Communication systems: EEL 3514

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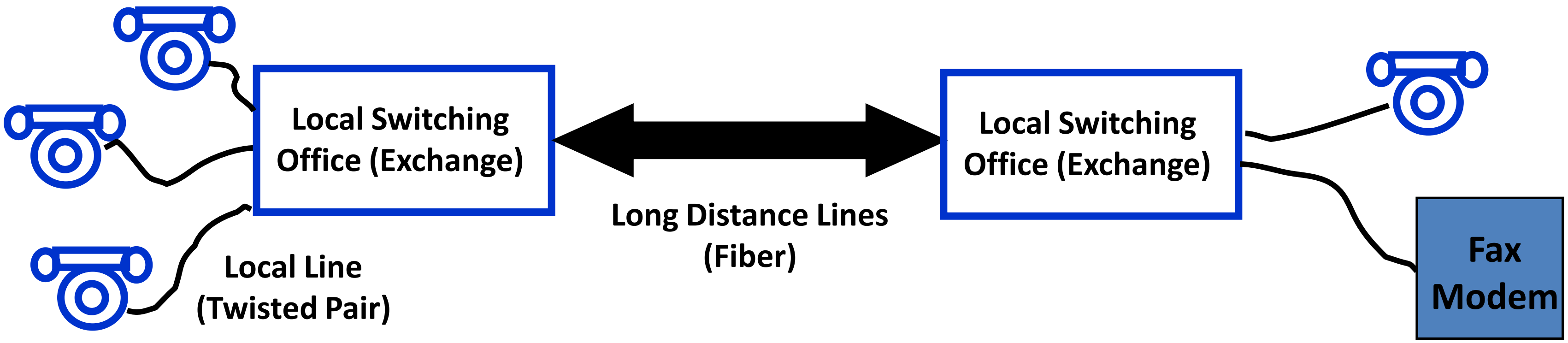
Communication Systems

- Provide for electronic exchange of multimedia data
  - Voice, data, video, music, email, web pages, etc.
- Communication Systems Today
  - Radio and TV broadcasting (will not be discussed)
  - Public Switched Telephone Network (voice,fax,modem)
  - Cellular Phones
  - Computer networks (LANs, WANs, and the Internet)
  - Satellite systems (pagers, voice/data, movie broadcasts)
  - Bluetooth

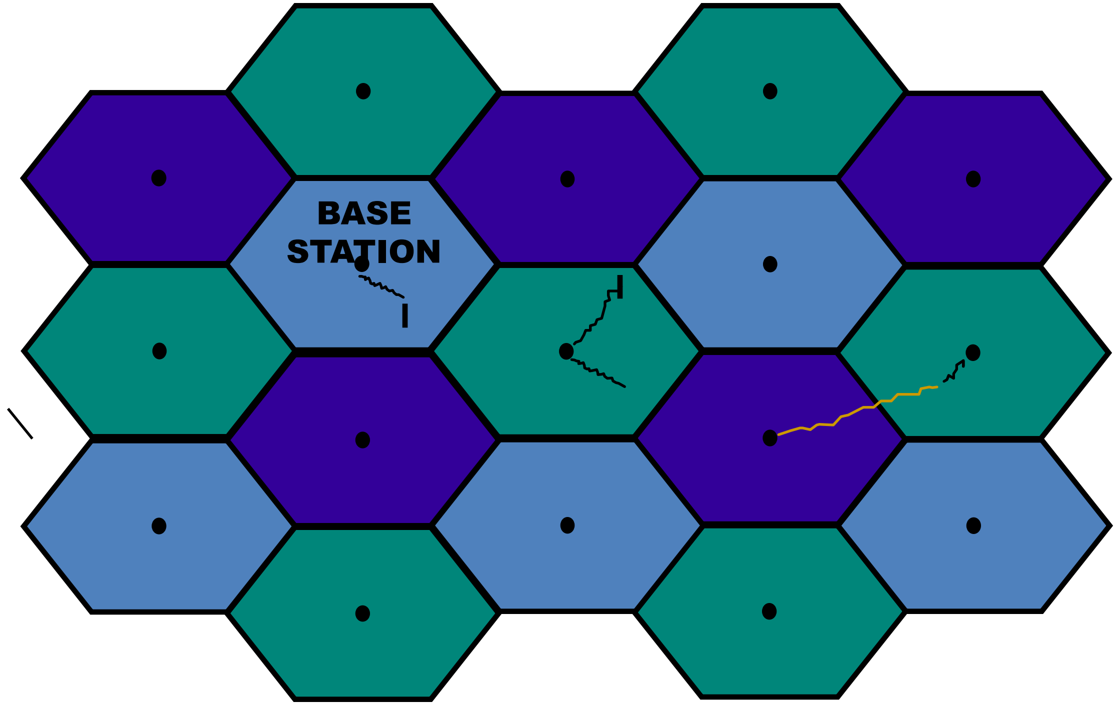
PSTN Design



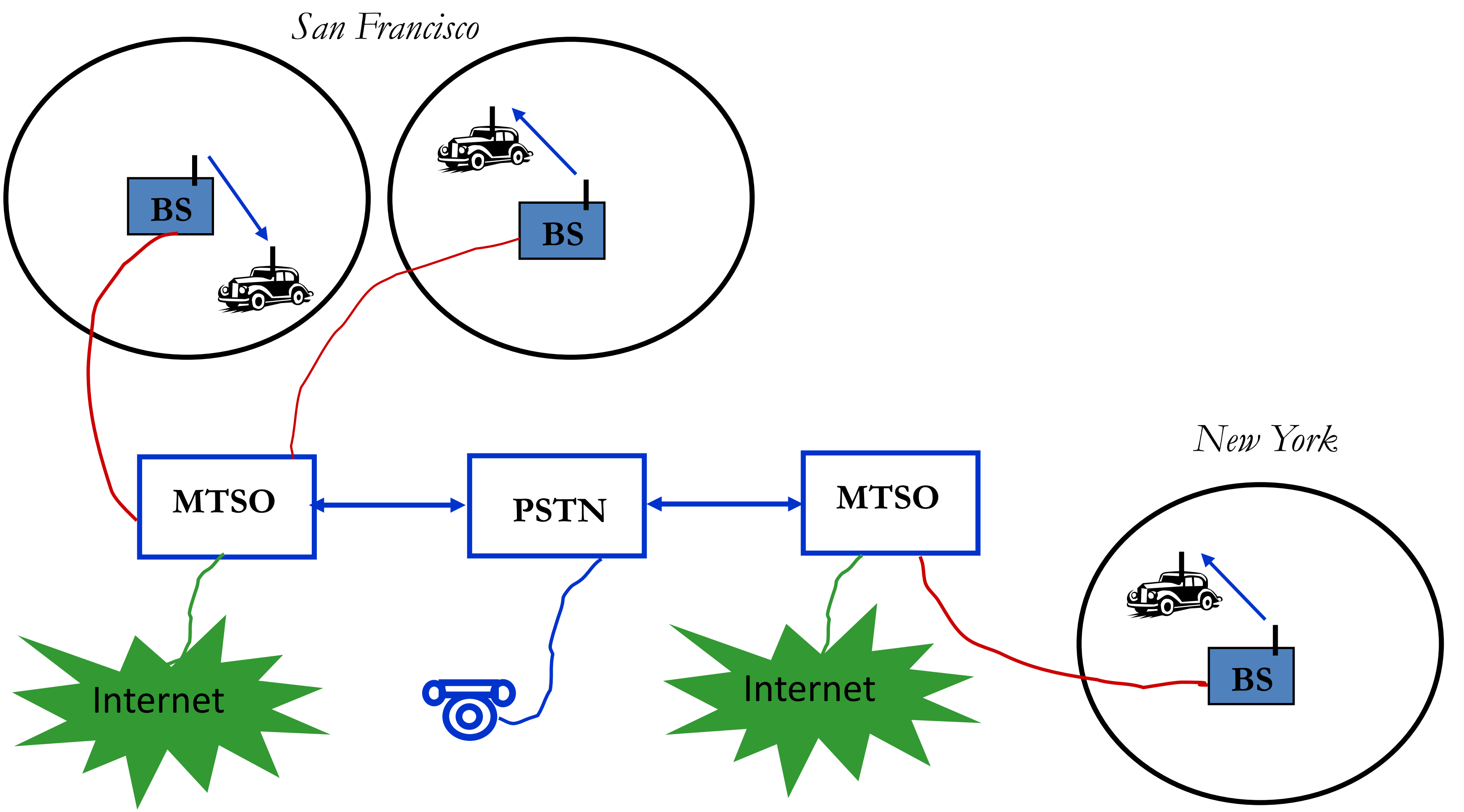
- Local exchange
  - Handles local calls
  - Routes long distance calls over high-speed lines
- Circuit switched network tailored for voice
- Faxes and modems modulate data for voice channel
- DSL uses advanced modulation to get 1.5 Mbps

Cellular System Basics

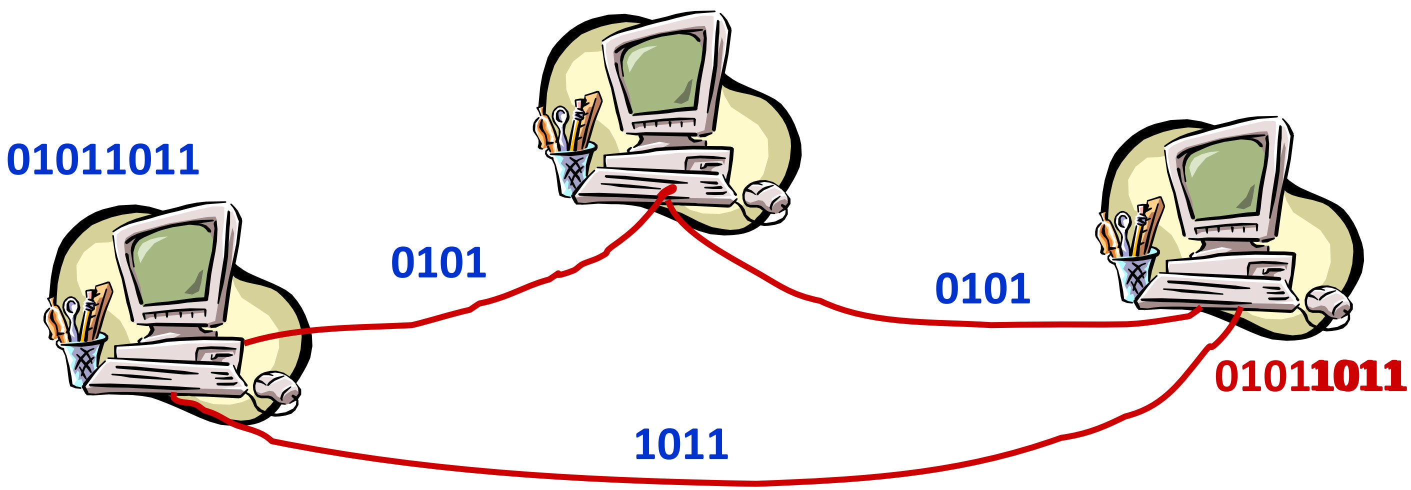
- Geographic region divided into cells
- Frequencies/timeslots/codes reused at spatially-separated locations (analog systems use FD, digital use TD or CD)
- Co-channel interference between same color cells.
- Handoff and control coordinated through cell base stations



Cell Phone Backbone Network

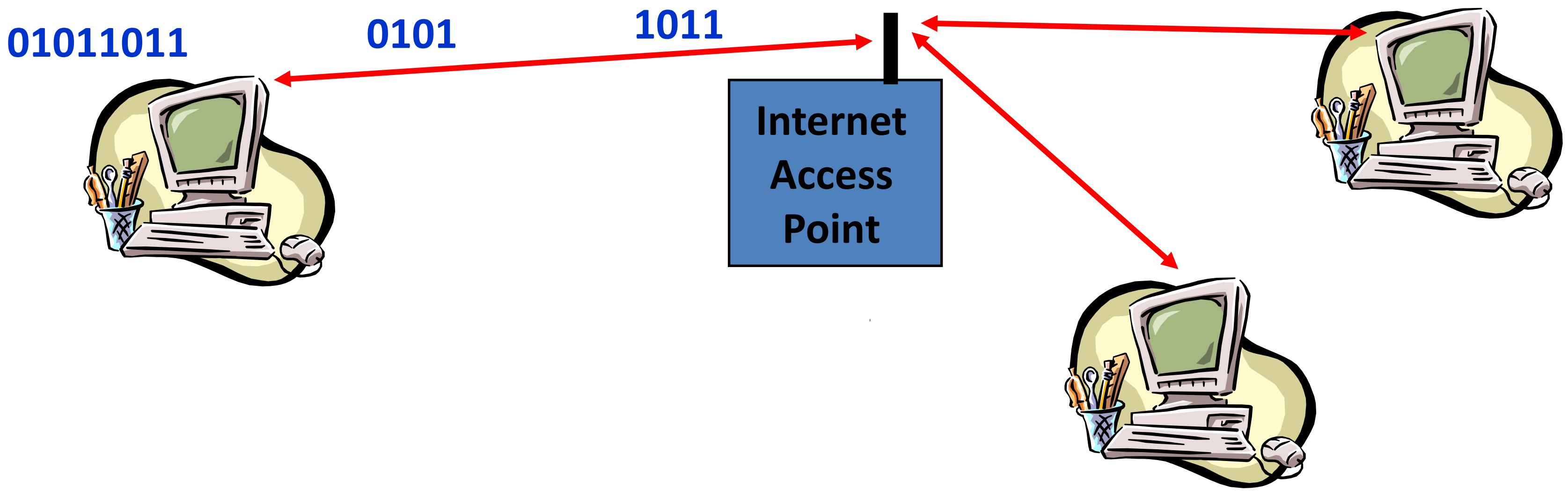


Local Area Networks (LANs)



- LANs connect “local” computers
- Breaks data into packets
- Packet switching (no dedicated channels)
- Proprietary protocols (access,routing, etc.)

Wireless Local Area Networks (WLANs)



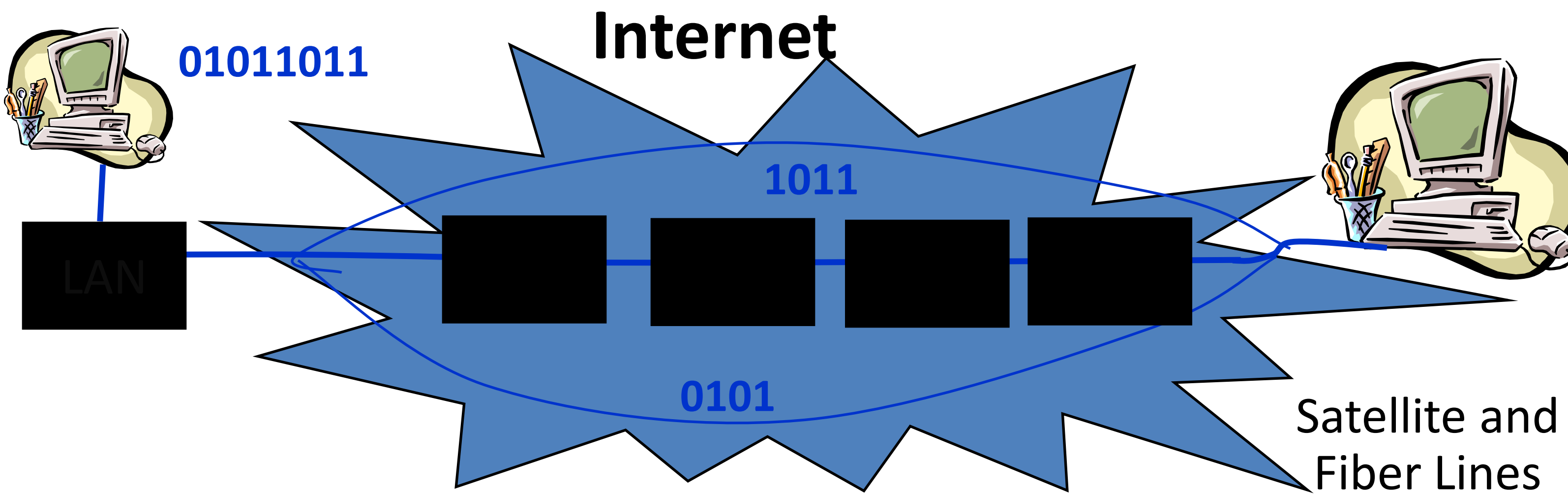
- WLANs connect "local" computers (100m range)
- Breaks data into packets
- Channel access is shared (random access)
- Backbone Internet provides best-effort service

Wireless LAN Standards

- 802.11b (Old – 1990s)
  - Standard for 2.4GHz ISM band (80 MHz)
  - Direct sequence spread spectrum (DSSS)
  - Speeds of 11 Mbps, approx. 500 ft range
- 802.11a/g (Middle Age– mid-late 1990s)
  - Standard for 5GHz NII band (300 MHz)
  - OFDM in 20 MHz with adaptive rate/codes
  - Speeds of 54 Mbps, approx. 100-200 ft range
- 802.11n (Hot stuff, standard completed in 2009)
  - Standard in 2.4 GHz and 5 GHzband
  - Adaptive OFDM /MIMO in 20/40 MHz (2-4 antennas)
  - Speeds up to 600Mbps, approx. 200 ft range
  - Other advances in packetization, antenna use, etc.

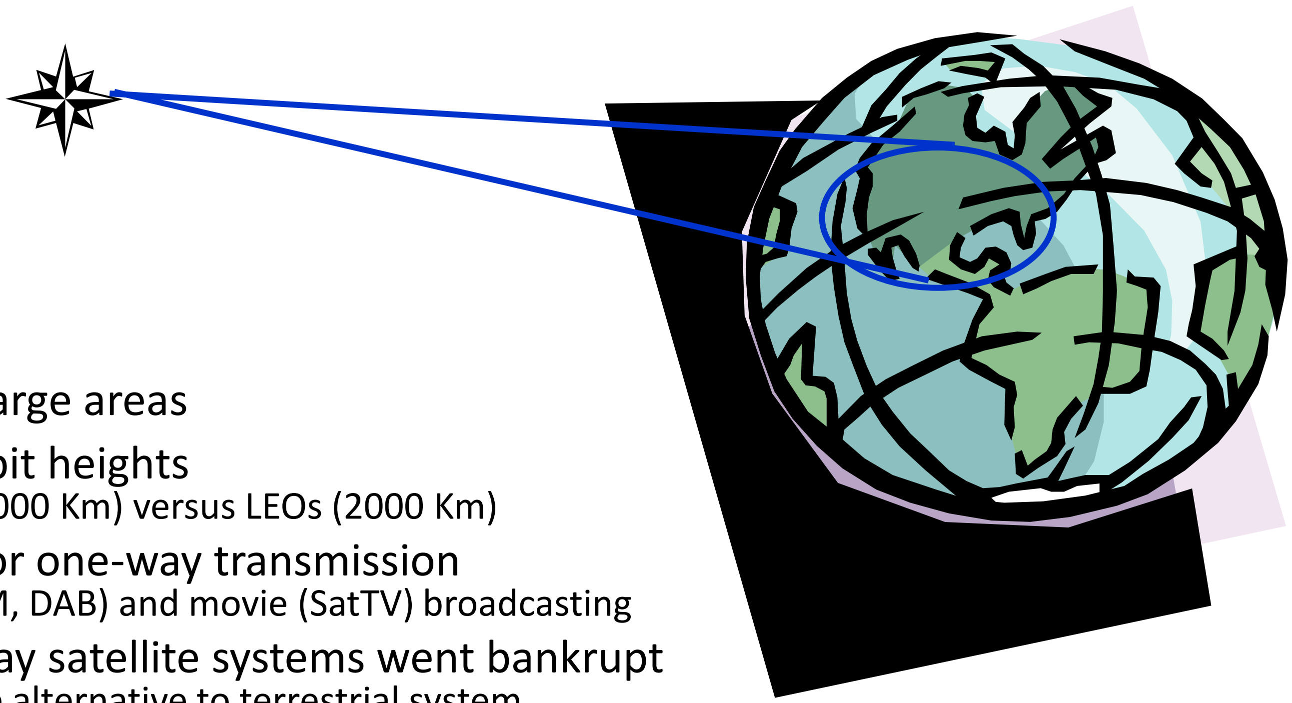
Many WLAN cards have all 4 (a/b/g/n)

Wide Area Networks: The Internet



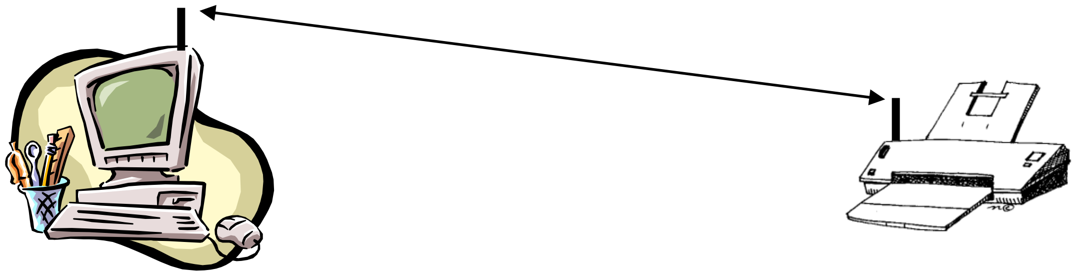
- Many LANs and MANs bridged together
- Universal protocol: TCP/IP (packet based).
- Guaranteed rates or delays cannot be provided.
- Hard to support user mobility.
- Highly scalable and flexible topology
- Much work in “reinventing” Internet for current uses

Satellite Systems



- Cover very large areas
- Different orbit heights
  - GEOs (39000 Km) versus LEOs (2000 Km)
- Optimized for one-way transmission
  - Radio (XM, DAB) and movie (SatTV) broadcasting
- Most two-way satellite systems went bankrupt
  - Expensive alternative to terrestrial system
  - Niche applications (airplane Wifi; paging; etc.)

