Metric Spaces

One important function of signal processing is approximation. We need to measure how close is the approximation. We have to define a way to measure this proximity. Let's introduce the notion of distance between signals that is both mathematically practically and physically meaningful.

Definition : X: set of objects (could be functions, signals ...)

A metric d is a function from $X \times X \longrightarrow \mathbb{R}^+$ such that.

1) $d(x,y) = d(y,x)$

2) $d(x,y) \geq 0$

3) $d(x,y) = 0$ if and only if $x = y$

4) $x, y, z \in X$

$$d(x,z) \leq d(x,y) + d(y,z)$$

Example 1

$$x \in \mathbb{R} \quad , y \in \mathbb{R}$$

$$d(x,y) = |x-y|$$

1) $d(y,x) = |y-x| = |x-y| = d(x,y)$

2) $d(x,y) = |x-y| \geq 0$

3) $d(x,y) = |x-y| = 0 \iff x = y$

4 $d(x,z) = |x-z| = |(x-y)+(y-z)|$

$\leq |x-y| + |y-z|$

$\leq d(x,y) + d(y,z)$

Example 2

$X = \mathbb{R}^n \qquad \vec{x} \in \mathbb{R}^n \quad , \vec{y} \in \mathbb{R}^n$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

distance $d_1 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$d_1(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

is called l_1 metric also known as the Manhattan metric

2. ~~med~~ metric $d_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$d_2(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

l_2 metric: Euclidean distance

3. generalizing d_p $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^p \right)^{1/p}$$

l_p metric

4 $p \to +\infty$

$$d_\infty(\vec{x}, \vec{y}) = \max_{i=1,\dots n} |x_i - y_i|$$

l_∞ metric

Definition: A metric space (X, d) is a set X together with a metric d

$(\mathbb{R}^n, d_1)$ is a metric space

$(\mathbb{R}^n, d_2)$ is another metric space

~~outlook.com/owa/fiu365.fiu.edu~~

$l_p = l_p(0, \infty)$ set of consisting of all infinite sequences of real or complex numbers $[x_0, x_1, x_2 \cdots]$ such that $\sum_{i=0}^{\infty} |x_i|^p < +\infty$

We will take $1 \le p < +\infty$. The function

$$d_p(x,y) = \left[\sum_{i=0}^{\infty} |x_i - y_i|^p \right]^{1/p}$$

defines a metric on l_p which we will call the l_p space. This is an infinite-dimensional space known as sequence space.

The set of two-sided sequences $[\cdots x_{-2}, x_{-1}, x_0, x_1 \cdots]$ with the metric d_p is the metric space $l_p(-\infty, +\infty)$.

In discrete-time signal processing applications we deal most frequently with l_1 space or with l_2 space, the former because absolute values are easy to compute, and the latter because the quadratic metric function is easily differentiable

3. The space $l_\infty(0, \infty)$ consists of all sequences of numbers $[x_0, x_1 \dots x_n \dots]$ such that $|x_n| \le M$ for some finite bound M, with the metric

$$d_\infty(x, y) = \sup_n |x_n - y_n|$$

~~max~~

Sup and Inf.

Let $S \subset \mathbb{R}$ (subset of $\mathbb{R}$), the least upper bound (LUB) is the smallest number z such that $z \ge x$ for every $x \in S$. The LUB of a set set S is called the sup (supremum) of the set. If there is ~~there is~~ ~~there~~

is no numbers that is greater than all elements of S, then $\sup(S) = \infty$. Similarly the ~~great~~ greatest lower bound (GLB) of a set is the largest number w such that $w \le x$ for every $x \in S$. The GLB is called the inf (infimum) of S. If there is no number less than all the elements of S, then $\inf(S) = -\infty$

The inf an sup are generalizations of min and max, respectively. Generally the inf and sup are used when there is a conitinuum of values over which to find the max or min, or where the extrema ~~is~~ may be infinite

Examples

$S = (2,5) \subset \mathbb{R}$ (This is an open set, and does not contain the endpoints.)

