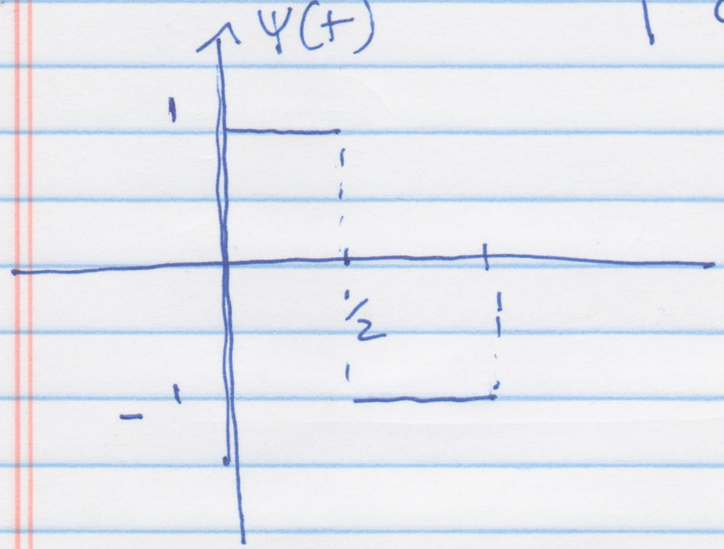
<u>Wavelet Bases</u> :

Wavelet Bases, like Fourier bases reveal the signal regularity through the amplitude of coefficients.

However, wavelets are well localized and few coefficients are needed to represent local transient structures.

Let's look at the Haar function

$$\psi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$



the Dilations and translations of which generate an ~~orthonormal~~ orthogonal bases.

$$\left\{ \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right) \right\}_{j,n \in \mathbb{Z}}$$

of $L^2(\mathbb{R})$, set of finite energy signals

$$\|f\|^2 = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt < +\infty$$

$$f(t) = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}(t)$$

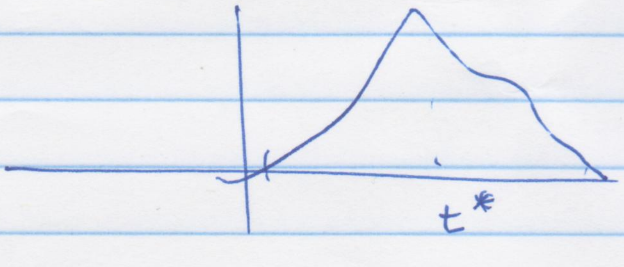
<u>Haar. Wavelets</u> :

Analyzing signals at different scales (frequencies) and times (positions)

<u>Time- Frequency Analysis</u>

D : Window function

$\phi(t)$



center

$$t^* = \frac{1}{\|\phi\|^2} \int_{-\infty}^{+\infty} t \, |\phi(t)|^2 dt$$

Radius $\Delta_\phi = \frac{1}{\|\phi\|} \left[\int_{-\infty}^{+\infty} (t-t^*)^2 |\phi(t)|^2 dt \right]^{1/2}$

$$\omega^* = \frac{1}{\|\hat{\phi}\|^2} \int_{-\infty}^{+\infty} \omega \, |\hat{\phi}(\omega)|^2 \, d\omega$$

$$\Delta_{\hat{\phi}} = \frac{1}{\|\hat{\phi}\|} \left[\int_{-\infty}^{+\infty} (\omega - \omega^*)^2 \, |\hat{\phi}(\omega)|^2 \, d\omega \right]^{1/2}$$

Figure of merit for the time - frequency window is its time-frequency width product $\Delta_\phi \Delta_{\hat{\phi}}$, which is bounded by the uncertainty principle and is given by

$$\Delta_\phi \Delta_{\hat{\phi}} \geq 1/2$$

The equality holds only when we have a Gaussian window