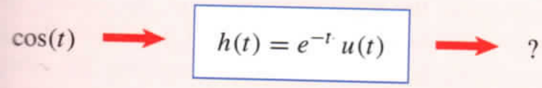
Exercise 2-12:



Answer: The output is $(1/\sqrt{2})\cos(t-45°)$. (See S²)

Exercise 2-13:

Initial conditions are zero. Is this system LTI?

Answer: No. An LTI cannot create a sinusoid at a frequency different from that of its input. (See S²)

Exercise 2-14:

Can we say that the system is not LTI?

Answer: No. An LTI can make an amplitude = 0. (See S²)

2-8 Impulse Response of Second-Order LCCDEs

Many physical systems are described by second-order LCCDEs of the form

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2\, y(t) = b_1\frac{dx}{dt} + b_2\, x(t), \qquad (2.121)$$

where a_1, a_2, b_1, and b_2 are constant coefficients. In this section, we examine how to determine the impulse response function $h(t)$ for such a differential equation, and in Section 2-9, we demonstrate how we use that experience to analyze a spring-mass-damper model of an automobile suspension system.

2-8.1 LCCDE with No Input Derivatives

For simplicity, we start by considering a version of Eq. (2.121) without the dx/dt term, and then we use the result to treat the more general case in the next subsection.

For $b_1 = 0$ and $b_2 = 1$, Eq. (2.121) becomes

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2\, y(t) = x(t). \qquad (2.122)$$

Step 1: Roots of characteristic equation

Assuming $y(t)$ has a general solution of the form $y(t) = Ae^{st}$, substitution in the homogeneous form of Eq. (2.122)—i.e., with $x(t) = 0$—leads to the ***characteristic equation***:

$$s^2 + a_1 s + a_2 = 0. \qquad (2.123)$$

If p_1 and p_2 are the roots of Eq. (2.123), then

$$s^2 + a_1 s + a_2 = (s - p_1)(s - p_2), \qquad (2.124)$$

which leads to

$$p_1 + p_2 = -a_1, \qquad p_1 p_2 = a_2, \qquad (2.125)$$

and

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,,$$
$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,. \qquad (2.126)$$

Roots p_1 and p_2 are

(a) real if $a_1^2 > 4a_2$,

(b) complex conjugates if $a_1^2 < 4a_2$, or

(c) identical if $a_1^2 = 4a_2$.

Step 2: Two coupled first-order LCCDEs

The original differential equation given by Eq. (2.122) now can be rewritten as

$$\frac{d^2y}{dt^2} - (p_1 + p_2)\frac{dy}{dt} + (p_1 p_2)\, y(t) = x(t), \qquad (2.127a)$$

which can in turn be cast in the form

$$\left[\frac{d}{dt} - p_1\right]\left[\frac{d}{dt} - p_2\right] y(t) = x(t). \qquad (2.127b)$$

Furthermore, we can split the ***second-order differential equation*** into ***two coupled first-order equations*** by introducing an intermediate variable $z(t)$:

$$\frac{dz}{dt} - p_1\, z(t) = x(t) \qquad (2.128a)$$

and

$$\frac{dy}{dt} - p_2\, y(t) = z(t). \qquad (2.128b)$$

These coupled first-order LCCDEs represent a ***series (or cascade) connection*** of LTI systems, each described by a first-order LCCDE. In symbolic form, we have

$$
\begin{aligned}
x(t) &\longrightarrow \boxed{h_1(t)} \longrightarrow z(t) \\
z(t) &\longrightarrow \boxed{h_2(t)} \longrightarrow y(t),
\end{aligned}
\tag{2.129}
$$

where $h_1(t)$ and $h_2(t)$ are the impulse responses corresponding to Eqs. (2.128a and b), respectively.

Step 3: Impulse response of cascaded LTI systems

By comparison with Eq. (2.10) and its corresponding impulse response, Eq. (2.17), we conclude that

$$h_1(t) = e^{p_1 t}\, u(t) \tag{2.130a}$$

and

$$h_2(t) = e^{p_2 t}\, u(t). \tag{2.130b}$$

Using convolution property #2 in **Table 2-1**, the impulse response of the series connection of two LTI systems is the convolution of their impulse responses. Utilizing entry #3 in **Table 2-2**, the combined impulse response becomes

$$
\begin{aligned}
h_c(t) &= h_1(t) * h_2(t) = e^{p_1 t}\, u(t) * e^{p_2 t}\, u(t) \\
&= \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t).
\end{aligned}
\tag{2.131}
$$