Then $\sup(S) = 5$ and $\inf(S) = 2$

Let $T = [4,7) \Rightarrow \inf(T) = 4$, $\sup(T) = 7$

$U = (1, \infty) \Rightarrow \inf(U) = 1$, $\sup(U) = \infty$

There are also many useful metric spaces defined over functions. These infinite dimensional spaces are called function spaces

<u>The metric space $(C_{[a,b]}, d_p)$</u>

Let $X = C_{[a,b]}$ be the set of all real-valued (or complex valued) functions defined on the interval $[a,b]$ with $b > a$

We can define a metric on functions

x and y in X by

$$d_p(x,y) = \left[\int_a^b |x(t)-y(t)|^p \, dt \right]^{1/p}$$

where $1 \le p < +\infty$. This gives the metric space $(C_{[a,b]}, d_p)$. The metric d_p between functions is referred to as the L_p metric

$$p=2 \qquad d_2(x,y) = \left[\int_a^b |x(t)-y(t)|^2 \, dt \right]^{1/2}$$

L_2 metric

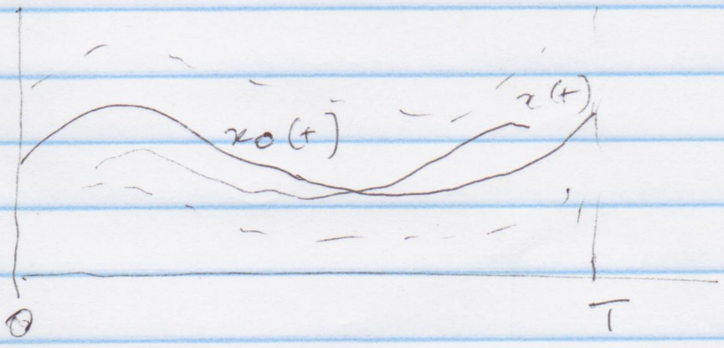
The metric space $(C_{[a,b]}, d_\infty)$

$$d_\infty(x,y) = \sup\{ |x(t)-y(t)| : a \le t \le b \}$$

In other words, the distance between the functions is obtained at the point where the functions are farthest apart

The difference between the metric spaces $(C_{[a,b]}, d_\infty)$ and $\mathcal{E}$ $(C_{[a,b]}, d_p)$ can be appreciated by considering the following example

$$X = C_{[0,T]} \qquad x_0(t) \in X$$



$$d_\infty(x_0, x) < \varepsilon$$

$$\sup |x_0(t) - x(t)| < \varepsilon$$

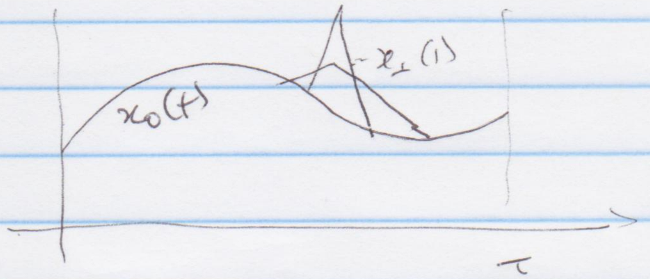
$$|x(t) - x_0(t)| < \varepsilon$$

$$-\varepsilon < x(t) - x_0(t) < \varepsilon$$

$$x_0(t) - \varepsilon < x(t) < x_0(t) + \varepsilon$$

$$d_2(x_0, x) < +\infty$$

$$d_2(x_0, x) = \int_0^T (x_0(t) - x(t))^2 \, dt$$



At any time to ~~x(t)~~ there may be significant deviation from $x_0(t)$, as long as long as the region over which the deviation occurs is not too long.

If $x(t)$ is an approximation to $x_0(t)$, using the d_∞ metric in expressing the approximation criterion provides an upper bond to the approximation error $|x(t) - x_0(t)|$ that cannot be obtained when using the d_p metric

Metric Space $L_p[a,b]$

$L_p[a,b]$ set of functions such that $\int_a^b |x(t)|^p dt < +\infty$

$1 \le p < \infty$

This set with d_p is

$(L_p[a,b], d_p)$ or simply noted

$L_p[a,b]$

Metric Space $L_\infty[a,b]$

$L_\infty[a,b]$: set of function such that $\sup_{t \in [a,b]} |x(t)| < \infty$

This set equipped with metric d_∞ is a metric Space

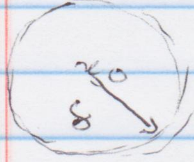
~~Some Topological~~

Some Topological terms

(X, d)