Bluetooth



- Cable replacement for electronic devices
  - Cell phones, laptops, PDAs, etc.
- Short range connection (10-100 m)
- 1 data (721 Kbps) and 3 voice (56 Kbps) channels
- Rudimentary networking capabilities

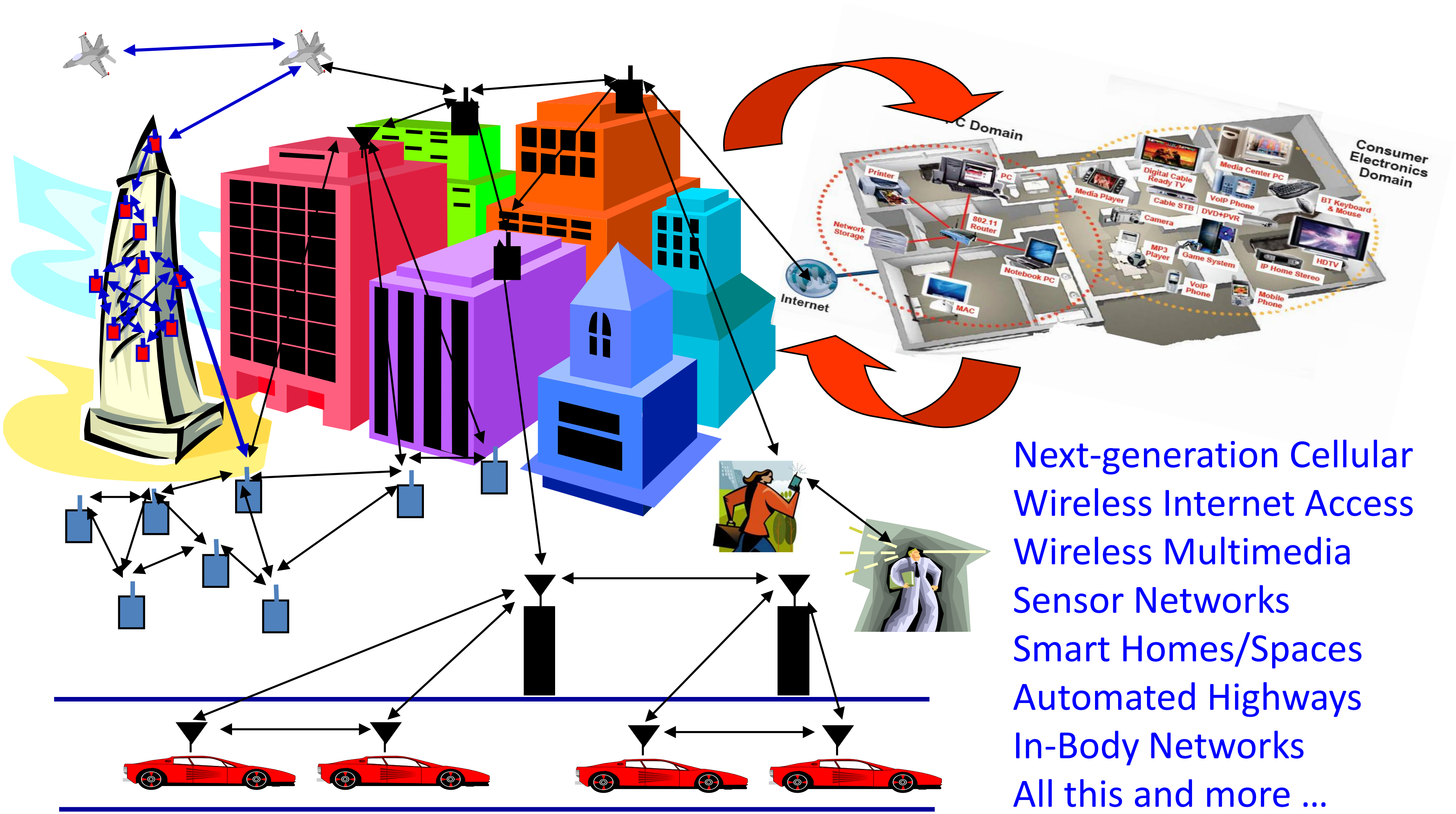
IEEE 802.15.4 / ZigBee Radios

- Low-Rate WPAN
- Data rates of 20, 40, 250 Kbps
- Support for large mesh networking or star clusters
- Support for low latency devices
- CSMA-CA channel access
- Very low power consumption
- Frequency of operation in ISM bands

Focus is primarily on low power RFID and sensor networks

Future Wireless Networks

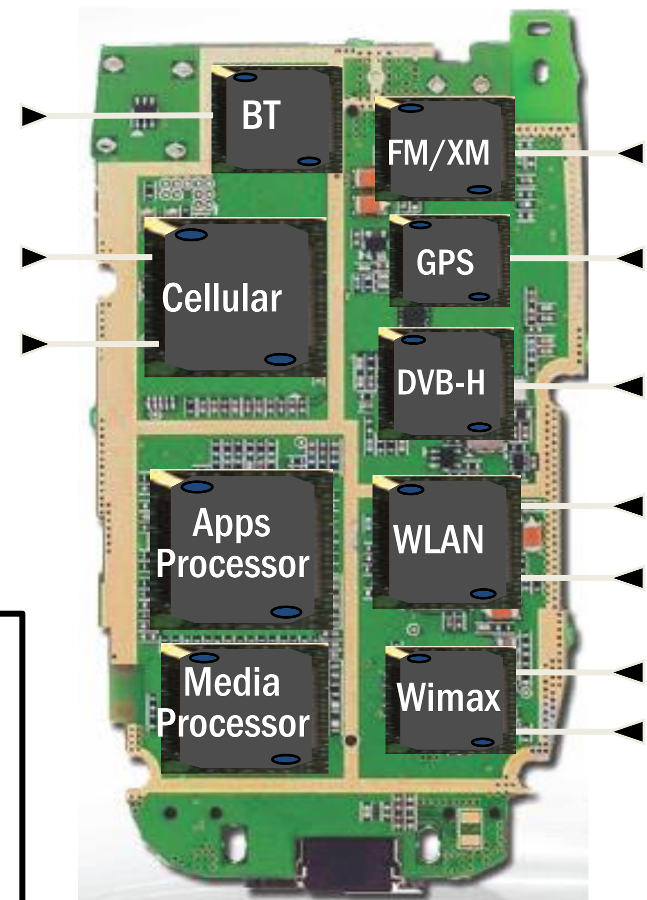
Ubiquitous Communication Among People and Devices



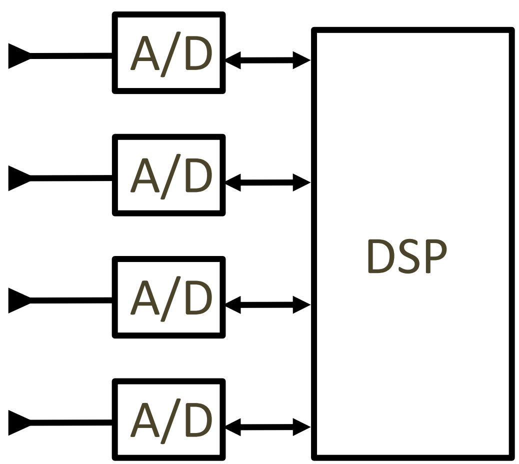
Next-generation Cellular
Wireless Internet Access
Wireless Multimedia
Sensor Networks
Smart Homes/Spaces
Automated Highways
In-Body Networks
All this and more …

Device Challenges

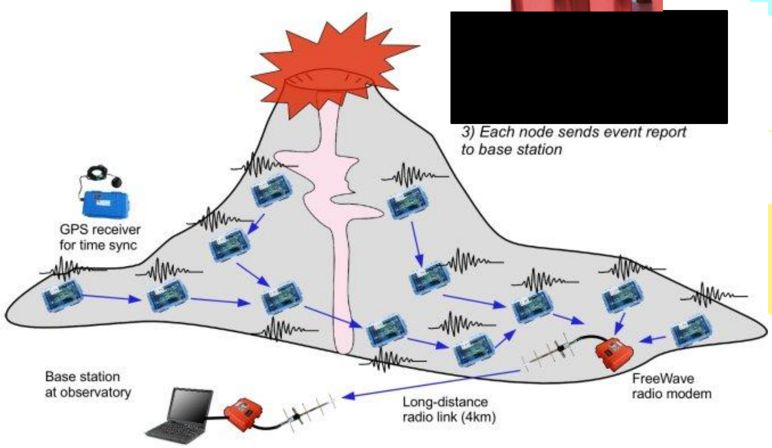
- Analog and RF Components
- A/D Converters
- Size, Power, Cost
- Multiple Antennas
- Multiradio Coexistence

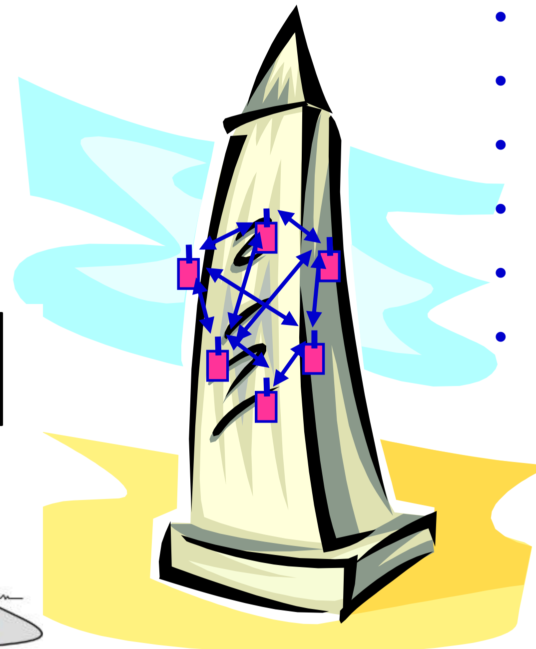


These challenges may someday be solved by a software-defined radio

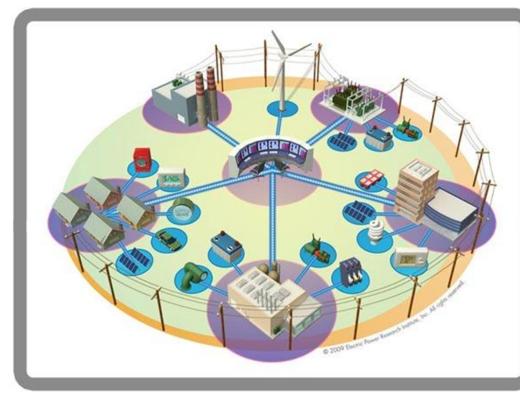


Wireless Sensor Networks



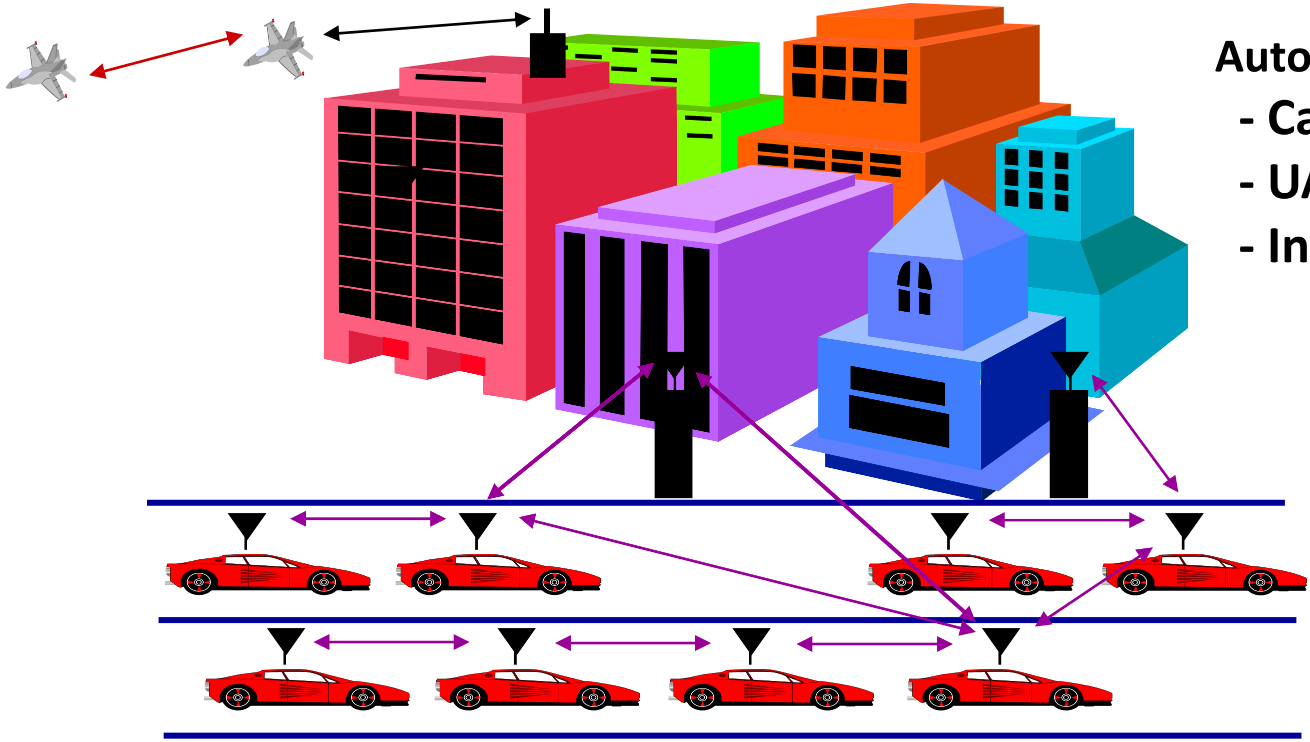


- Smart homes/buildings
- Smart grid
- Search and rescue
- Homeland security
- Event detection
- Surveillance



- Energy (transmit and processing) is the driving constraint
- Data flows to centralized location (joint compression)
- Low per-node rates but tens to thousands of nodes
- Intelligence is in the network rather than in the devices

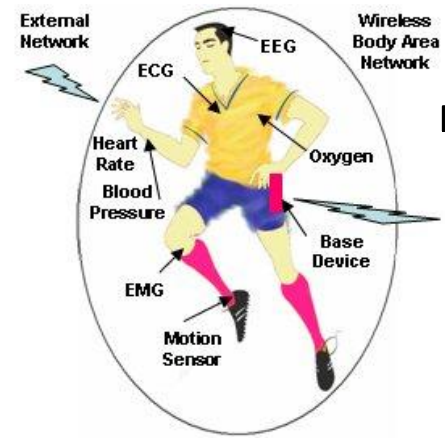
Distributed Control over Wireless Links



Automated Vehicles
- **Cars**
- **UAVs**
- **Insect flyers**

- **Different design principles**
  - **Control requires fast, accurate, and reliable feedback.**
  - **Networks introduce delay and loss for a given rate.**
- **Controllers must be robust and adaptive to random delay/loss.**
- **Networks must be designed with control as the design objective.**

Comm in Health, Biomedicine and Neuroscience

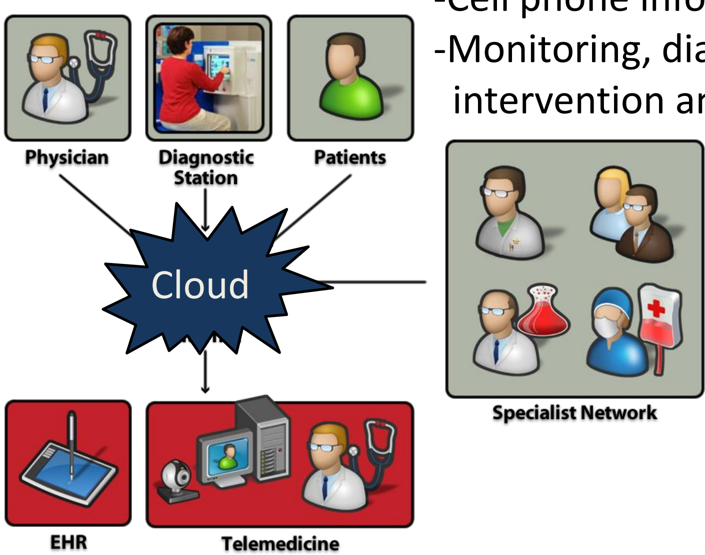


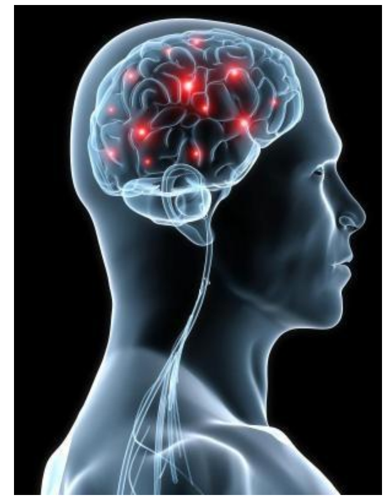
Body-Area Networks

Doctor-on-a-chip

-Cell phone info repository

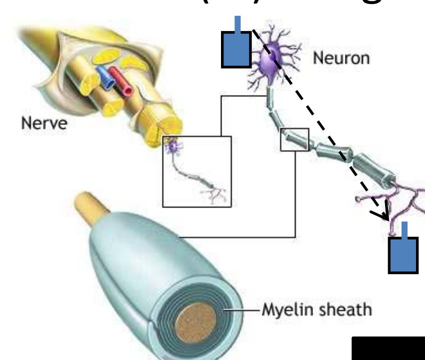
-Monitoring, diagnosis, intervention and services





The brain as a wireless network

- EKG signal reception/modeling
- Signal encoding and decoding
- Nerve network (re)configuration



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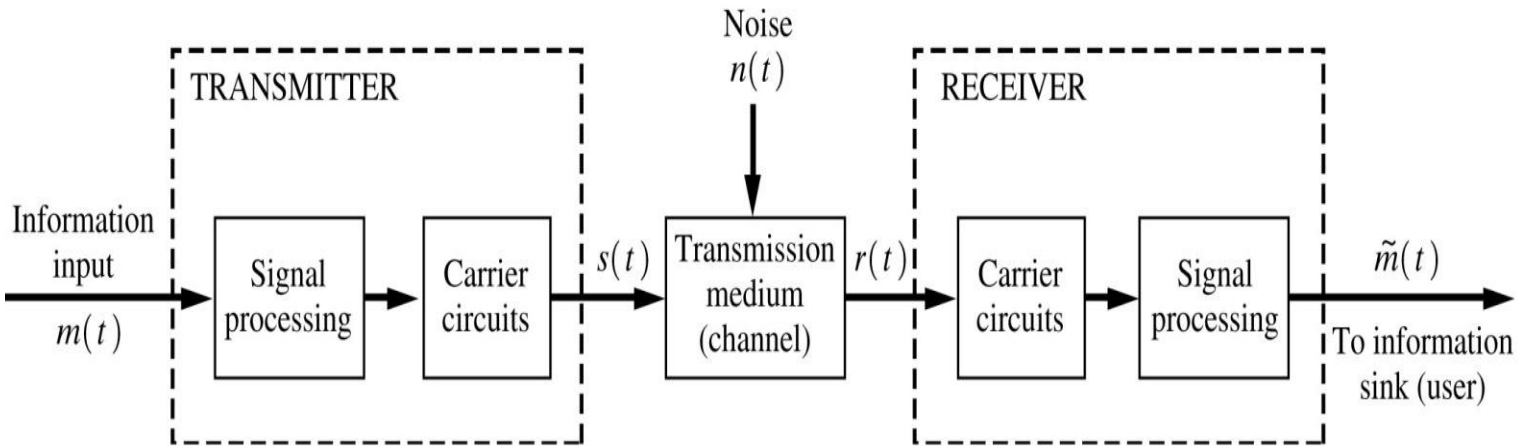
Design Challenges

- Hardware Design
  - Precise components
  - Small, lightweight, low power
  - Cheap
  - High frequency operation
- System Design
  - Converting and transferring information
  - High data rates
  - Robust to noise and interference
  - Supports many users
- Network Design
  - Connectivity and high speed
  - Energy and delay constraints

Figure 1–1 Communication system.