2-8.2 LCCDE with Input Derivative

We now consider the more general case of a second-order LCCDE that contains a first-order derivative on the input side of the equation

$$\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2\, y(t) = b_1 \frac{dx}{dt} + b_2\, x(t). \tag{2.132}$$

By defining the right-hand side of Eq. (2.132) as an intermediate variable $w(t)$, the system can be represented as

$$
\begin{aligned}
x(t) &\longrightarrow \boxed{w(t) = b_1 \frac{dx}{dt} + b_2\, x(t)} \longrightarrow w(t) \\
w(t) &\longrightarrow \boxed{h_c(t)} \longrightarrow y(t),
\end{aligned}
\tag{2.133}
$$

where $h_c(t)$ is the impulse response given by Eq. (2.131) for the system with $b_1 = 0$ and $b_2 = 1$.

To determine the impulse response of the overall system, we need to compute the convolution of $h_c(t)$ with the (yet to be determined) impulse response representing the other box in Eq. (2.133). A more expedient route is to use convolution property #1 in **Table 2-1**. Since convolution is commutative, we can reverse the order of the two LTI systems in Eq. (2.133):

$$
\begin{aligned}
x(t) &\longrightarrow \boxed{h_c(t)} \longrightarrow \upsilon(t) \\
\upsilon(t) &\longrightarrow \boxed{y(t) = b_1 \frac{d\upsilon}{dt} + b_2\, \upsilon(t)} \longrightarrow y(t),
\end{aligned}
\tag{2.134}
$$

where $\upsilon(t)$ is another intermediate variable created for the sake of convenience. By definition, when $x(t) = \delta(t)$, the output $y(t)$ becomes the impulse response $h(t)$ of the overall system. That is, if we set $x(t) = \delta(t)$, which results in $\upsilon(t) = h_c(t)$ and $y(t) = h(t)$, the system becomes

$$
\begin{aligned}
\delta(t) &\longrightarrow \boxed{h_c(t)} \longrightarrow h_c(t) \\
h_c(t) &\longrightarrow \boxed{h(t) = b_1 \frac{dh_c}{dt} + b_2\, h_c(t)} \longrightarrow h(t).
\end{aligned}
\tag{2.135}
$$

Finally, the impulse response $h(t)$ of the overall system is

$$
\begin{aligned}
h(t) &= b_1 \frac{dh_c}{dt} + b_2\, h_c(t) \\
&= \left[b_1 \frac{d}{dt} + b_2\right] \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t) \\
&= \frac{b_1 p_1 + b_2}{p_1 - p_2}\, e^{p_1 t}\, u(t) - \frac{b_1 p_2 + b_2}{p_1 - p_2}\, e^{p_2 t}\, u(t).
\end{aligned}
\tag{2.136}
$$

Having established in the form of Eq. (2.136) an explicit expression for the impulse response of the general LCCDE given by Eq. (2.132), we can now determine the response $y(t)$ to any causal input excitation $x(t)$ by evaluating

$$y(t) = u(t) \int_0^t h(\tau)\, x(t - \tau)\, d\tau. \tag{2.137}$$

2-8.3 Parameters of Second-Order LCCDE

Mathematically, our task is now complete. However, we can gain much physical insight into the nature of the system's response by examining scenarios associated with the three states of roots p_1 and p_2 [as noted earlier in connection with Eq. (2.126)].

(a) p_1 and p_2 are real and distinct (different).

(b) p_1 and p_2 are complex conjugates of one another.

(c) p_1 and p_2 are real and equal.

Recall that p_1 and p_2 are defined in terms of coefficients a_1 and a_2 in the LCCDE, so different systems characterized by LCCDEs with identical forms but different values of a_1 and a_2 may behave quite differently.

Before we examine the three states of p_1 and p_2 individually, it will prove useful to express p_1 and p_2 in terms of physically meaningful parameters. To start with, we reintroduce the expressions for p_1 and p_2 given by Eq. (2.126):

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,, \tag{2.138a}$$

$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}\,. \tag{2.138b}$$

Based on the results of Section 2-6.4, in order for the system described by Eq. (2.136) to be BIBO stable, it is necessary that the real parts of both p_1 and p_2 be negative. We now show that this is true if and only if $a_1 > 0$ and $a_2 > 0$. Specifically, we have the following:

(a) If both p_1 and p_2 are real, distinct, and negative, Eq. (2.138) leads to the conclusion that $a_1^2 > 4a_2$, $a_1 > 0$, and $a_2 > 0$.