Ball or sphere centered at x_0 with radius δ is the set of points such that

$$B(x_0, \delta) = \{ x \in X : d(x_0, x) < \delta \}$$



Such a ball is ~~also~~ ~~called~~ also said to be a <u>neighborhood</u> of x_0

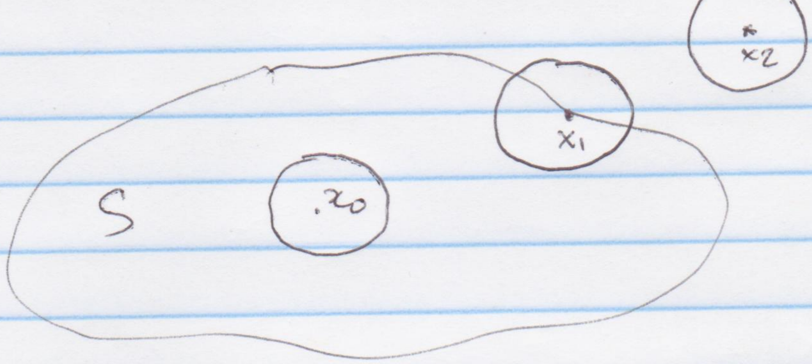
it is the set of points that live close to x_0

<u>Definition</u>: A point $x_0 \in X$ is <u>interior</u> to a set $S \subset X$ if all point sufficiently near x_0 are in S. That is there is ~~some~~ some $\delta > 0$ such that

$$B(x_0, \delta) \subset S$$

The interior of set S is ~~all~~ the set of all points that are interior to the set

$x_0 \notin S$ is exterior to if there is a neighborhood of x_0 that is not outside (does not intereset) S.



<u>Definition</u> A set X is <u>open</u> if every point in X is interior

Example $\quad X = (0,1) \subset \mathbb{R}$

$x_0 \in X \quad 0 < x_0 < 1$

$B(x_0, x_0/2) \qquad |x - x_0| < \frac{x_0}{2}$

$\cancel{x_0 - \frac{x_0}{2} < x < x_0 + \frac{x_0}{2}}$

<u>Definition</u> A set $S \subset X$ is said to be closed if the complement of S is open

<u>Definition</u> A boundary point of a set S is a point x_0 such that every neighborhood

of x_0 contains elements both in S and not in S. A boundary point is not necessarily an element of S

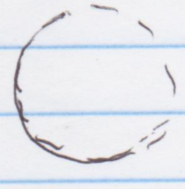
The boundary set of S is the collection of all boundary points of S. It is sometimes noted bdy(S)

<u>Definition</u>: the closure of X is the union of X and bdy(X) and is noted closure(X) or $\overline{X}$

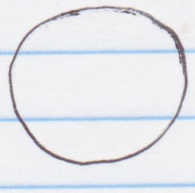
$$\text{Closure}(X) = \overline{X} = X \cup \text{bdy}(X)$$

Example $\quad X = [0,1)$

$\overline{X} = [0,1] \quad \overline{X}$ is closed



open set

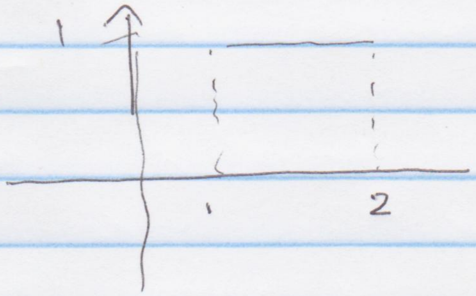


closed set

Definition x ~~is~~ X is said to be a cluster point if every neighborhood around x contains infinitely many pts of X

Definition : The support of a function $f: A \longrightarrow B$ is the closure of the set of elements $a \in A$ where $f(A) \neq 0$

Example



$$f(x) = \begin{cases} 1 & 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

support of f: closure$((1,2)) = [1,2]$

Sequences, Cauchy sequences and ~~Comple~~ Completeness

sequences can be generated many ways

Most frequently in signal processing is

through iterations. ~~for example~~

$$x_{n+1} = f(x_n, u_n)$$

Example : car loan

x_0 : initial loan

x_n : balance on month n

u_n : payment on month n

I : annual interest.

$$x_1 = \left(1 + \frac{I}{12}\right) x_0 - u_1$$

$$x_2 = \left(1 + \frac{I}{12}\right) x_1 - u_2$$

$$\vdots$$

$$x_n = \left(1 + \frac{I}{12}\right) x_{n-1} - u_n.$$

sequence $[x_0, x_1 \ldots\ldots]$

$n \to$ large $x_n \to 0$ (~~pa~~ paid)