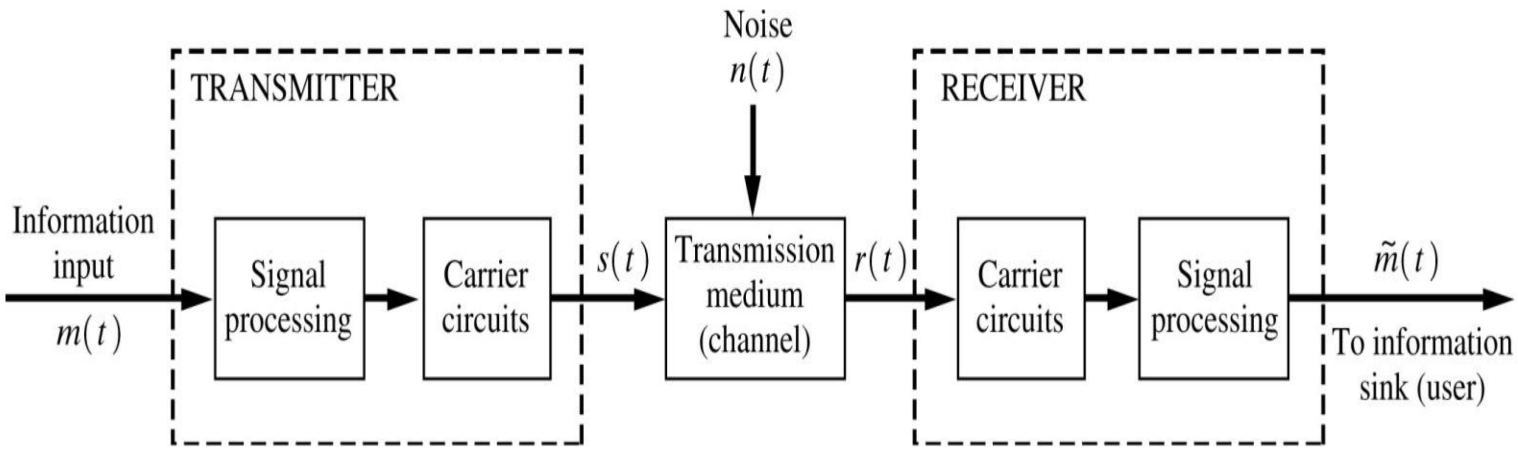
Main Points

- Communication systems send information electronically over communication channels
- Many different types of systems which convey many different types of information
- Design challenges include hardware, system, and network issues
- Communication systems recreate transmitted information at receiver with high fidelity
- Focus of this class is design and performance of analog and digital communication systems

Information Representation

- Communication systems convert information into a format appropriate for the transmission medium.
  - Channels convey electromagnetic waves (signals).
- Analog communication systems convert (modulate) analog signals into modulated (analog) signals
- Digital communication systems covert information in the form of bits into binary/digital signals
- Types of Information:
  - Analog Signals: Voice, Music, Temperature readings
  - Analog signals or bits: Video, Images
  - Bits: Text, Computer Data
  - Analog signals can be converted into bits by quantizing/digitizing

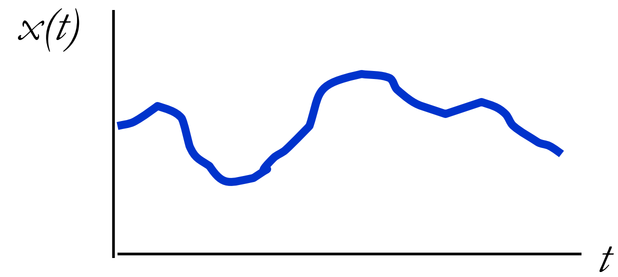
Communication System Block Diagram



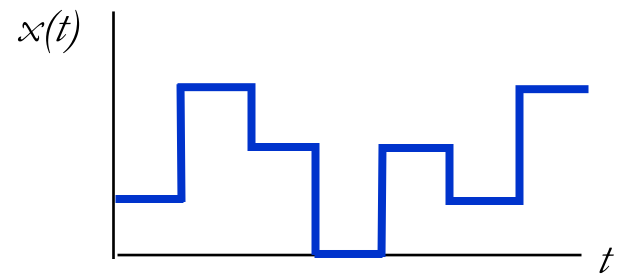
- Source encoder converts message into message signal or bits.
- Transmitter converts message signal or bits into format appropriate for channel transmission (analog/digital signal).
- Channel introduces distortion, noise, and interference.
- Receiver decodes received signal back to message signal.
- Source decoder decodes message signal back into original message.

Analog vs. Digital Systems

- Analog signals
  - Value varies continuously

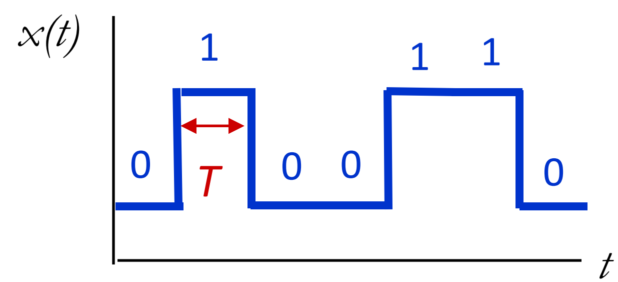


- Digital signals
  - Value limited to a finite set



Digital systems more robust

- Binary signals
  - Has at most 2 values
  - Used to represent bit values
  - Bit time *T* needed to send 1 bit
  - Data rate *R=1/T* bits per second



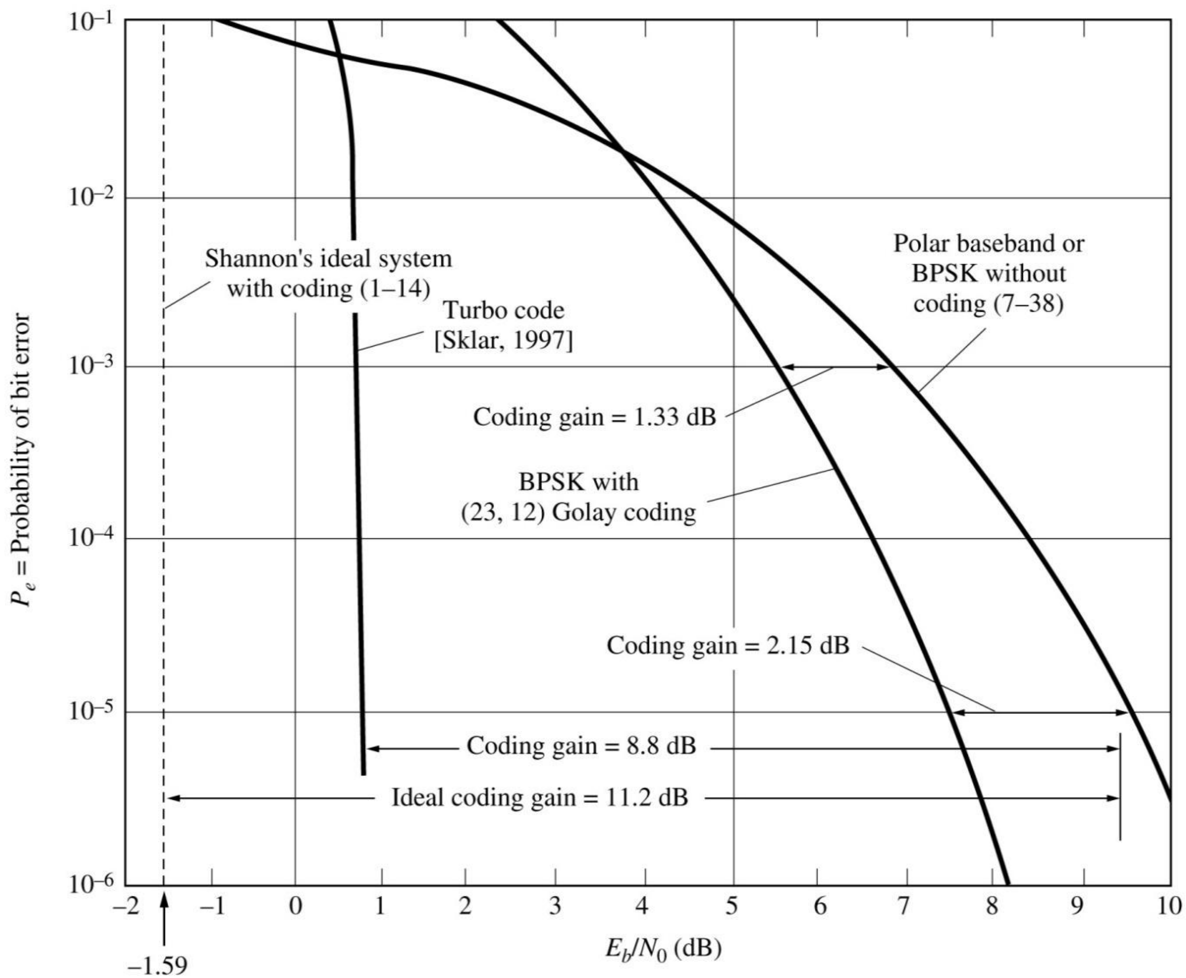
Performance Metrics

- Analog Communication Systems
  - Metric is fidelity
  - Want $m(t) \approx \hat{m}(t)$
- Digital Communication Systems
  - Metrics are data rate (R bps) and probability of bit error ($P_b = p(b \neq \hat{b})$)
  - Without noise, never make bit errors
  - With noise, P_b depends on signal and noise power, data rate, and channel characteristics.

Data Rate Limits

- Data rate R limited by signal power, noise power, distortion, and bit error probability
- Without distortion or noise, can have infinite data rate with $P_b=0$.
- Shannon capacity defines maximum possible data rate for systems with noise and distortion
  - Rate achieved with bit error probability close to zero
  - In white Gaussian noise channels, $C=B\log(1+\text{SNR})$
  - Does not show how to design real systems
- Shannon obtained C=32 Kbps for phone channels
  - Get higher rates with modems/DSL (use more BW)
  - Nowhere near capacity in wireless systems

Figure 1–8 Performance of digital systems—with and without coding.



Main Points

- Communication systems modulate analog signals or bits for transmission over channel.
- The building blocks of a communication system convert information into an electronic format for transmission, then convert it back to its original format after reception.
- Goal of transmitter (modulator) and receiver (demodulator) is to mitigate distortion/noise from the channel.
- Digital systems are more robust to noise and interference.
- Performance metric for analog systems is fidelity, for digital it is rate and error probability.
- Data rates over channels with noise have a fundamental capacity limit.

Fourier Representation of Signals and Systems

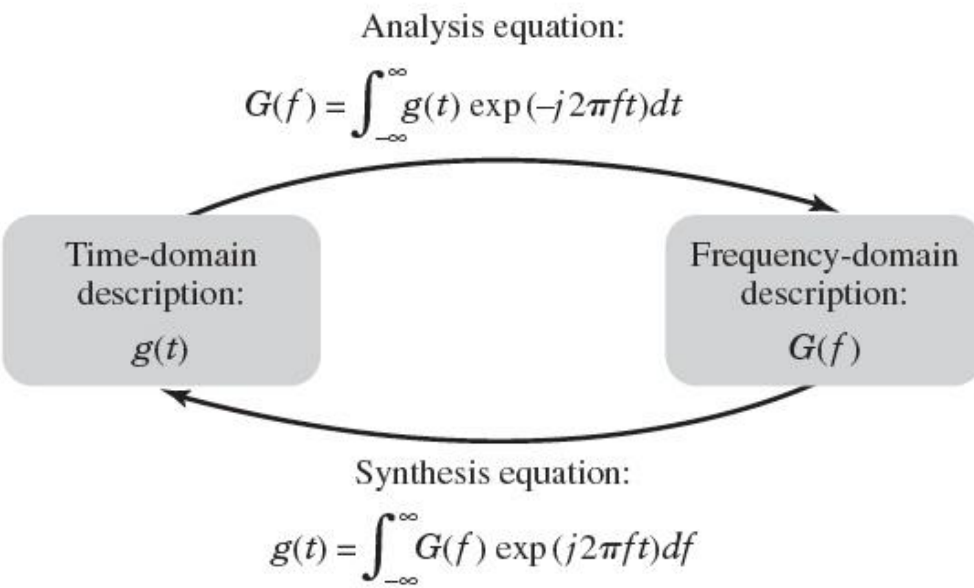


FIGURE 2.1 Sketch of the interplay between the synthesis and analysis equations embodied in Fourier transformation.

02_01

Signal Energy and Power

- The energy in a signal *g(t)* is

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} g^2(t) dt \quad \text{for } g(t) \text{ real}$$

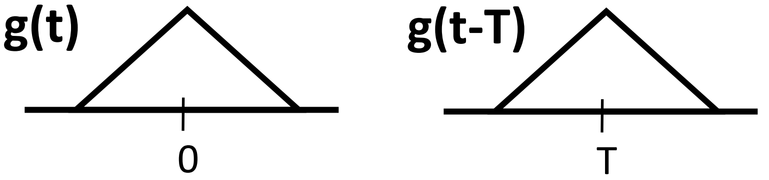
- The power in a signal *g(t)* is

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)^2 dt \quad \text{for } g(t) \text{ real}$$

- Power is often expression in dBw or dBm
  - [10 $\log_{10}$ P] dBW is dB power relative to Watts
  - [10 $\log_{10}$ (P/.001)] dBm is dB power relative to mWatts
  - Signal power/energy determines its resistance to noise

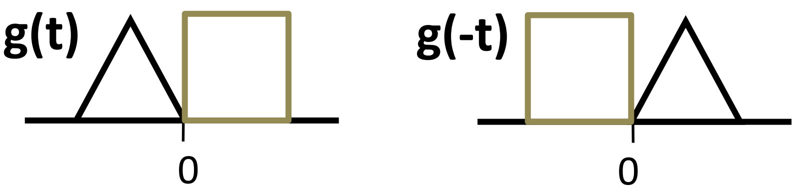
Signal Operations

Time-shifting: g(t-T):



- Time-scaling: g(t/a)

- Time-inversion: g(-t)



- Correlation of two signals x(t) and g(t)

$$\rho = \frac{1}{\sqrt{E_g E_x}} \int_{-\infty}^{\infty} g(t)x^*(t)dt$$

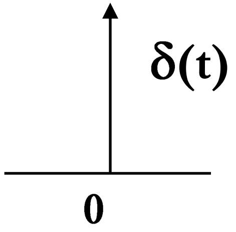
g(t) **and** ***x(t)*** **are orthogonal if** ***ρ=0.***

Unit Impulse Response $\delta(t)$ (Dirac Delta Function)

- Defined by two properties

$$1.\quad \delta(t) = 0$$

$$2.\quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



  - Also limit of unit area pulse with vanishing width

- Properties:

$$1.\quad \phi(t)\delta(t) = \phi(0)\delta(t)$$

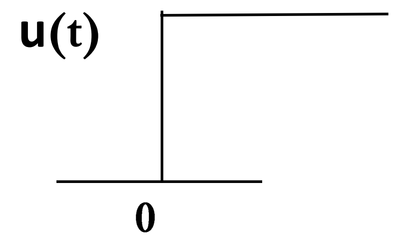
$$2.\quad \phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

$$3.\quad \int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)$$

Unit Step Function *u(t)*

- Defined as

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



- Properties

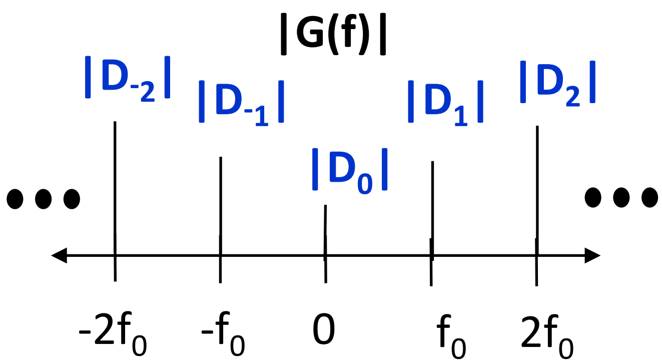
$$1. \quad \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

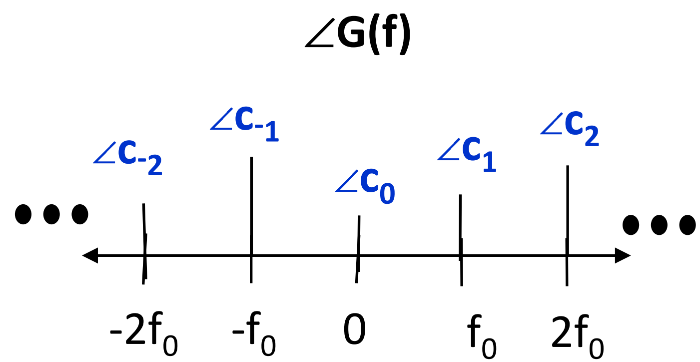
$$2. \quad \frac{du}{dt} = \delta(t)$$

Fourier Series Transform Pair

Let *g(t)* be a periodic signal with period $T_0=1/f_0=1/(2\pi\omega_0)$

$$g(t)=\sum_{n=-\infty}^{\infty} D_n e^{j2\pi n f_0 t} \qquad D_n=\frac{1}{T_0}\int_{T_0} g(t)e^{-j2\pi n f_0 t}\,dt$$

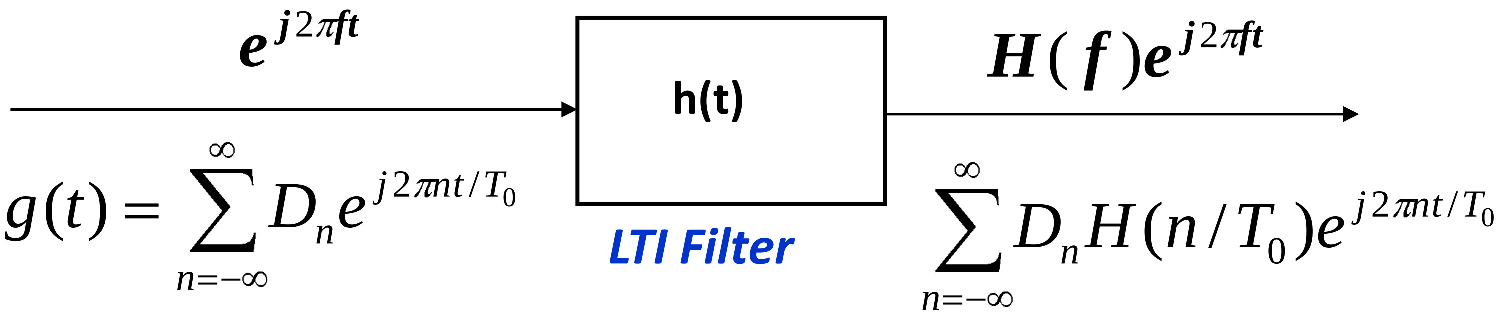




$$g(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(2\pi n f_0 t)+b_n\sin(2\pi n f_0 t)\right)$$

$$a_n=\frac{2}{T_0}\int_{T_0} g(t)\cos(2\pi n f_0 t)\,dt \qquad b_n=\frac{2}{T_0}\int_{T_0} g(t)\sin(2\pi n f_0 t)\,dt$$