(b) If p_1 and p_2 are complex conjugates with negative real parts, it follows that $a_1^2 < 4a_2$, $a_1 > 0$, and $a_2 > 0$.

(c) If p_1 and p_2 are real, equal, and negative, then $a_1^2 = 4a_2$, $a_1 > 0$, and $a_2 > 0$.

> ► The LTI system described by the LCCDE Eq. (2.132) is BIBO stable if and only if $a_1 > 0$ and $a_2 > 0$. ◄

We now introduce three new non-negative, physically meaningful parameters:

$$\alpha = \frac{a_1}{2} = \textit{attenuation coefficient} \quad \text{(Np/s)}, \tag{2.139a}$$

$$\omega_0 = \sqrt{a_2} = \textit{undamped natural frequency} \quad \text{(rad/s)}, \tag{2.139b}$$

and

$$\xi = \frac{\alpha}{\omega_0} = \frac{a_1}{2\sqrt{a_2}} = \textit{damping coefficient} \quad \text{(unitless)}. \tag{2.139c}$$

The unit Np is short for nepers, named after the inventor of the logarithmic scale, John Napier. In view of Eq. (2.139), p_1 and p_2 can be written as

$$p_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = \omega_0\left[-\xi + \sqrt{\xi^2 - 1}\right] \tag{2.140a}$$

and

$$p_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = \omega_0\left[-\xi - \sqrt{\xi^2 - 1}\right] \tag{2.140b}$$

The damping coefficient ξ plays a critically important role, because its value determines the character of the system's response to any input $x(t)$. The system exhibits markedly different responses depending on whether

(a) $\xi > 1$ ⟶ ***overdamped*** response,

(b) $\xi = 1$ ⟶ ***critically damped*** response, or

(c) $\xi < 1$ ⟶ ***underdamped*** response.

The three names, overdamped, underdamped, and critically damped, refer to the shape of the system's response. **Figure 2-22** displays three system step responses, each one of which starts at zero at $t = 0$ and rises to $y = 1$ as $t \to \infty$, but the shapes of their waveforms are quite different. The overdamped response exhibits the slowest path towards $y = 1$; the underdamped response is very fast, but it includes an oscillatory component; and the critically damped response represents the fastest path without oscillations.

2-8.4 Overdamped Case ($\xi > 1$)

For convenience, we rewrite Eq. (2.136) as

$$h(t) = A_1 e^{p_1 t}\, u(t) + A_2 e^{p_2 t}\, u(t) \tag{2.141}$$

(overdamped impulse response)

with

$$A_1 = \frac{b_1 p_1 + b_2}{p_1 - p_2} \quad \text{and} \quad A_2 = \frac{-(b_1 p_2 + b_2)}{p_1 - p_2}\,. \tag{2.142}$$

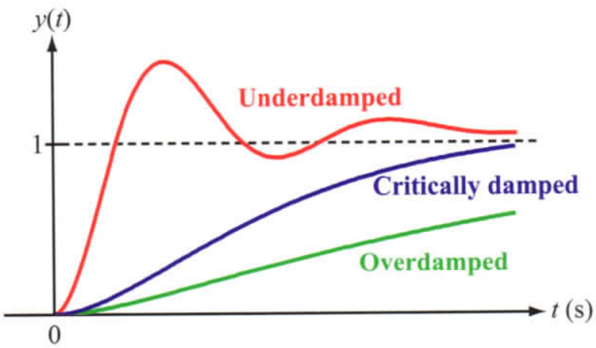


Figure 2-22: Comparison of overdamped, underdamped, and critically damped responses. In each case, the response starts at zero at $t = 0$ and approaches 1 as $t \to \infty$, but the in-between paths are quite different.

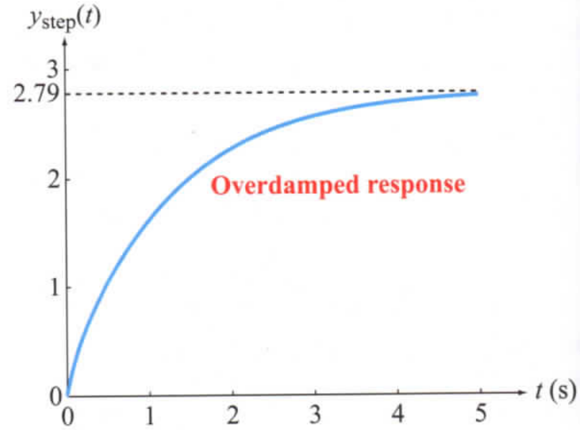


Figure 2-23: Step response of overdamped system in Example 2-14.