- Deposit (with interest)

y_0 : intial deposit

y_n : balance on month n

u_n : deposit on ~~mo~~ month n

I : annual interest

$$y_1 = \left(1 + \frac{I}{12}\right) y_0 + u_1$$

$$y_2 = \left(1 + \frac{I}{12}\right) y_1 + u_2$$

$$y_n = \left(1 + \frac{I}{12}\right) y_{n-1} + u_n$$

Definition

If for every $\delta > 0$ there is an n_0 such that $d(x_n, x^*) < \delta$ for every $n > n_0$ for some fixed value x^*, then the sequence $\{x_n\}$ is said to converge to x^*. In this case we write

$$x_n \longrightarrow x^*$$

x^* is the limit of x_n

For every neighborhood N of X^*, there is

an n_0 such that $x_n \in N$, when $n \geq n_0$

Examples:

$a_n = n^2$? converge.

$b_n = 1 + (-1)^n$? converge

Properties of convergent sequences

1) Let (X, d) a metric space

The closure of a set $A \subset X$ is the set of all limits of converging sequences of points from A

2) A set $A \subset X$ is closed if it contains the limit of every converging sequence $\{x_n\}$ whose points lies in A

Example
$\{x_n\} = [1, 1.41, 1.414, 1.4142, \ldots]$

each number in this sequence is a rational number $\in \mathbb{Q}$ however.

$n \to +\infty$ it converges to $\sqrt{2} \notin \mathbb{Q}$ so

$\mathbb{Q}$ is not a closed set $\overline{\mathbb{Q}} = \mathbb{R}$

$\mathbb{R}$ is closed because every convergent.

~~eq~~ sequence in $\mathbb{R}$ has its limit in $\mathbb{R}$

limit point

$$b_n = 1 + (-1)^n$$

$$n \to +\infty \qquad b_n \begin{cases} \to 2 & n \text{ even} \\ \to 0 & n \text{ odds} \end{cases}$$

both 0 and 2 are limit points but not limits.

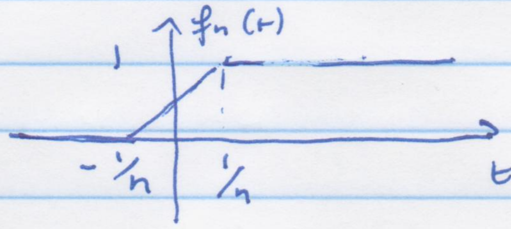
<u>Definition</u>: A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for any $\varepsilon > 0$ there is an $N > 0$ (may depend on ε) such that $d(x_m, x_n) < \varepsilon$ for $m, n > N$

$\{x_n\}$ converges $\Longrightarrow$ $\{x_n\}$ is an ~~the~~ Cauchy sequence. but a Cauchy sequence does not ~~~~ ~~~~ necessarily converge.

Example

Let $C_{[-1,1]}$ be the set of continuous functions defined in the interval $[-1, 1]$ and consider the sequence of functions defined by

$$f_n(t) = \begin{cases} 0 & t < -\frac{1}{n} \\ \frac{nt}{2} + \frac{1}{2} & -\frac{1}{n} \le t \le \frac{1}{n} \\ 1 & t > \frac{1}{n} \end{cases}$$



$(C_{[-1,1]}, d_2)$

$$d_2(f,g) = \int_{-1}^{1} (f(t) - g(t))^2 \, dt$$

$$d_2(f_n, f_m) = \frac{1}{6m^3 n}\left(m^3 + 4m^2 n + mn^2 + 2n^3\right) \quad m > n$$

$$= \frac{1}{6}\left(\frac{1}{n} + \frac{4}{m} + \frac{n}{m^2} + \frac{2n^2}{m^3}\right)$$

n, m large $\quad d_2(f_n, f_m) = 0$

It is a Cauchy sequence.

$$\lim f_n = \frac{1}{}$$

$$f(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0 \end{cases}$$

is a discontinuous function $\notin C[-1,1]$

~~does~~ not converge in $C[-1,1]$

The ~~factor~~ failure of a Cauchy sequence to converge is a deficiency - a "hole" in the underlying metric space.

<u>Definition</u>: A metric space (X, d) is complete if every Cauchy sequence in X is convergent in X

Cauchy seq $\Longleftrightarrow$ convergent sequence

when the space is complete