Filtering and Power of Periodic Signals



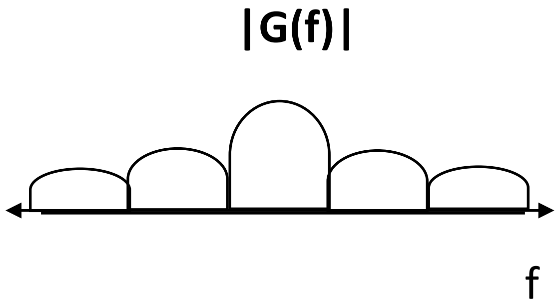
- Exponentials are filter eigenfunctions
  - An exponential input yields a scaled output at the frequency of the exponential
  - By linearity, we can use this property to determine the output of a filter to a periodic input
  - This can be used to derive the convolution operation associated with filtering
- Parseval's Relation:

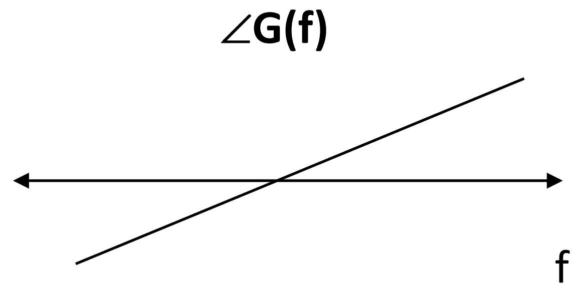
$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Fourier Transform Pair

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi n f t} dt$$

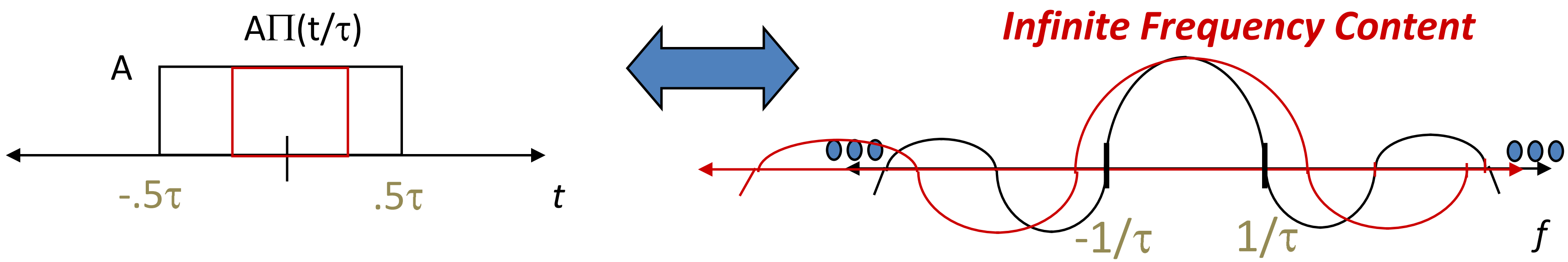
$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$





Real signals have |G(f)|=|G(-f)| and ∢G(f)=-<G(-f)

Rectangular Pulse Example



$$g(t) = A\Pi(t/\tau) \Leftrightarrow G(f) = A\tau\,\mathrm{sinc}(f\tau)$$

- Rectangular pulse is a time window
- Shrinking time axis causes stretching of frequency axis
- Signals cannot be both time-limited and bandwidth-limited
- Bandwidth of rectangular pulse defined as first null

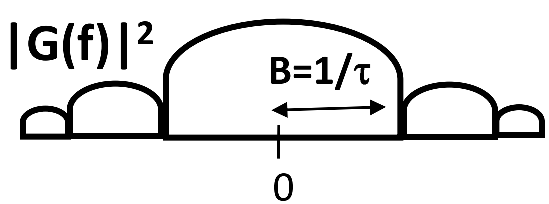
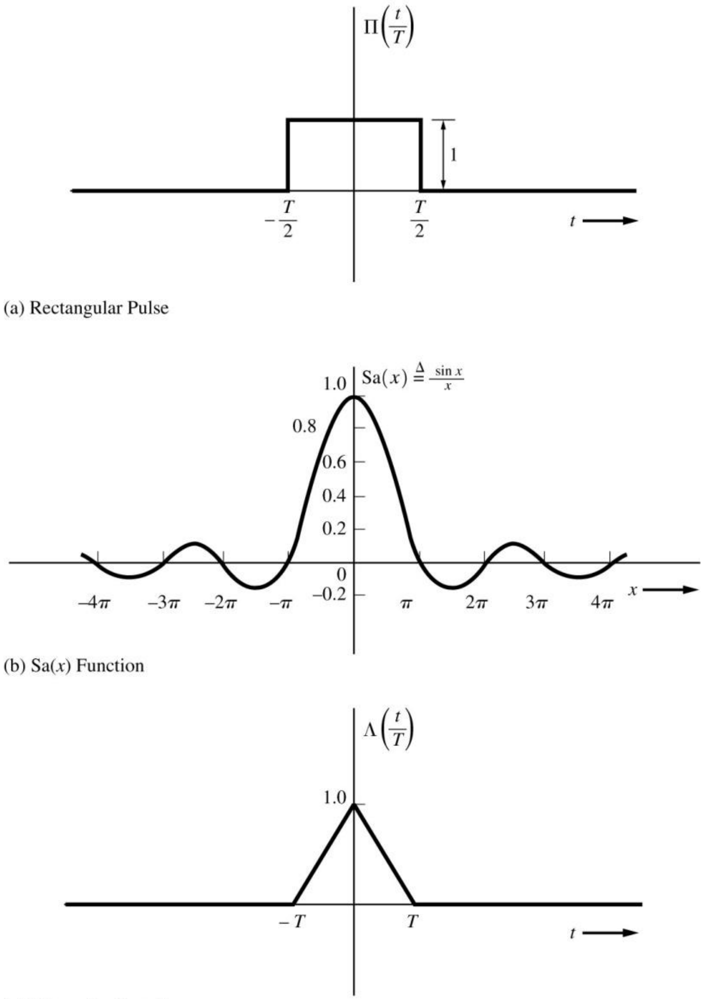


Figure 2–5 Waveshapes and corresponding symbolic notation.



(a) Rectangular Pulse

(b) Sa(x) Function

(c) Triangular Function

Figure 2–6 Spectra of rectangular, (sin *x*)/*x*, and triangular pulses.

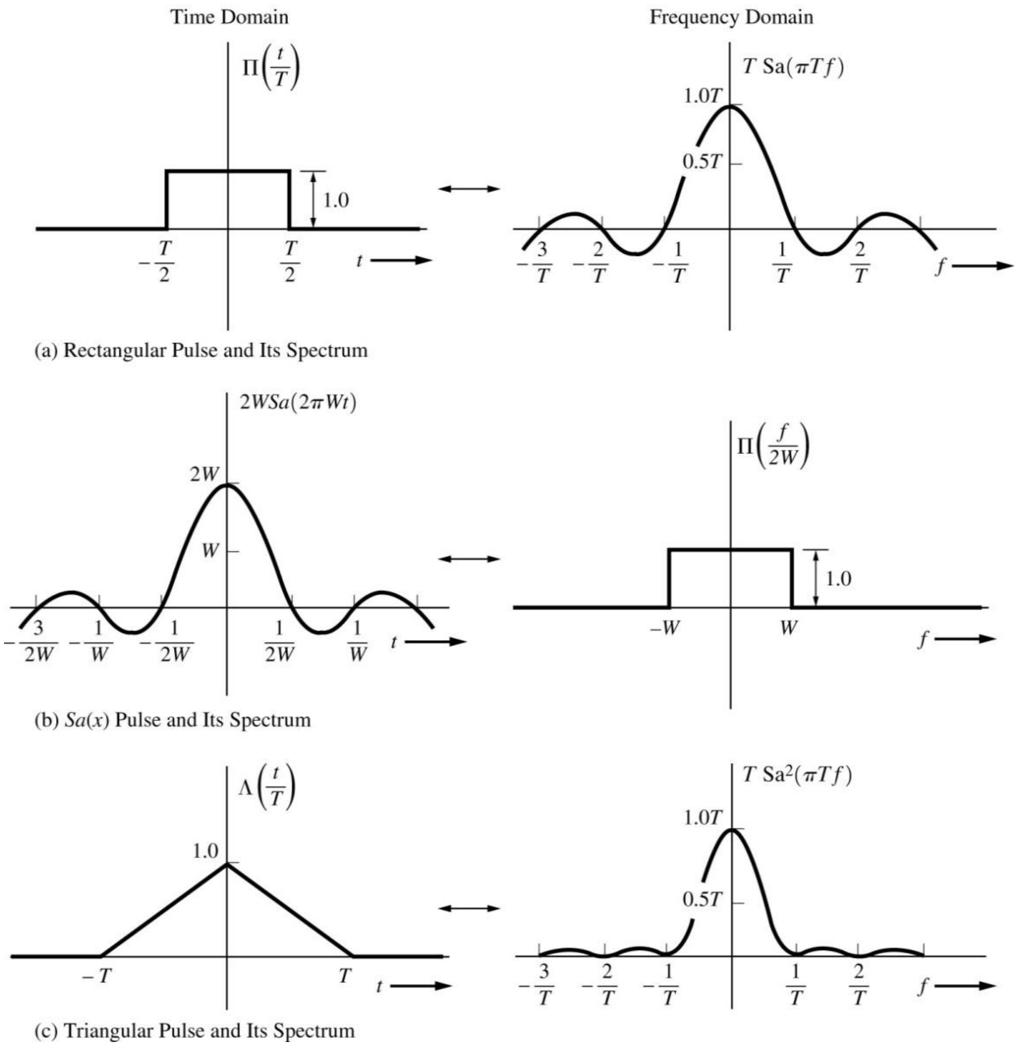
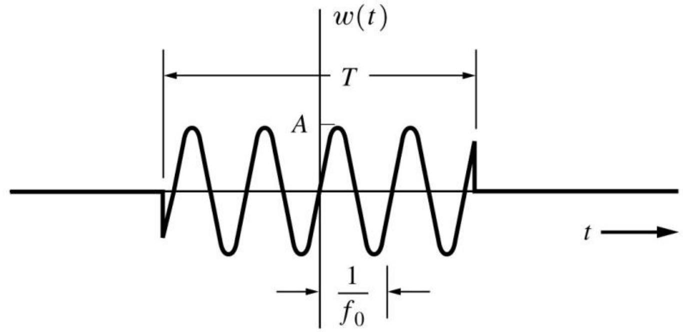
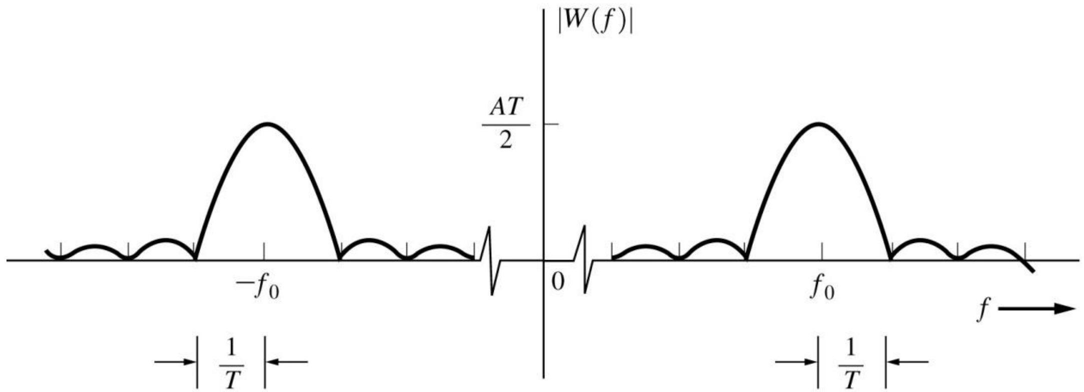


Figure 2–8 Waveform and spectrum of a switched sinusoid.

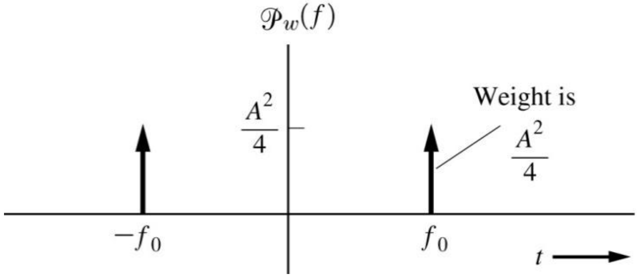


(a) Time Domain



(b) Frequency Domain (Magnitude Spectrum)

Figure 2–9 Power spectrum of a sinusoid.



Main Points

- Data rates over channels with noise have a fundamental capacity limit.
- Signal energy and power determine resistance to noise
- Communication system shift, scale, and invert signals
- Unit impulse and step functions important for analysis
- Fourier series represents periodic signals in terms of exponential or sinusoidal basis functions
- Exponentials are eigenfunctions of LTI filters
- Fourier transform is the spectral components of a signal
- Rectangle in time is sinc in frequency; Time-limited signals are not bandlimited and vice versa

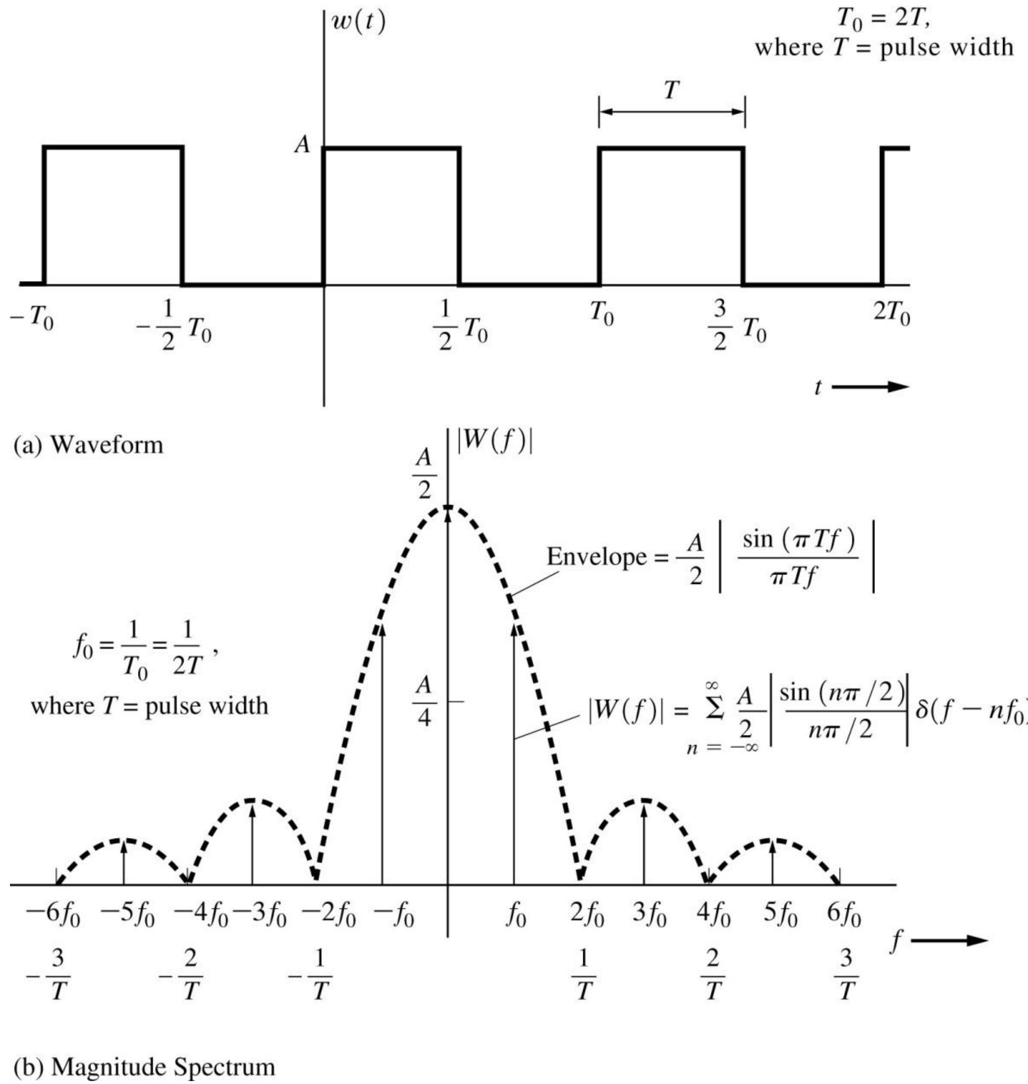
Important Transform Pairs

- $1 \Leftrightarrow \delta(f)$
- $\delta(t) \Leftrightarrow 1$
- $\cos(2\pi f_0 t) \Leftrightarrow .5[\delta(f+f_0)+\delta(f-f_0)]$
- $\sin(2\pi f_0 t) \Leftrightarrow .5j[\delta(f+f_0)-\delta(f-f_0)]$
- $\exp(j2\pi f_0 t) \Leftrightarrow \delta(f-f_0)$
- Rectangular pulse: $\Pi(t/\tau) \Leftrightarrow \tau \mathrm{sinc}(\pi f \tau)$
- Sinc pulse: $2B\mathrm{sinc}(2\pi Bt) \Leftrightarrow \Pi(f/(2B))$

Key Transform Properties

- Duality
  - Operations in time lead to dual operations in frequency
  - Fourier transform pairs are duals of each other
- Time scaling
  - Contracting in time yields expansion in frequency
- Delay
  - Leads to a linear phase shift
- Frequency shifting
  - Multiplying in time by an exponential leads to a frequency shift.
- Convolution and Multiplication
  - Multiplication in time leads to convolution in frequency
  - Convolution in time leads to multiplication in frequency

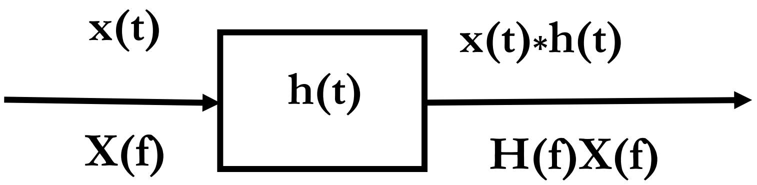
Figure 2–12 Periodic rectangular wave used in Example 2–12.