If $\xi > 1$—which corresponds to $\alpha^2 > \omega_0^2$, or equivalently, $a_1^2 > 4a_2$—roots p_1 and p_2 are both negative real numbers with $|p_2| > |p_1|$. The step response $y_{\text{step}}(t)$ is obtained by inserting Eq. (2.141) into Eq. (2.137) and setting $x(t-\tau) = u(t-\tau)$ and $y(t) = y_{\text{step}}(t)$:

$$y_{\text{step}}(t) = \int_0^t [A_1 e^{p_1\tau}\, u(\tau) + A_2 e^{p_2\tau}\, u(\tau)]\, u(t-\tau)\, d\tau. \tag{2.143}$$

Over the range of integration $(0, t)$, $u(\tau) = 1$ and $u(t-\tau) = 1$. Hence,

$$y_{\text{step}}(t) = \left[\int_0^t (A_1 e^{p_1\tau} + A_2 e^{p_2\tau})\, d\tau \right] u(t),$$

which integrates to

$$y_{\text{step}}(t) = \left[\frac{A_1}{p_1}\,(e^{p_1 t} - 1) + \frac{A_2}{p_2}\,(e^{p_2 t} - 1) \right] u(t).$$

(overdamped step response) (2.144)

Example 2-14: Overdamped Response

Compute and plot the step response $y_{\text{step}}(t)$ of a system described by the LCCDE

$$\frac{d^2y}{dt^2} + 12\,\frac{dy}{dt} + 9y(t) = 4\,\frac{dx}{dt} + 25x(t).$$

Solution: From the LCCDE, $a_1 = 12$, $a_2 = 9$, $b_1 = 4$, and $b_2 = 25$. When used in Eqs. (2.139), (2.140), and (2.142), we obtain the values

$$\alpha = \frac{a_1}{2} = \frac{12}{2} = 6 \text{ Np/s},$$

$$\omega_0 = \sqrt{a_2} = \sqrt{9} = 3 \text{ rad/s}, \qquad \xi = \frac{\alpha}{\omega_0} = \frac{6}{3} = 2,$$

$$p_1 = -0.8 \text{ Np/s}, \qquad p_2 = -11.2 \text{ Np/s},$$

$$\frac{A_1}{p_1} = -2.62, \text{ and } \frac{A_2}{p_2} = -0.17.$$

Since $\xi > 1$, the response is overdamped, in which case Eq. (2.144) applies:

$$y_{\text{step}}(t) = \left[2.62(1 - e^{-0.8t}) + 0.17(1 - e^{-11.2t}) \right] u(t).$$

The plot of $y_{\text{step}}(t)$ is shown in Fig. 2-23.

2-8.5 Underdamped Case ($\xi < 1$)

If $\xi < 1$, or equivalently, $\alpha^2 < \omega_0^2$, the square root in Eq. (2.140) becomes negative, causing roots p_1 and p_2 to become complex numbers. For reasons that will become apparent shortly, this condition leads to an ***underdamped*** step response with a waveform that oscillates at a ***damped natural frequency*** ω_d defined as

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \xi^2}\,. \tag{2.145}$$

In terms of ω_d, roots p_1 and p_2 [Eq. (2.140)] become

$$p_1 = -\alpha + j\omega_d \quad \text{and} \quad p_2 = -\alpha - j\omega_d. \qquad (2.146)$$

Inserting these expressions into the impulse response given by Eq. (2.141) leads to

$$\begin{aligned} h(t) &= [A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}]\, u(t) \\ &= [A_1(\cos\omega_d t + j\sin\omega_d t) \\ &\quad + A_2(\cos\omega_d t - j\sin\omega_d t)]e^{-\alpha t}\, u(t) \\ &= [(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]e^{-\alpha t}\, u(t), \end{aligned}$$

which can be contracted into

$$h(t) = [B_1\cos\omega_d t + B_2 \sin\omega_d t]e^{-\alpha t}\, u(t)$$

(underdamped impulse response) (2.147)

by introducing two new coefficients, B_1 and B_2, given by

$$B_1 = A_1 + A_2 = \frac{b_1 p_1 + b_2}{p_1 - p_2} - \frac{b_1 p_2 + b_2}{p_1 - p_2} = b_1 \qquad (2.148a)$$

and

$$B_2 = j(A_1 - A_2) = \frac{b_2 - b_1\alpha}{\omega_d}. \qquad (2.148b)$$

The negative exponential $e^{-\alpha t}$ in Eq. (2.147) signifies that $h(t)$ has a damped waveform, and the sine and cosine terms signify that $h(t)$ is oscillatory with an angular frequency ω_d and a corresponding ***time period***

$$T = \frac{2\pi}{\omega_d}. \qquad (2.149)$$

Example 2-15: Underdamped Response

Compute and plot the step response $y_{\text{step}}(t)$ of a system described by the LCCDE

$$\frac{d^2y}{dt^2} + 12\,\frac{dy}{dt} + 144y(t) = 4\,\frac{dx}{dt} + 25x(t).$$

Solution: From the LCCDE, $a_1 = 12$, $a_2 = 144$, $b_1 = 4$, and $b_2 = 25$.