Filtering

- Filter response to $\delta(t)$ is impulse response

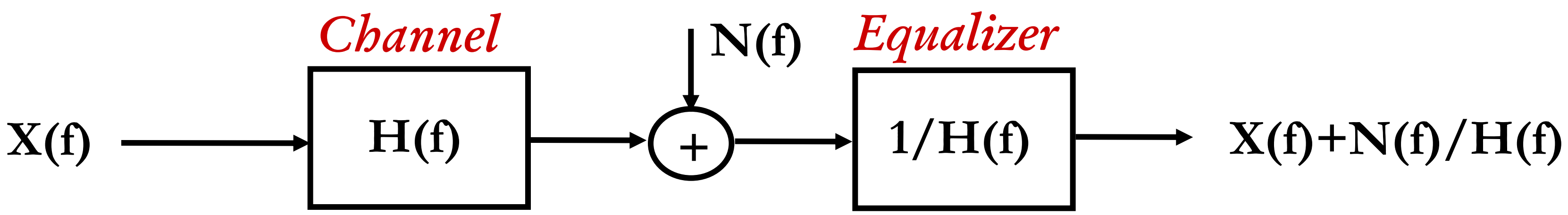
- For any input x(t), filter output is x(t)*h(t)



Much easier to study filtering in the frequency domain

Channel Distortion

- Channels introduce linear distortion
  - Electronic components introduce nonlinear distortion
- Simple equalizers invert channel distortion
  - Can enhance noise power



Main Points

- Fourier transform is spectral components of a signal
- Rectangle in time is sinc in frequency
- Time-limited signals not bandlimited and vice versa
- Duality, time-scaling, time-delay, freq. shifting, multiplication, and convolution are key FT properties
- Easier to study filtering in the frequency domain
- Channels introduce distortion, can be compensated by an equalizer

Ideal Filters

- Low Pass Filter (linear phase)

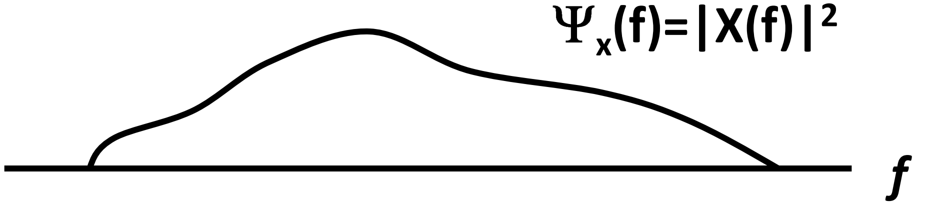
- Band Pass Filter (linear phase)

- Most filtering (and other signal processing) is done digitally (A/D followed by DSP)

Energy Spectral Density (ESD)

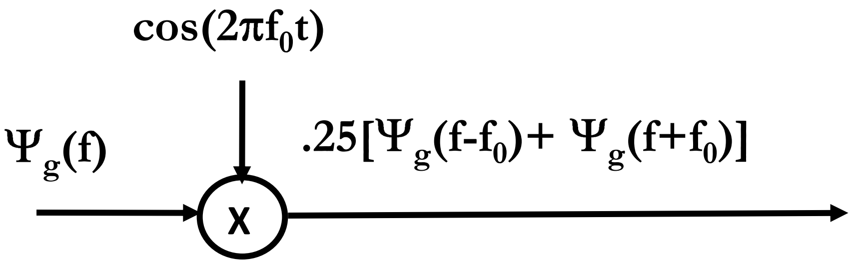
- Signal energy: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df$
- ESD measures signal energy per unit Hz.

$$E_g = \int |g(t)|^2 dt = \int \Psi_x(f) df$$



Contains less information than Fourier Transform (no phase)

- ESD of a modulation signal

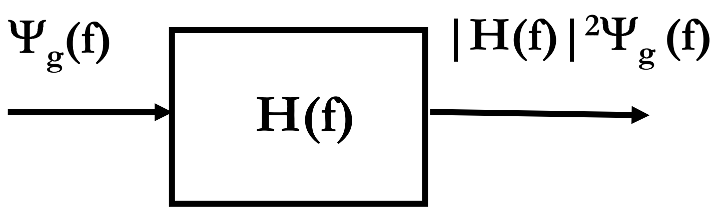


Autocorrelation

- Defined for real signals as $\psi_g(\tau)=g(\tau)*g(-\tau)$
  - Measures signal self-similarity at τ
  - Can be used for synchronization
- ESD and autocorrelation FT pairs: $\psi_g(\tau) \Leftrightarrow \Psi_g(f)$

$$\Psi_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt = g(\tau) * g(-\tau) \Leftrightarrow G(f)G^*(f) = |G(f)|^2 = \Psi_x(f)$$

- Filtering based on ESD



1.4.1 Autocorrelation of an Energy Signal

Correlation is a matching process; *autocorrelation* refers to the matching of a signal with a delayed version of itself. The autocorrelation function of a real-valued energy signal $x(t)$ is defined as

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt \qquad \text{for } -\infty < \tau < \infty \tag{1.21}$$

The autocorrelation function $R_x(\tau)$ provides a measure of how closely the signal matches a copy of itself as the copy is shifted τ units in time. The variable τ plays the role of a scanning or searching parameter. $R_x(\tau)$ is not a function of time; it is only a function of the time difference τ between the waveform and its shifted copy.

The autocorrelative function of a real-valued *energy* signal has the following properties:

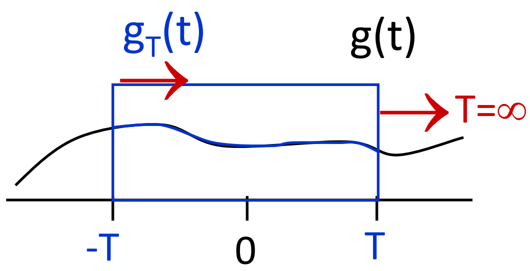
1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $\lvert R_x(\tau)\rvert \le R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow \psi_x(f)$	autocorrelation and ESD form a Fourier transform pair, as designated by the double-headed arrows
4. $R_x(0) = \int_{-\infty}^{\infty} x^2(t)\,dt$	value at the origin is equal to the energy of the signal

Power Spectral Density

- Similar to ESD but for power signals (P=E/t)
- Distribution of signal power over frequency

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 \, dt$$

$$S_g(f) = \lim_{T\to\infty} \frac{1}{2T} \left|G_T(f)\right|^2$$



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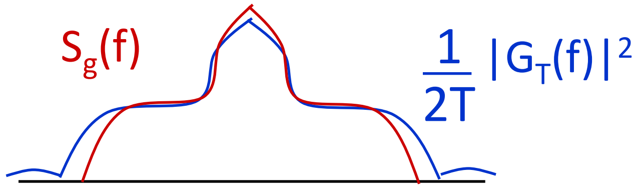


$$P_g = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt = \int_{-\infty}^{\infty} S_g(f) df$$

1.4.2 Autocorrelation of a Periodic (Power) Signal

The autocorrelation function of a real-valued power signal $x(t)$ is defined as

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.22}$$

When the power signal $x(t)$ is periodic with period T_0, the time average in Equation (1.22) may be taken over a *single period* T_0, and the autocorrelation function can be expressed as

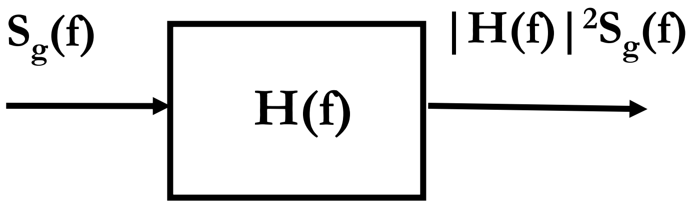
$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)x(t+\tau)\, dt \qquad \text{for } -\infty < \tau < \infty \tag{1.23}$$

The autocorrelation function of a real-valued *periodic* signal has properties similar to those of an energy signal:

1. $R_x(\tau) = R_x(-\tau)$	symmetrical in τ about zero
2. $\lvert R_x(\tau)\rvert \le R_x(0)$ for all τ	maximum value occurs at the origin
3. $R_x(\tau) \leftrightarrow G_x(f)$	autocorrelation and PSD form a Fourier transform pair
4. $R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)\, dt$	value at the origin is equal to the average power of the signal

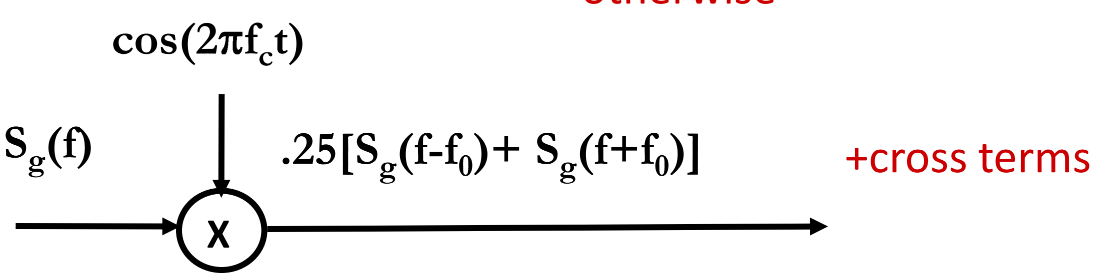
Filtering and Modulation

- Filtering



- Modulation
  - When $S_g(f)$ has bandwidth $B<f_0$ (Sec 7.8.2),

otherwise

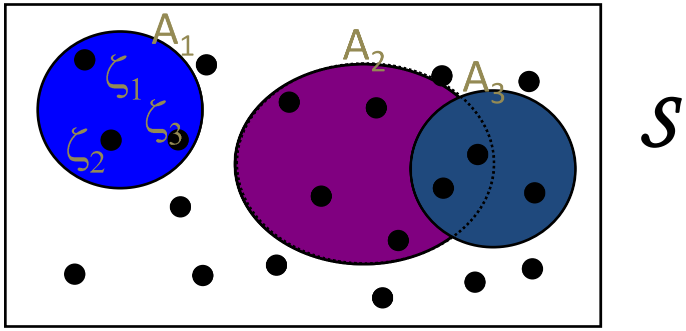


Main Points

- Channels cause distortion, compensated by equalizer
- Ideal filters approximate real implementations
- Most filtering and other SP done digitally
- Energy spectral density measures signal energy distribution across frequency domain
- A signal's autocorrelation and ESD are FT pairs
- Power signals often don't have FTs: characterized with PSD and autocorrelation, which are FT pairs.
- Filtering and modulation can be analyzed using PSD

Probability Theory

- Mathematically characterizes random events.
- Defined on a probability space: $(\mathcal{S}, \{A_i\}, P(\bullet))$
  - Sample space of possible outcomes ζ_i.
- Sample space has a subset of events A_i
- Probability defined for these subsets.



Probability Measure

- $P(S)=1$
- $0 \leq P(A) \leq 1$ for all events A
- If $(A \cap B)=\varnothing$ then $P(A \cup B)=P(A)+P(B)$.
- Conditional Probability:
  - $P(B|A)=P(A \cap B)/P(A)$
  - Bayes Rule: $P(B|A)=P(A|B)P(B)/P(A)$
- Independent Events:
  - A and B are independent if $P(A \cap B)=P(B)P(A)$
  - Independence is a property of $P(\bullet)$
  - For independent events, $P(B|A)=P(B)$.

Main Points

- Modulation of power signals can be analyzed using PSD
- Autocorrelation and PSD are FT pairs
- Random signals analyzed by PSD
- Random events defined on a probability space, with events as subsets and a probability measure.
- Conditional probability characterizes the effect of one event on another.
- Events are independent if their joint probability equals the product of their probabilities.

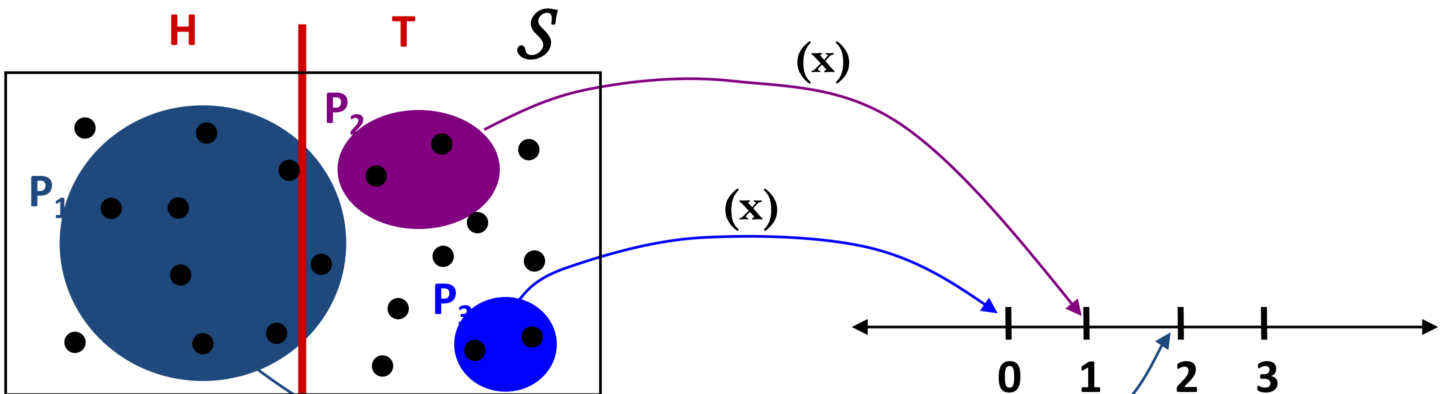
Independence and Bernoulli Trails

- Independent Events:
  - A and B are independent if $P(A \cap B)=P(B)P(A)$
  - Independence is a property of P(•)
  - For independent events, $P(B|A)=P(B)$.
- Bernoulli Trails
  - Probability of *k* successes in *n* trails
  - *p* is probability of success
  - Successes can occur in different orders
  - Probability given by:

$$\binom{n}{k} p^{k}(1-p)^{n-k}$$

Random Variables

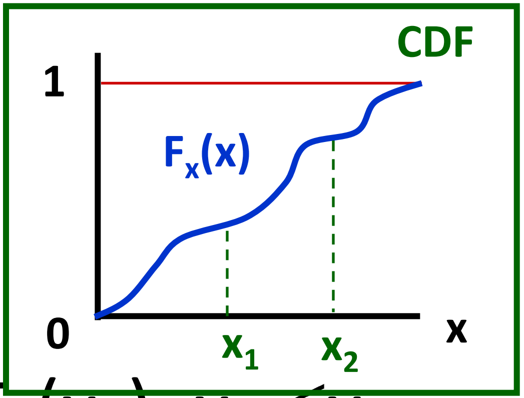
- Defined on Probability Space ($\mathcal{S}$,{A_i},P(•))
  - Random variable (x) maps subsets of $\mathcal{S}$ to real line.
  - Example: (x)="Coin Toss Outcome", H=1, T=0.

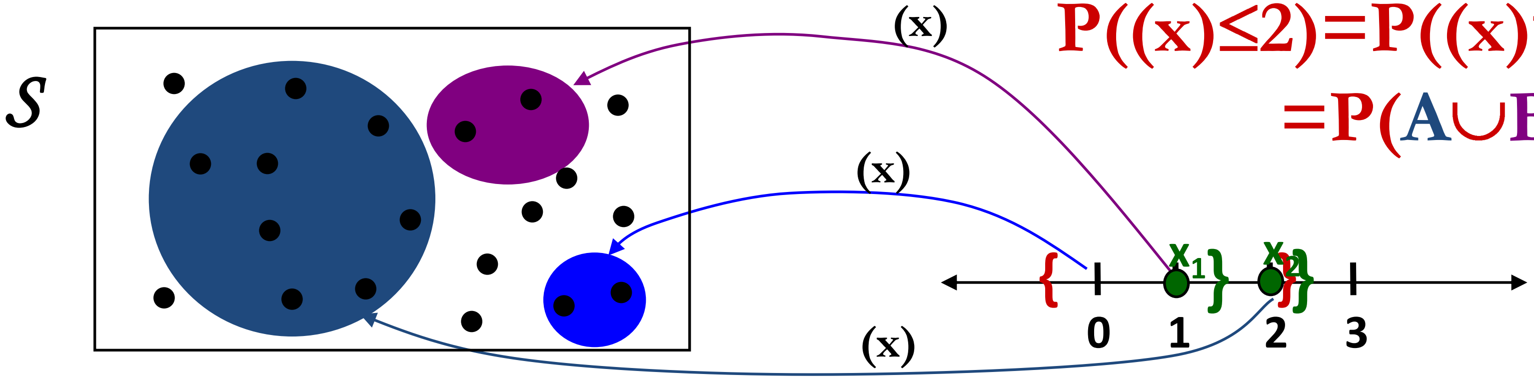


- If (x) takes on discrete values it is a discrete RV
- If (x) takes continuous values it is a continuous RV

Cumulative Distribution Function (CDF)

- CDF of (x): $F_x(x)=P((x)\leq x)$
- CDF satisfies $0\leq F_x(x)\leq 1$
- CDF is nondecreasing: $F_x(x_1) \leq F_x(x_2)$, $x_1 \leq x_2$





$P((x)\leq 2)=P((x)^{-1}(-\infty,2))$
$=P(A\cup B\cup C)$

1.5.1 Random Variables

Let a *random variable* $X(A)$ represent the functional relationship between a random event A and a real number. For notational convenience, we shall designate the random variable by X, and let the functional dependence upon A be implicit. The random variable may be discrete or continuous. The *distribution function* $F_X(x)$ of the random variable X is given by

$$F_X(x) = P(X \leq x) \tag{1.24}$$

where $P(X \leq x)$ is the probability that the value taken by the random variable X is less than or equal to a real number x. The distribution function $F_X(x)$ has the following properties:

1. $0 \le F_X(x) \le 1$
2. $F_X(x_1) \le F_X(x_2)$ if $x_1 \le x_2$
3. $F_X(-\infty) = 0$
4. $F_X(+\infty) = 1$

Probability Density Function

- The pdf defined by $f_x(x)=dF_x(x)/dx$
- Defines probability (x)lies in a given range

$$P(x_1 \le (x) \le x_2) = \int_{x_1}^{x_2} f_X(x)dx$$

- The pdf integrates to 1

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Mean, Moments, Variance

- Mean of x is $E[x]=\int x p_X(x)dx$
- $E[g(X)]= \int g(x)p_X(x)dx$
- nth Moment: $E[x^n]$, $E[x^2]$ is mean square
- $Var[x]=\sigma^2=E[(x- E[x])^2]=E[x^2]- E[x]^2$

1.5.1.1 Ensemble Averages

The *mean value* m_X, or *expected value* of a random variable X, is defined by

$$m_X = \mathbf{E}\{X\} = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{1.26}$$

where $\mathbf{E}[\cdot]$ is called the *expected value operator.* The *nth moment* of a probability distribution of a random variable X is defined by

$$\mathbf{E}\{X^n\} = \int_{-\infty}^{\infty} x^n p_X(x)\, dx \tag{1.27}$$

For the purposes of communication system analysis, the most important moments of X are the first two moments. Thus, $n = 1$ in Equation (1.27) gives m_X as discussed above, whereas $n = 2$ gives the mean-square value of X, as follows:

$$\mathbf{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{1.28}$$

We can also define *central moments,* which are the moments of the difference between X and m_X. The second central moment, called the *variance* of X, is defined as

$$\text{var}(X) = \mathbf{E}\{(X - m_X)^2\} = \int_{-\infty}^{\infty} (x - m_X)^2 p_X(x)\, dx \tag{1.29}$$

The variance of X is also denoted as σ_X^2, and its square root, σ_X, is called the *standard deviation* of X. Variance is a measure of the "randomness" of the random variable X. By specifying the variance of a random variable, we are constraining the width of its probability density function. The variance and the mean-square value are related by

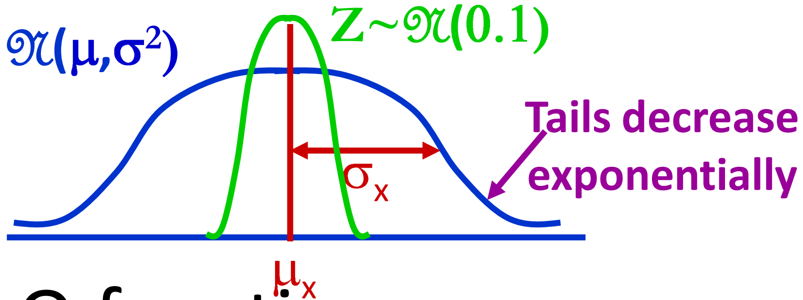
$$\begin{aligned}
\sigma_X^2 &= \mathbf{E}\{X^2 - 2m_X X + m_X^2\} \\
&= \mathbf{E}\{X^2\} - 2m_X \mathbf{E}\{X\} + m_X^2 \\
&= \mathbf{E}\{X^2\} - m_X^2
\end{aligned}$$

Thus, the variance is equal to the difference between the mean-square value and the square of the mean.

Gaussian Random Variables

- pdf defined in terms of mean and variance

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-[(x-\mu)^2/\sigma^2]}$$



- Gaussian CDF defined by Q function:

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2} dx$$

$$p(X \le x) = F_X(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right), \quad Q(x) = .5\,\mathrm{erfc}(x/\sqrt{2})$$

Several Random Variables

- Let X and Y be defined on ($\mathcal{S}$,{A_i},P(•))
- Joint CDF $F_{X,Y}$(x,y)=P(X≤x,Y≤y)
- Joint pdf: $p_{XY}(x,y): \;\; F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p_{XY}(\xi,\nu)d\xi d\nu$
- Conditional densities:
- Independent RVs:
- Sums of RVs:
  - Mean of sum is sum of means
  - Variance of sum is sum of variances

$$p_Y(y \mid X = x) = p_{XY}(x,y) / p_X(x)$$

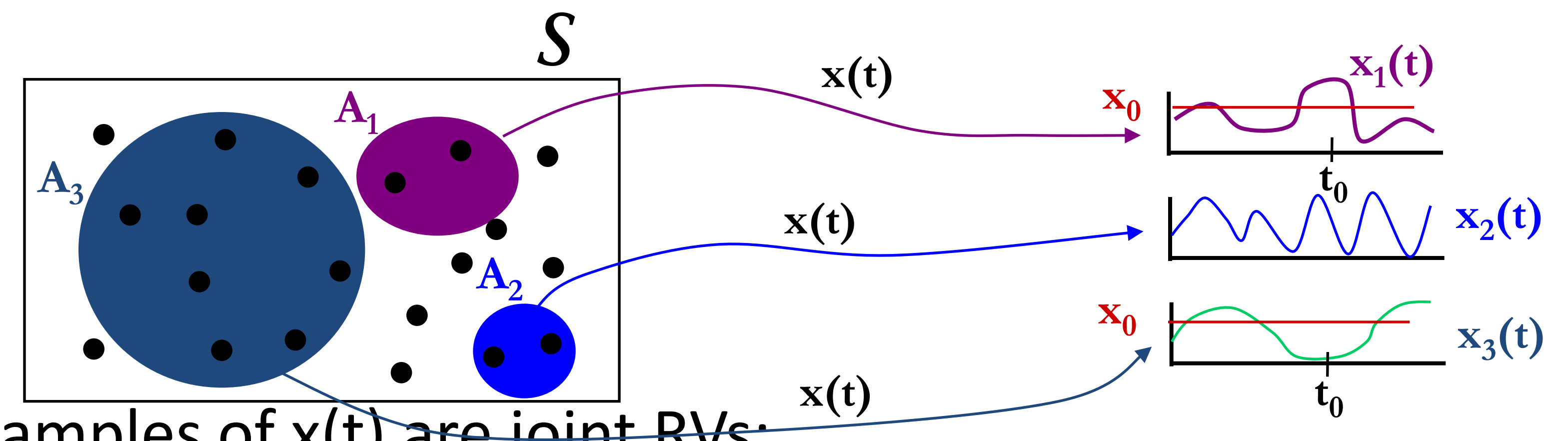
$$p_{XY}(x,y) = p_X(x) p_Y(y)$$

Main Points

- Events are independent if their joint probability equals the product of their probabilities.
- Random variables are functions mapping from subsets of the probability space to the real line
- The CDF and pdf of a random variable are derived from its underlying probability space.
- Mean of a RV is its average value. Variance is the 2^{nd} moment minus the mean squared.
- Gaussian RV a common model for noise
- Several random variables have a joint pdf. Their sum has a pdf that is the convolution of individual pdfs

Random Processes

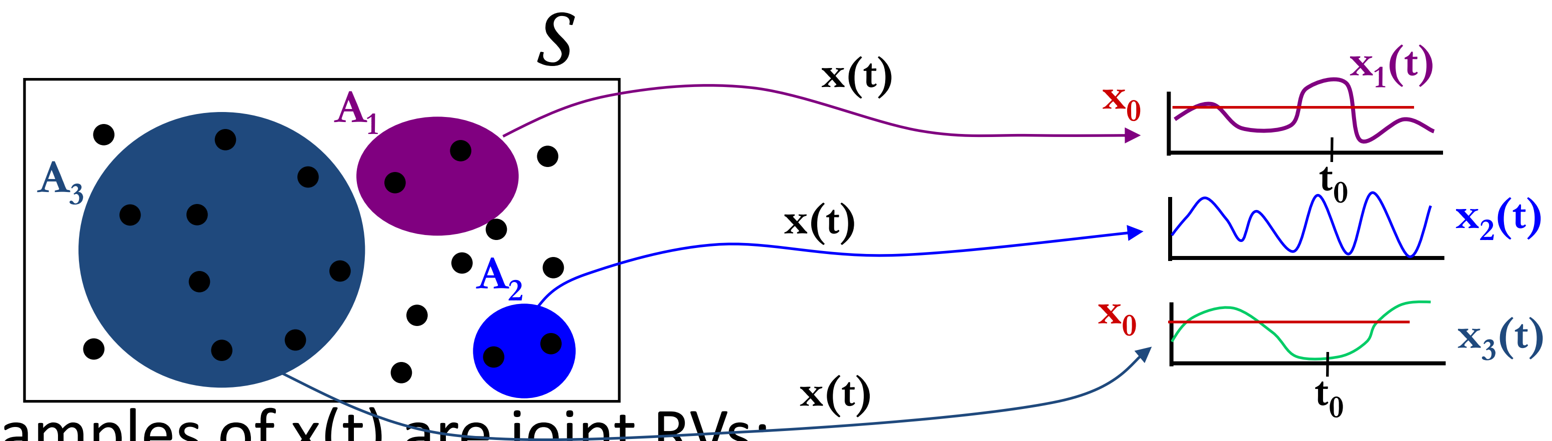
- Defined on Probability Space ($\mathcal{S}$,{A_i},P(•))
  - Random process X maps $\mathcal{S}$ to a set of functions.



- Samples of x(t) are joint RVs:
  - $P(x(t_1) \leq x_1) = P(\cup A_i : x_i(t_i) \leq x_i)$
  - $P(x(t_1) \leq x_1, x(t_2) \leq x_2, \ldots, x(t_n) \leq x_n) = F_{x(t1)x(t2)\ldots X(tn)}(x_0, \ldots x_n)$

Random Processes

- Defined on Probability Space ($\mathcal{S}$,{A_i},P(•))
  - Random process X maps $\mathcal{S}$ to a set of functions.



- Samples of x(t) are joint RVs:
  - $P\,(x(t_1) \leq x_1) = P(\cup A_i : x_i(t_i) \leq x_i)$
  - $P(x(t_1) \leq x_1, x(t_2) \leq x_2, \ldots, x(t_n) \leq x_n) = F_{x(t1)x(t2)\ldots X(tn)}(x_0, \ldots x_n)$

1.5.2 Random Processes

A random process $X(A, t)$ can be viewed as a function of two variables: *an event A* and *time*. Figure 1.5 illustrates a random process. In the figure there are N *sample functions* of time, $\{X_j(t)\}$. Each of the sample functions can be regarded as the output of a different noise generator. For a specific event A_j, we have a single time function $X(A_j, t) = X_j(t)$ (i.e., a sample function). The totality of all sample functions is called an *ensemble*. For a specific time t_k, $X(A, t_k)$ is a *random variable* $X(t_k)$ whose value depends on the event. Finally, for a specific event, $A = A_j$ and a specific time $t = t_k$, $X(A_j, t_k)$ is simply a *number*. For notational convenience we shall designate the random process by $X(t)$, and let the functional dependence upon A be implicit.

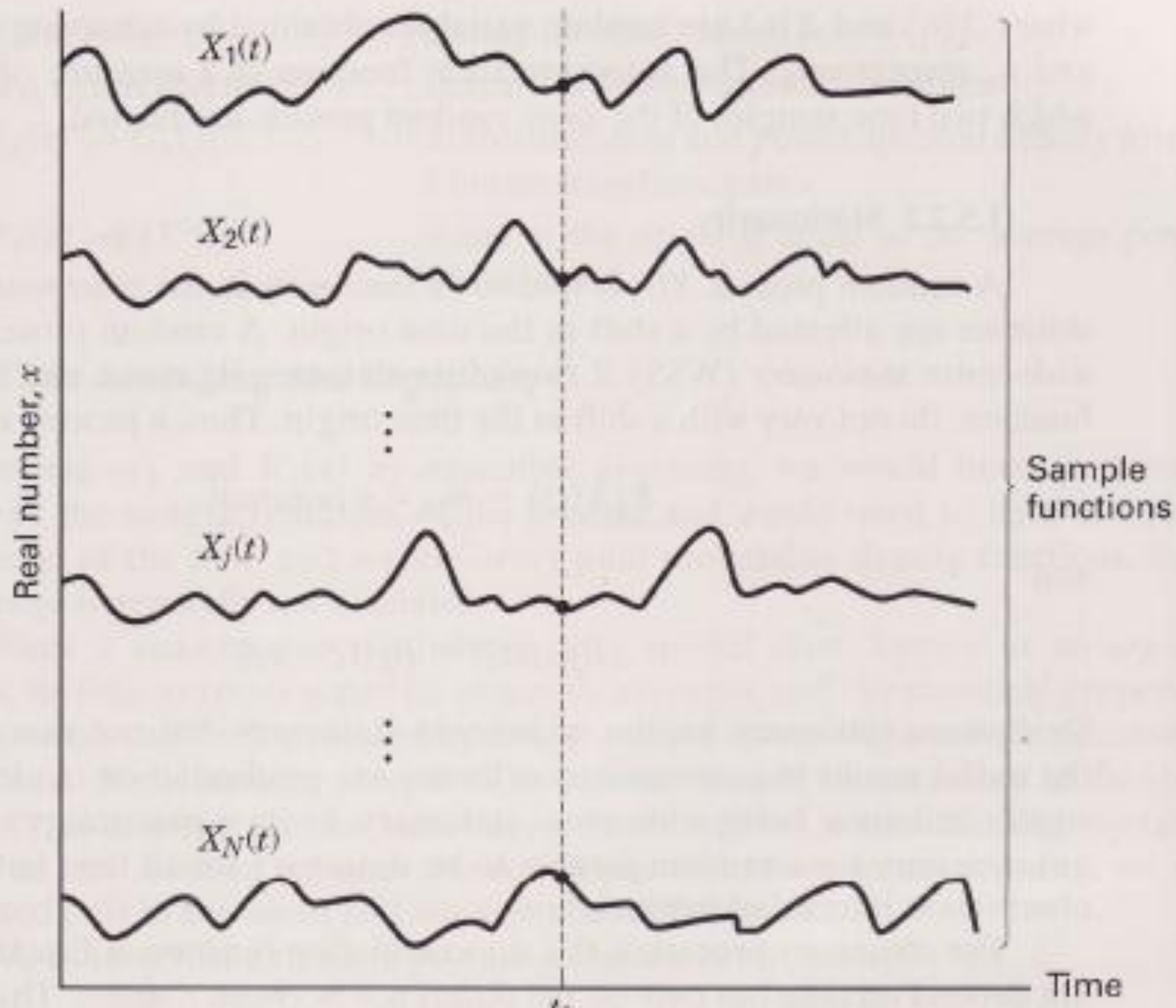


Figure 1.5 Random noise process.

1.5.2.1 Statistical Averages of a Random Process

Because the value of a random process at any future time is unknown (since the identity of the event A is unknown), a random process whose distribution functions are continuous can be described statistically with a probability density function (pdf). In general, the form of the pdf of a random process will be different for different times. In most situations it is not practical to determine empirically the probability distribution of a random process. However, a partial description consisting of the mean and autocorrelation function are often adequate for the needs of communication systems. We define the mean of the random process $X(t)$ as

$$\mathbf{E}\{X(t_k)\} = \int_{-\infty}^{\infty} x p_{X_k}(x)\, dx = m_X(t_k) \tag{1.30}$$

where $X(t_k)$ is the random variable obtained by observing the random process at time t_k and the pdf of $X(t_k)$, the density over the ensemble of events at time t_k, is designated $p_{X_k}(x)$.

We define the autocorrelation function of the random process $X(t)$ to be a function of two variables, t_1 and t_2, given by

$$R_X(t_1, t_2) = \mathbf{E}\{X(t_1)X(t_2)\} \tag{1.31}$$

Stationarity, Mean, Autocorrelation

- A random process is (strictly) stationary if time shifts don't change probability:
  - $P(x(t_1) \leq x_1, x(t_2) \leq x_2, \ldots, x(t_n) \leq x_n) =$
    $P(x(t_1+T) \leq x_1, x(t_2+T) \leq x_2, \ldots, x(t_n+T) \leq x_n)$
  - True for all T and all sets of sample times
- Mean of random process: $E[x(t)] =$ $\overline{x(t)}$
  - Stationary process: $E[X(t)] =$ $\bar{x}$
- Autocorrelation of a random process:
  - Defined as $R_X(t_1,t_2) = E[x(t_1)x(t_2)]]$
  - Stationary process: $R_x(t_1,t_2) = R_X(t_2-t_2)$
  - Correlation of process samples over time

1.5.2.2 Stationarity

A random process $X(t)$ is said to be *stationary* in the *strict sense* if none of its statistics are affected by a shift in the time origin. A random process is said to be *wide-sense stationary* (WSS) if two of its statistics, its mean and autocorrelation function, do not vary with a shift in the time origin. Thus, a process is WSS if

$$\mathbf{E}\{X(t)\} = m_X = \text{a constant} \tag{1.32}$$

and

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \tag{1.33}$$

Strict-sense stationary implies wide-sense stationary, but not vice versa. Most of the useful results in communication theory are predicated on random information signals and noise being wide-sense stationary. From a practical point of view, it is not necessary for a random process to be stationary for all time but only for some observation interval of interest.

For stationary processes, the autocorrelation function in Equation (1.33) does not depend on time but only on the difference between t_1 and t_2. That is, all pairs of values of $X(t)$ at points in time separated by $\tau = t_1 - t_2$ have the same correlation value. Thus, for stationary systems, we can denote $R_X(t_1, t_2)$ simply as $R_X(\tau)$.

1.5.2.3 Autocorrelation of a Wide-Sense Stationary Random Process

Just as the variance provides a measure of randomness for random variables, the autocorrelation function provides a similar measure for random processes. For a wide-sense stationary process, the autocorrelation function is only a function of the *time difference* $\tau = t_1 - t_2$; that is,

$$R_X(\tau) = \mathbf{E}\{X(t)X(t + \tau)\} \qquad \text{for } -\infty < \tau < \infty \tag{1.34}$$

For a zero mean WSS process, $R_X(\tau)$ indicates the extent to which the random values of the process separated by τ seconds in time are statistically correlated. In other words, $R_X(\tau)$ gives us an idea of the frequency response that is associated with a random process. If $R_X(\tau)$ changes slowly as τ increases from zero to some value, it indicates that, on average, sample values of $X(t)$ taken at $t = t_1$ and $t = t_1 + \tau$ are nearly the same. Thus, we would expect a frequency domain representation of $X(t)$ to contain a preponderance of low frequencies. On the other hand, if $R_X(\tau)$ decreases rapidly as τ is increased, we would expect $X(t)$ to change rapidly with time and thereby contain mostly high frequencies.

Properties of the autocorrelation function of a real-valued wide-sense stationary process are as follows:

1. $R_X(\tau) = R_X(-\tau)$	symmetrical in τ about zero
2. $\lvert R_X(\tau)\rvert \le R_X(0)$ for all τ	maximum value occurs at the origin
3. $R_X(\tau) \leftrightarrow G_X(f)$	autocorrelation and power spectral density form a Fourier transform pair
4. $R_X(0) = \mathbf{E}[X^2(t)]$	value at the origin is equal to the average power of the signal

1.5.3 Time Averaging and Ergodicity

To compute m_X and $R_X(\tau)$ by ensemble averaging, we would have to average across all the sample functions of the process and would need to have complete knowledge of the first- and second-order joint probability density functions. Such knowledge is generally not available.

When a random process belongs to a special class, known as an *ergodic process*, its time averages equal its ensemble averages, and the statistical properties of the process can be determined by *time averaging over a single sample function* of the process. For a random process to be ergodic, it must be stationary in the strict sense. (The converse is not necessary.) However, for communication systems, where we are satisfied to meet the conditions of wide-sense stationarity, we are interested only in the mean and autocorrelation functions.

We can say that a random process is *ergodic in the mean* if

$$m_X = \lim_{T \to \infty} 1/T \int_{-T/2}^{T/2} X(t)\, dt \tag{1.35}$$

and it is *ergodic in the autocorrelation function* if

$$R_X(\tau) = \lim_{T \to \infty} 1/T \int_{-T/2}^{T/2} X(t)X(t + \tau)\, dt \tag{1.36}$$

Testing for the ergodicity of a random process is usually very difficult. In practice one makes an intuitive judgment as to whether it is reasonable to interchange the time and ensemble averages. A reasonable assumption in the analysis of most communication signals (in the absence of transient effects) is that the random waveforms are ergodic in the mean and the autocorrelation function. Since time averages equal ensemble averages for ergodic processes, fundamental electrical engineering parameters, such as dc value, rms value, and average power can be related to the moments of an ergodic random process. Following is a summary of these relationships:

1. The quantity $m_X = \mathbf{E}\{X(t)\}$ is equal to the dc level of the signal.
2. The quantity m_X^2 is equal to the normalized power in the dc component.
3. The second moment of $X(t)$, $\mathbf{E}\{X^2(t)\}$, is equal to the total average normalized power.

Wide Sense Stationary (WSS)

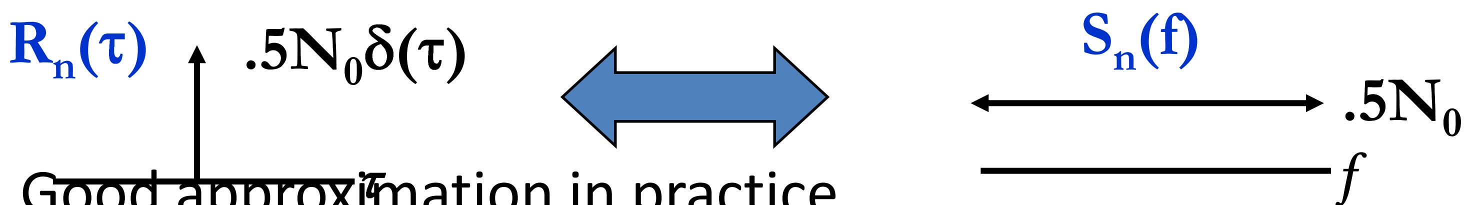
- A process is WSS if
  - E[x(t)] is constant
  - $R_X(t_1,t_2)= E[X(t_1)X(t_2)]]=R_X(t_2-t_1)= R_X(\tau)$
  - Intuitively, stationary in 1^{st} and 2^{nd} moments
- Ergodic WSS processes
  - Have the property that time averages equal probabilistic averages
  - Allow probability characteristics to be obtained from a single sample over time

Main Points

- Several random variables have a joint pdf. Their sum has a pdf that is the convolution of individual pdfs
- Gaussian RV a common model for noise
- Sums of i.i.d. shifted, normalized RVs converge to a $\mathcal{N}(0,1)$ Gaussian RV.
- Random process x(t) maps $\mathcal{S}$ to a set of functions
- Samples of x(t) are joint RVs
- A process is stationary if time shifts don't affect probability characteristics of process.
- WSS process has constant mean and autocorrelation that depends on time difference

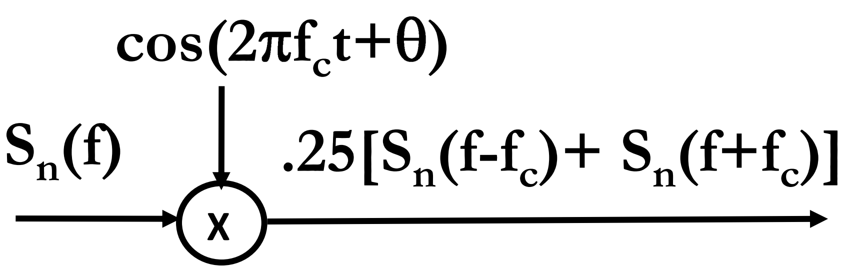
Power Spectral Density (PSD)

- Defined only for WSS processes
- FT of autocorrelation function: $R_X(\tau) \Leftrightarrow S_X(f)$
- $E[X^2(t)] = \int S_X(f)df$
- White Noise: Flat PSD



  - Good approximation in practice

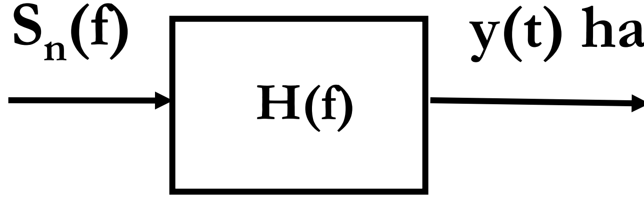
- Modulation:



Filtering a WSS Process

- Same PSD effect as for deterministic signals

n(t) with PSD $S_n(f)$ **y(t) has PSD $|H(f)|^2 S_n(f)$**



Multiple Random Processes

- Cross Correlation

$$R_{xy}(t_1, t_2) = E[x(t_1)y(t_2)]$$

- Processes are jointly stationary if
- WSS processes uncorrelated if

$$R_{xy}(t_1, t_2) = R_{xy}(t_2 - t_1)$$

- WSS processes orthogonal if

$$R_{xy}(t_1, t_2) = \overline{x}\overline{y}$$

$$R_{xy}(t_1, t_2) = 0$$

- For orthogonal random processes, autocorrelation of sum is sum of autocorrelation, and PSD of sum is sum of PSDs

Gaussian Processes

- z(t) is a Gaussian process if its samples are jointly Gaussian
- Filtering a Gaussian process results in a Gaussian process
- Integrating a Gaussian process results in a Gaussian random variable

$$Y_g = \int_0^T g(t)x(t)dt$$

Examples of noise in Communication Systems

- Gaussian processes
  - Filtering a Gaussian process yields a Gaussian process.
  - Sampling a Gaussian process yields jointly Gaussian RVs
  - If the autocorrelation at the sample times is zero, the RVs are independent.
- The signal-to-noise power ratio (SNR) is obtained by integrating the PSD of the signal and integrating the PSD of the noise.
- In digital communications, the bit value is obtained by integrating the signal, and the probability of error by integrating Gaussian noise.

Main Points

- PSD of a WSS process is the average power of a random processes, equals the FT of its autocorrelation
- White noise has flat PSD
- Modulation and filtering lead to WSS processes with same properties as for deterministic functions
- Gaussian random processes integrate to Gaussian RVs
- Can use random process analysis in studying noise in communication systems:
  - SNR obtained for analog systems by measuring power of signal and power of noise at receiver output
  - Probability of bit error uses properties of Gaussian random variables and processes.

Main Points

- Sums of i.i.d. shifted, normalized RVs converge to a $\mathcal{N}(0,1)$ Gaussian RV.
- Random process x(t) maps $\mathcal{S}$ to a set of functions
- Samples of x(t) are joint RVs
- A process is stationary if time shifts don't affect probability characteristics of process.
- WSS process has constant mean and autocorrelation that depends on time difference
- PSD of a WSS process is the FT of its autocorrelation
- White noise has flat PSD
- Modulation lead to WSS processes with same properties as for deterministic functions

Introduction to Carrier Modulation

- Basic concept is to vary carrier signal relative to information signal or bits
  - The carrier frequency is allocated by a regulatory body like the FCC – spectrum is pretty crowded at this point.
- Analog modulation varies amplitude (AM), frequency (FM), or phase (PM) of carrier
- Digital modulation varies amplitude (MAM), phase (PSK), pulse (PAM), or amplitude/phase (QAM)

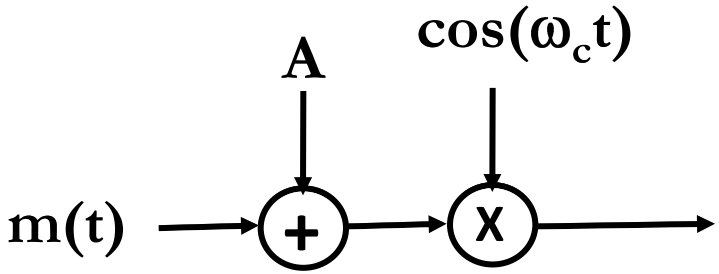
Double Sideband (Suppressed Carrier) Amplitude Modulation

- Modulated signal is $s(t)=m(t)\cos(2\pi f_c t)$
  - Called double-sideband suppressed carrier (DSBSC) AM

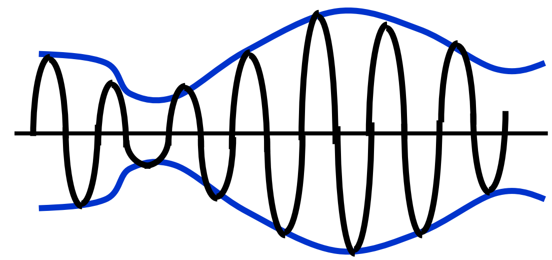
$$s(t) = m(t)\cos(2\pi f_c t) \Leftrightarrow .5[S(f - f_c) + S(f + f_c)]$$

- Generation of DSB-SC AM modulation
  - Direct multiplication (impractical)
  - Nonlinear modulators: Basic premise is to add m(t) and the carrier, then perform a nonlinear operation
  - Generates desired signal s(t) plus extra terms that are filtered out.
  - Examples include diode/transistor modulators, switch modulators, and ring modulators

Amplitude Modulation



$$s(t)=[A+m(t)]\cos\omega_c t$$

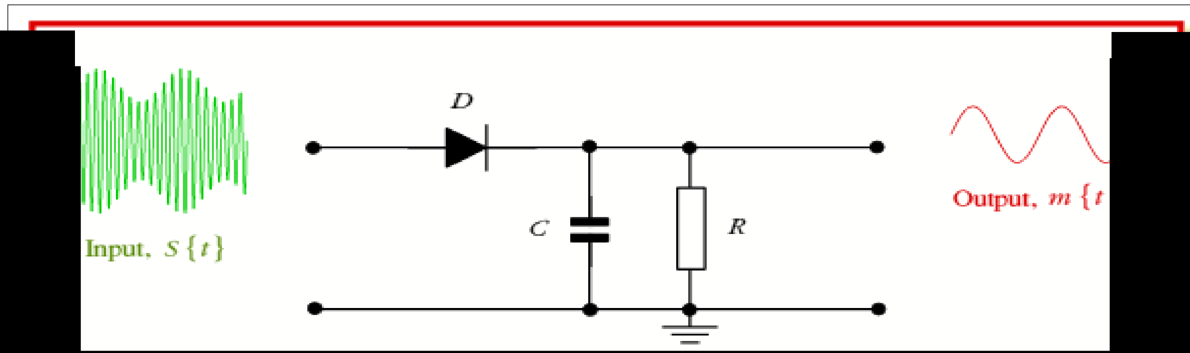


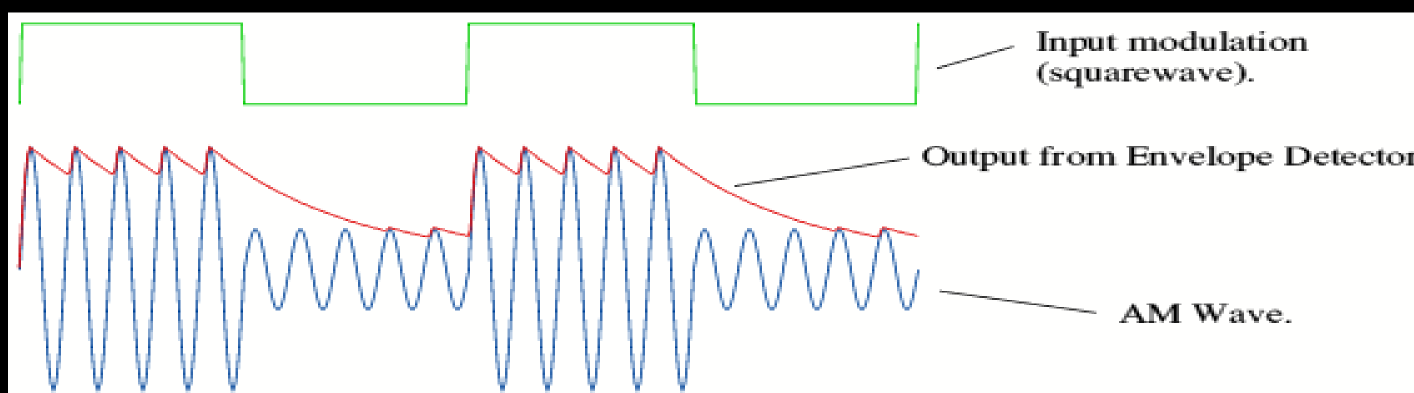
- Constant added to signal m(t)
  - Simplifies demodulation if $A>|m(t)|$
  - Demodulate based on envelope $|A+m(t)|=A+m(t)$
  - Constant is wasteful of power
- Modulated signal has twice bandwidth of m(t)
- Modulate same as DSBSC, except first add A to m(t)

Detection of AM Waves

- Entails tradeoff between performance and complexity (cost)
- Many techniques can be used
  - Square law detector , rectifier, envelope detector
- Rectifier “cuts off” negative components
  - Results in desired signal plus higher order terms that are filtered out
- Envelope detection is cheapest method
  - Used in most AM radios

Envelope Detector





- **Simple circuit**
- **Only works when $|m(t)|<A$ (lots of wasted power)**

Main Points

- Modulation is the process of encoding a message signal or bits into a carrier signal
- DSBSC multiplies the message signal and the carrier together.
- Coherent detection is needed for DSBSC: obtaining carrier phase is one of biggest challenges in *all* demodulators.
- AM modulation adds a constant term to message signal to simplify demodulation: wasteful of power and hurts SNR
- AM waves typically generated similar to DSBSC
- AM waves demodulated using rectifier or envelope detectors, which is the cheapest (and most common) method.

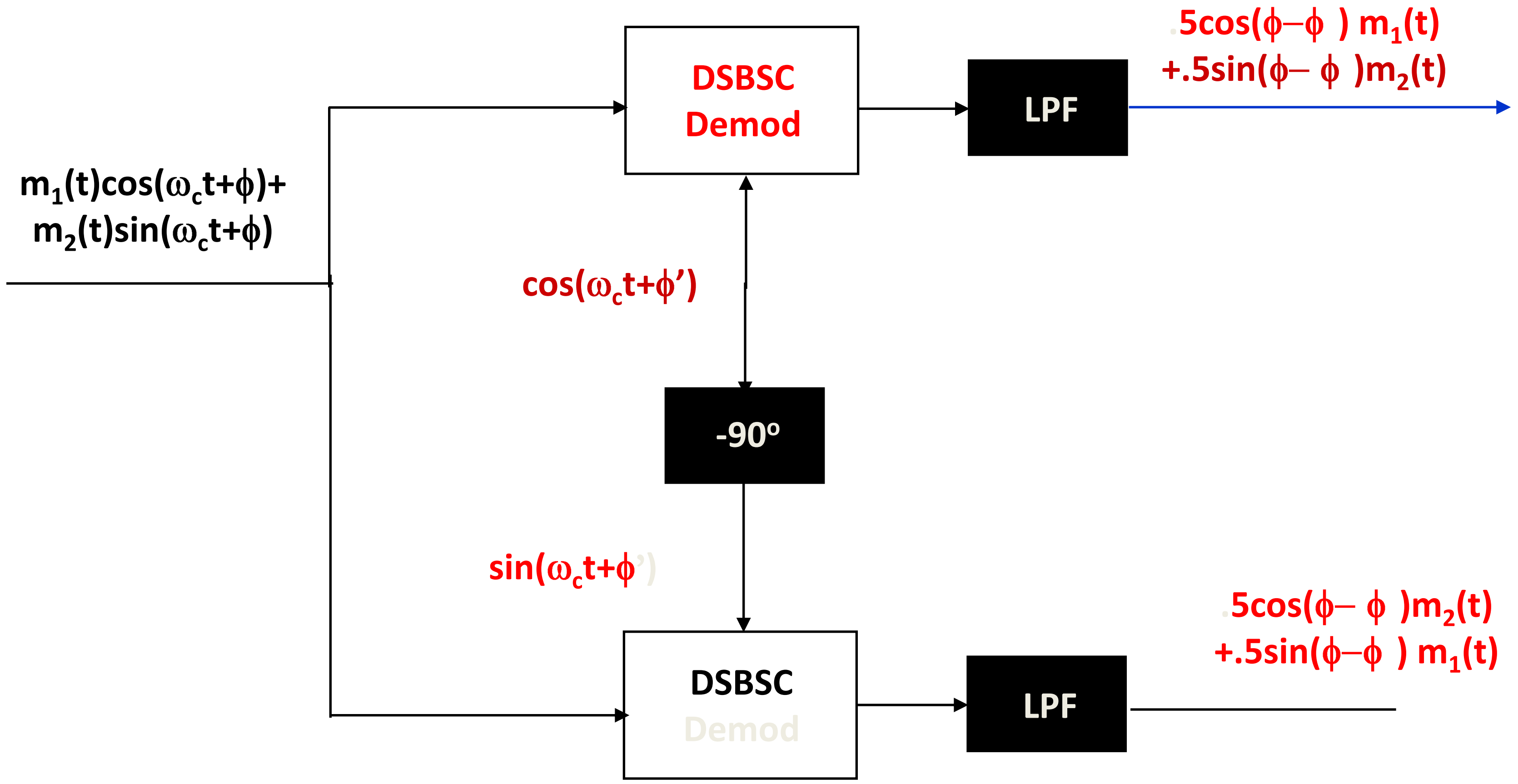
Single Sideband

- Only transmits upper or lower sideband of AM

- Reduces bandwidth by factor of 2
- Transmitted signal can be written in terms of Hilbert transform of m(t)
- SSB can introduce distortion at DC

Quadrature Modulation

Sends two info. signals on the cosine and sine carriers



Introduction to Angle Modulation and FM

- Information encoded in carrier freq./phase
- Modulated signal is $s(t)=A_c\cos(\theta(t))$
  - $\theta(t)=f(m(t))$

- Standard FM: $\theta(t)=2\pi f_c t+2\pi k_f\int m(\tau)d\tau$
  - Instantaneous frequency: $f_i=f_c+k_f m(t)$
  - Signal robust to amplitude variations
  - Robust to signal reflections and refractions

- Analysis nonlinear
  - Hard to analyze

FM Bandwidth and Carson's Rule

- Frequency Deviation: $\Delta f = k_f \max|m(t)|$
  - Maximum deviation of ω_i from ω_c: $\omega_i = \omega_c + k_f m(t)$
- Carson's Rule:

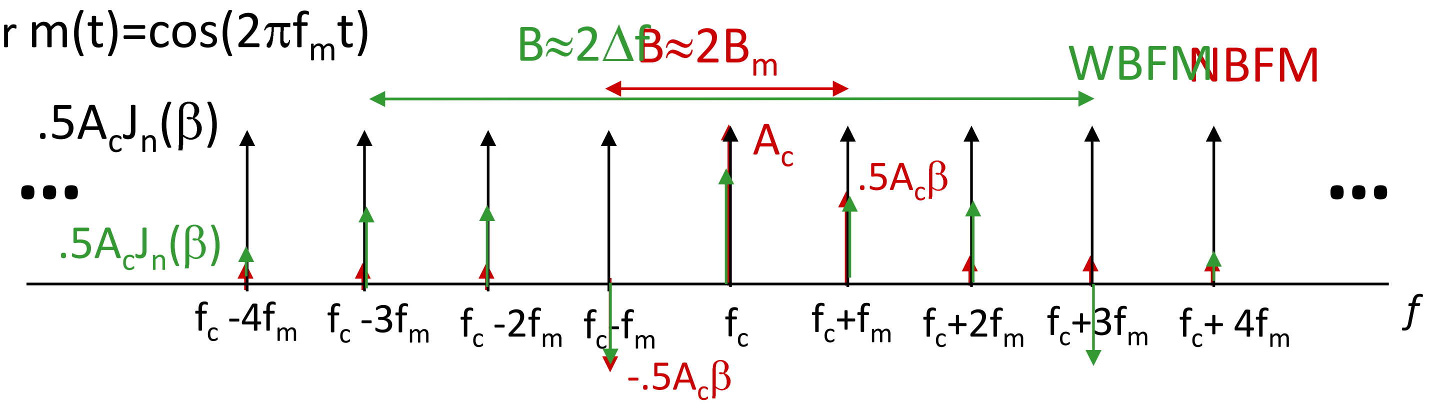
$$B_s \approx 2\Delta f + 2B_m$$

  - B_s depends on maximum deviation from ω_c AND how fast ω_i changes
- Narrowband FM: $\Delta f \ll B_m \Rightarrow B_s \approx 2B_m$
- Wideband FM: $\Delta f \gg B_m \Rightarrow B_s \approx 2\Delta f$

Spectral Analysis of FM

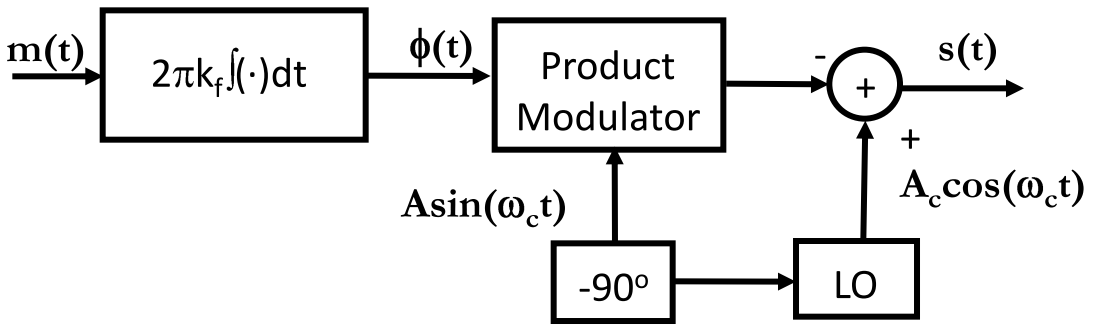
- $s(t)=A\cos(\omega_c t+k_f \int m(\alpha)d\alpha)$
  - Very hard to analyze for general m(t).
- Let $m(t)=\cos(\omega_m t)$: Bandwidth f_m
- S(f) sequence of δ functions at $f=f_c \pm nf_m$
  - If $\Delta f \ll f_m$, Bessel function small for $f \notin (f_c \pm f_m)$
  - If $\Delta f \gg f_m$, significant components up to $f_c \pm \Delta f$.

S(f) for $m(t)=\cos(2\pi f_m t)$

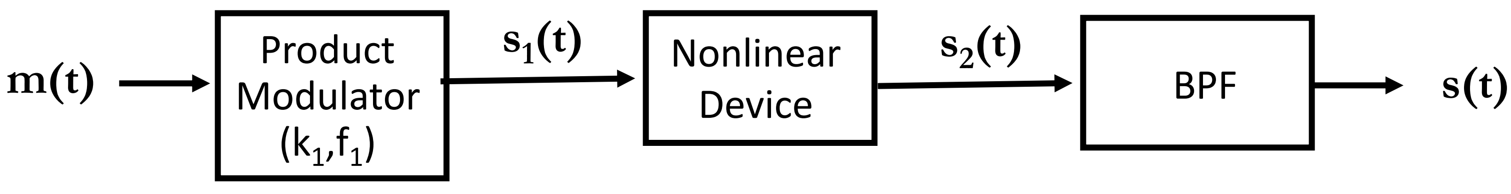


Generating FM Signals

- NBFM



- WBFM
  - Direct Method: Modulate a VCO with m(t)
  - Indirect Method



$$s_1(t) = A_c \cos(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau); \qquad s_2(t) = a_1 s_1(t) + \ldots + a_n s^n(t)$$

$$s_2(t) = a_n A_c^n \cos^n(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau) + \textbf{other terms}$$
$$= A\cos(2\pi n f_1 t + 2\pi n k_1 \int_0^t m(\tau)d\tau) + \textbf{other terms}$$

FM Detection

- Differentiator and Envelope Detector

$$\dot{s}(t) = A[\omega_c + k_f m(t)]\sin[\omega_c t + k_f \int_{-\infty}^{t} m(\alpha)d\alpha]$$

- Zero Crossing Detector
  - Uses rate of zero crossings to estimate ω_i
- Phase Lock Loop (PLL)
  - Uses VCO and feedback to extract m(t)

Main Points

- FM modulation encodes information in signal frequency. More robust to amplitude errors
- FM modulation nonlinear. Bandwidth approximate by Carson's rule: $B_s \approx 2\Delta f + 2B_m$
- Spectral analysis of FM difficult. For a simple cosine information signal, FM spectrum is discrete and infinite.
- NBFM is easier to analyze and generate (simple product modulator). WBFM more complicated to analyze and generate
- In theory just need differentiator and envelope detection for FM. Many techniques used in practice (mostly VCO).

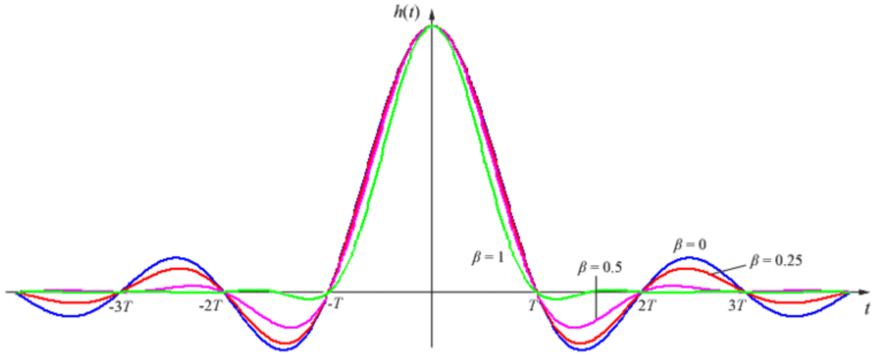
Introduction to Digital Modulation

- Most information today is in bits
- Baseband digital modulation converts bits into analog signals *y(t)* (bits encoded in amplitude)

$$\sum_{k=-\infty}^{\infty} a_k p(t-kT_b) = x(t) * p(t) \;\; for \;\; x(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t-kT_b)$$

Pulse Shaping

- Pulse shaping is the design of pulse *p(t)*
  - Want pulses that have zero value at sample times *nT*
- Rectangular pulses don't have good BW properties
- Nyquist pulses allow tradeoff of bandwidth characteristics and sensitivity to timing errors



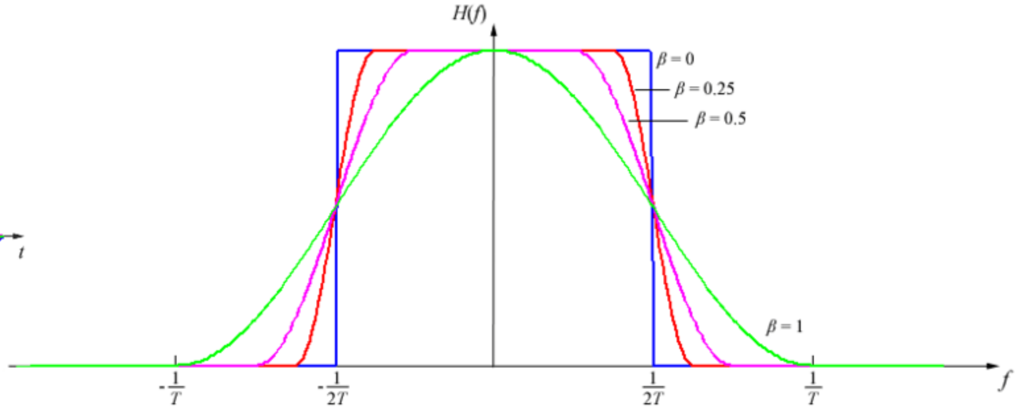


Figure 3–15 Binary signaling formats.

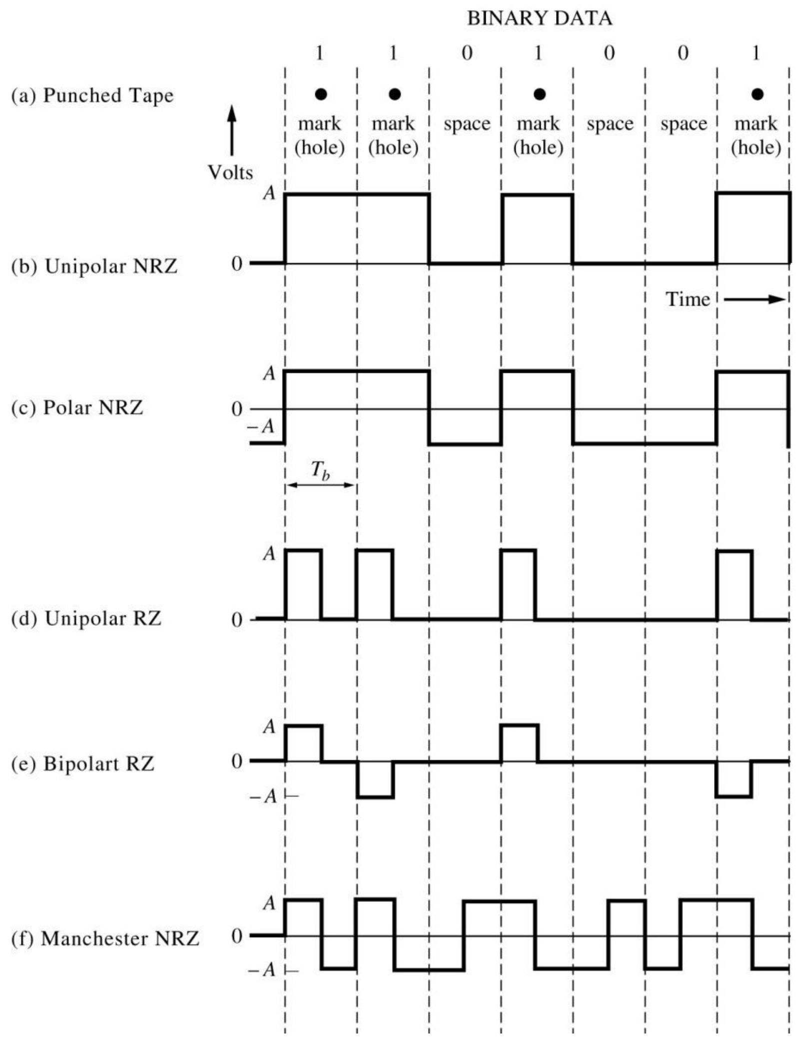
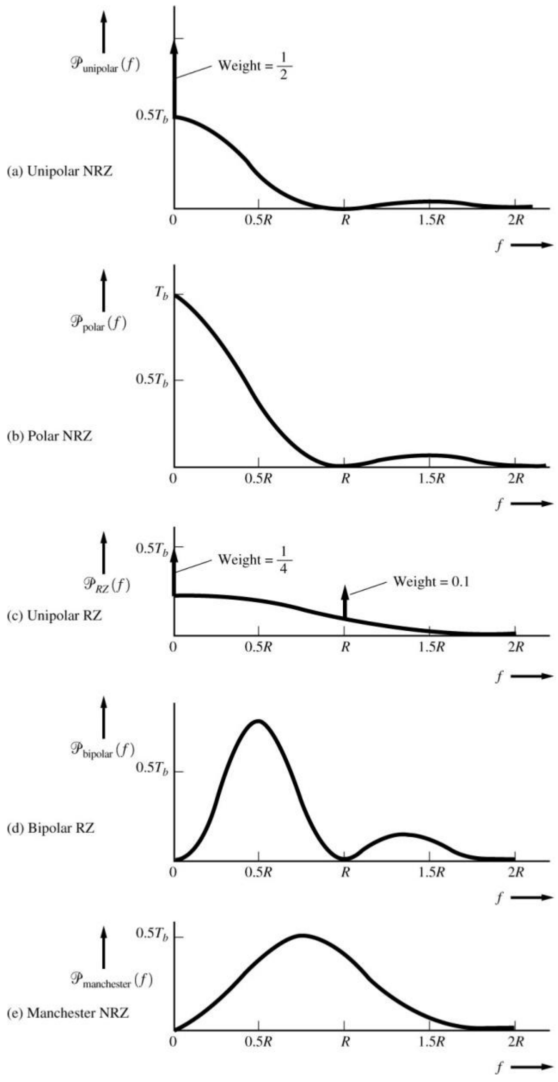


Figure 3–16 PSD for line codes (positive frequencies shown).



Passband Digital Modulation

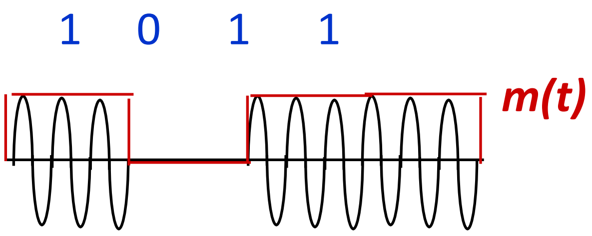
- Changes amplitude (ASK), phase (PSK), or frequency (FSK) of carrier relative to bits
- We use BB digital modulation as the information signal m(t) to encode bits, i.e. m(t) is on-off, etc.
- Passband digital modulation for ASK/PSK) is a special case of DSBSC; has form
- FSK is a special case of FM

$$s(t) = \sum_{k=-\infty}^{\infty} m(t)\cos(\omega_c t + \varphi)$$

ASK, PSK, and FSK

- Amplitude Shift Keying (ASK)

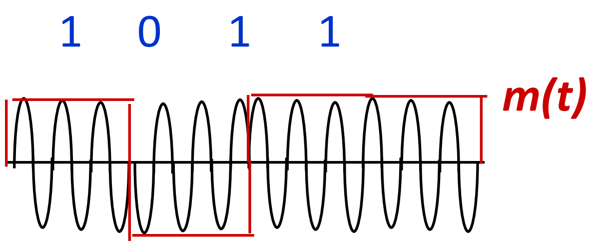
$$s(t) = m(t)\cos(\omega_c t) = \begin{cases} A\cos(\omega_c t) & m(nT_b) = A(\text{"1"}) \\ 0 & b(nT_b) = 0\,(\text{"0"}) \end{cases}$$



AM Modulation

- Phase Shift Keying (PSK)

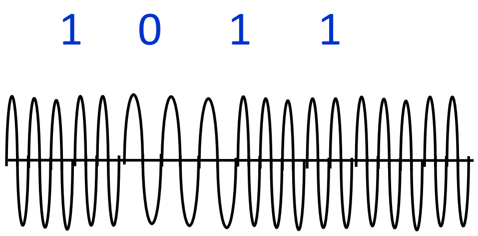
$$s(t) = m(t)\cos(\omega_c t) = \begin{cases} A\cos(\omega_c t) & m(nT_b) = A(\text{"1"}) \\ A\cos(\omega_c t + \pi) & m(nT_b) = -A(\text{"0"}) \end{cases}$$



Phase Modulation

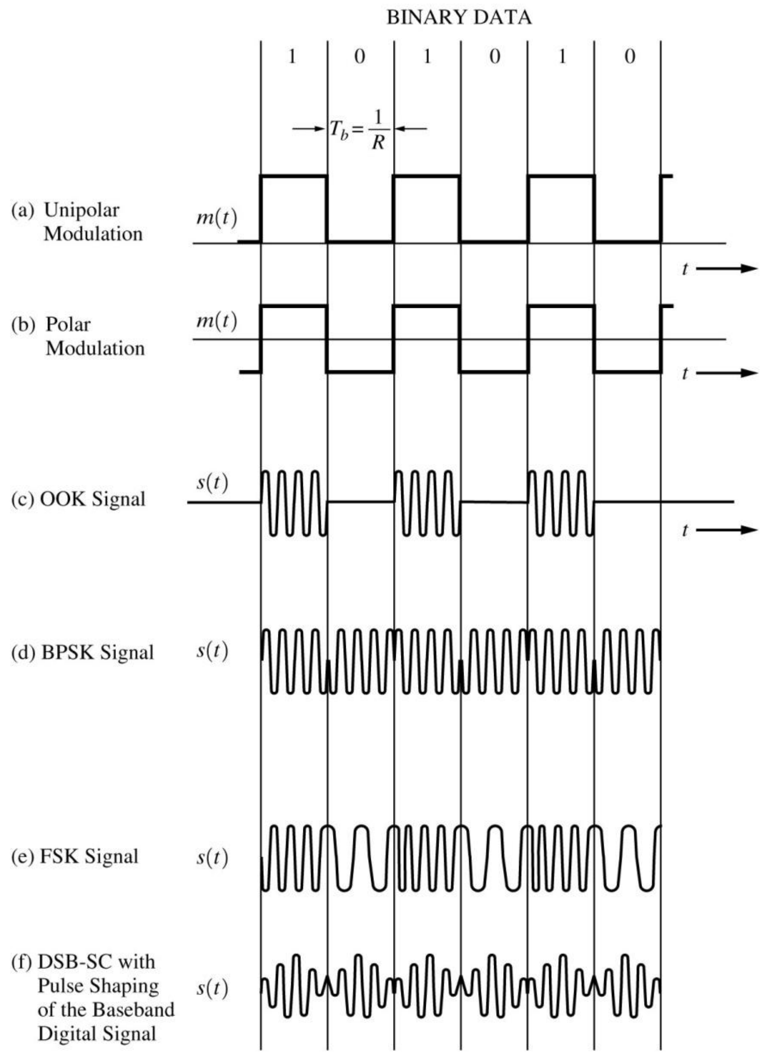
- Frequency Shift Keying

$$s(t) = \begin{cases} A\cos(\omega_1 t) & m(nT_b) = A \\ A\cos(\omega_0 t) & m(nT_b) = -A \end{cases}$$



FM Modulation

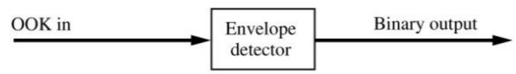
Figure 5–19 Bandpass digitally modulated signals.



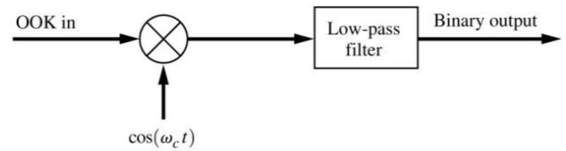
Main Points

- NBFM easy to generate and analyze. WBFM harder
- In theory just need differentiator and envelope detection for FM. Many techniques used in practice (mostly VCO).
- Digital baseband modulation encodes bits in analog signal, whose properties are dictated by the pulse shape
- Digital passband modulation encodes binary bits into the amplitude, phase, or frequency of the carrier.
- ASK/PSK special case of AM; FSK special case of FM

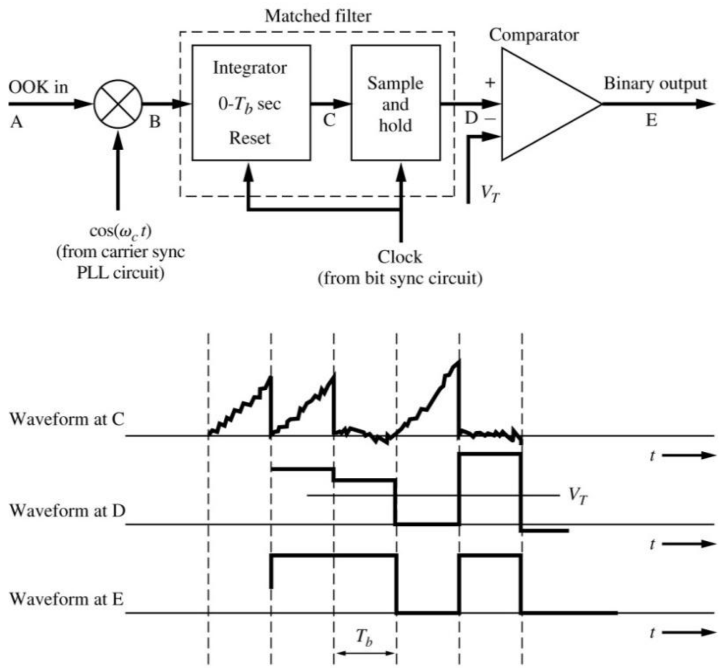
Figure 5–21 Detection of OOK.



(a) Noncoherent Detection

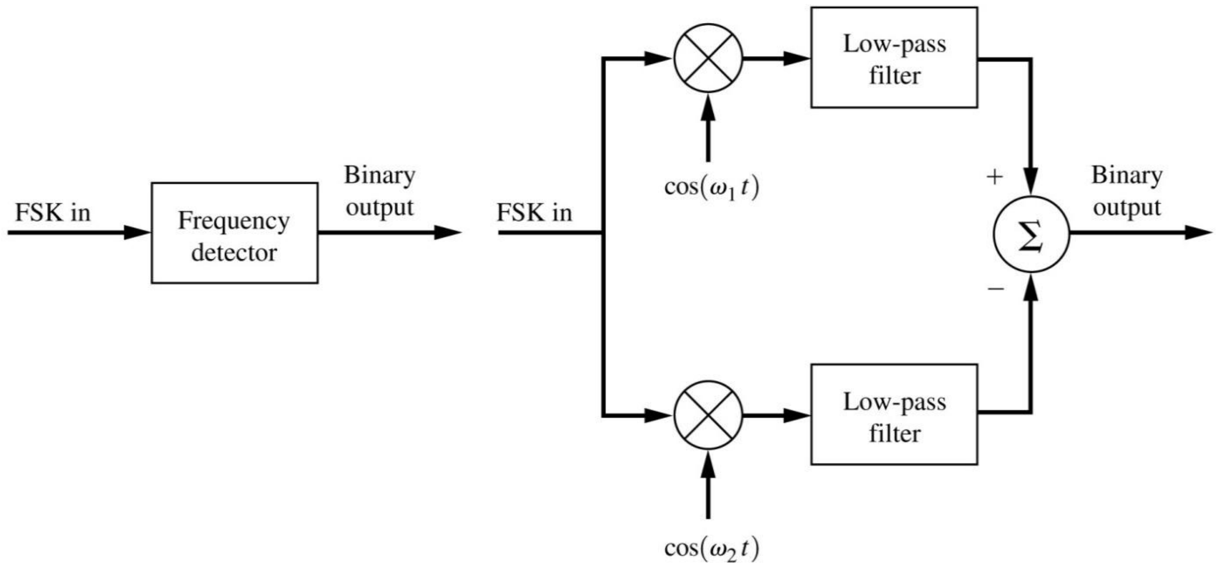


(b) Coherent Detection with Low-Pass Filter Processing



(c) Coherent Detection with Matched Filter Processing

Figure 5–28 Detection of FSK.



(a) Noncoherent Detection

(b) Coherent (Synchronous) Detection

Main Points

- Pulse shaping used in both baseband and passband modulation to determine signal BW and resistance to impairments.
- Digital passband modulation encodes binary bits into the amplitude, phase, or frequency of the carrier.
- ASK/PSK special case of AM; FSK special case of FM
- Noise immunity in receiver dictates how much noise reqd to make an error
- White Gaussian noise process causes a Gaussian noise term to be added to the decision device input
- Bit error probability with white noise is a function of the symbol energy to noise spectral density ratio.
- BPSK has lower error probability than ASK for same energy per bit.
- FSK same error prob. as ASK; less susceptible to amplitude fluctuations.

Multilevel Modulation

- m bits encoded in pulse of duration T_s ($R_b = m/T_s$)

$$s(t) = \sum_{n=-\infty}^{\infty} A\cos(\omega_c t + \theta_n(t))$$

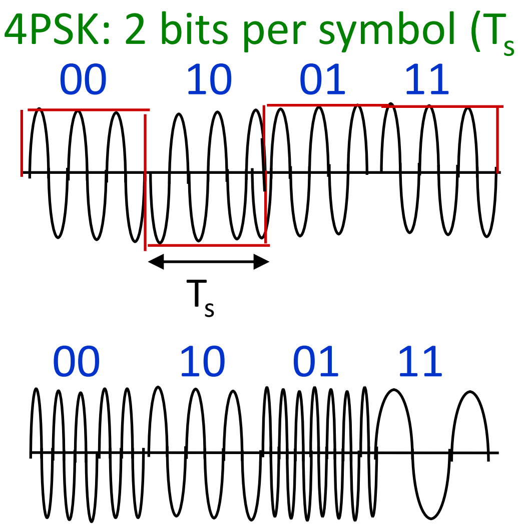
  - θ_n constant over a symbol time T_s, and can take $M=2^m$ different values on each pulse.

Higher data rate more susceptible to noise

- Phase Shift Keying (MPSK)

$$s(t) = \begin{cases} A_c\cos(\omega_c t) & \text{"00"} \\ A_c\cos(\omega_c t + \pi/2) & \text{"01"} \\ A_c\cos(\omega_c t + \pi) & \text{"10"} \\ A_c\cos(\omega_c t + 3\pi/2) & \text{"11"} \end{cases}$$

- Similar ideas in MFSK
- Demodulation similar to binary case



Main Points

- Bit error probability with white noise is a function of the symbol energy to noise spectral density ratio.
- BPSK has lower error probability than ASK for same energy per bit.
- FSK same error probability as ASK and less susceptible to amplitude fluctuations.
- Phase offset, timing offset, and biased noise can increase probability of error.
- Can encode more than one bit per symbol. This increases the data rate but makes signal more susceptible to noise