The damping coefficient is

$$\xi = \frac{a_1}{2\sqrt{a_2}} = \frac{12}{2\sqrt{144}} = 0.5.$$

Hence, this is an underdamped case and the appropriate impulse response is given by Eq. (2.147). The step response $y_{\text{step}}(t)$ is obtained by convolving $h(t)$ with $x(t) = u(t)$:

$$\begin{aligned} y_{\text{step}}(t) &= h(t) * u(t) \\ &= \left[\int_0^t (B_1\cos\omega_d\tau + B_2\sin\omega_d\tau)e^{-\alpha\tau}\,d\tau\right] u(t). \end{aligned}$$

Performing the integration by parts leads to

$$\begin{aligned} y_{\text{step}}(t) &= \frac{1}{\alpha^2 + \omega_d^2} \\ &\quad \cdot \{[-(B_1\alpha + B_2\omega_d)\cos\omega_d t \\ &\quad + (B_1\omega_d + B_2\alpha)\sin\omega_d t]e^{-\alpha t} \\ &\quad + (B_1\alpha + B_2\omega_d)\}\, u(t). \end{aligned} \qquad (2.150)$$

(underdamped step response)

For the specified constants,

$$\alpha = 6 \text{ Np/s}, \qquad \omega_0 = \sqrt{a_2} = 12 \text{ rad/s}, \qquad \xi = 0.5,$$

$$\omega_d = \omega_0\sqrt{1 - \xi^2} = 10.4 \text{ rad/s},$$

$$B_1 = 4, \qquad B_2 = 9.62 \times 10^{-2}, \text{ and}$$

$$y_{\text{step}}(t) = \Big\{[-0.17\cos 10.4t + 0.29\sin 10.4t]\,e^{-6t} + 0.17\Big\}\, u(t).$$

Figure 2-24 displays a plot of $y_{\text{step}}(t)$, which exhibits an oscillatory pattern superimposed on the exponential response. The oscillation period is $T = 2\pi/10.4 = 0.6$ s.

2-8.6 Critically Damped Case ($\xi = 1$)

According to Eq. (2.140), if $\xi = 1$, then $p_1 = p_2$. Repeated roots lead to a ***critically damped*** step response, so called because it provides the fastest path to the steady state that the system approaches as $t \to \infty$. With $p_1 = p_2$, the expression for $h(t)$ reduces to a single exponential, which is not a viable solution, because a second-order LCCDE should include two time-dependent functions in its solution. To derive an appropriate expression for $h(t)$, we take the following indirect approach.

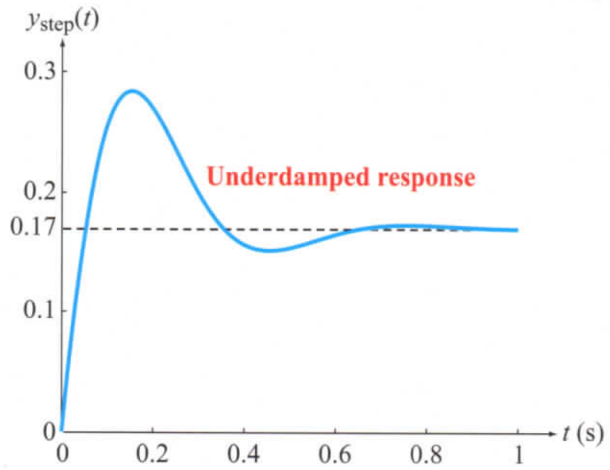


Figure 2-24: Underdamped response of Example 2-15. Note the oscillations.

Step 1: Start with a slightly underdamped system:

Suppose we have a slightly underdamped system with a very small damped natural frequency $\omega_d = \epsilon$ and roots

$$p_1 = -\alpha + j\epsilon \quad \text{and} \quad p_2 = -\alpha - j\epsilon.$$

According to Eq. (2.131), the impulse response $h_c(t)$ of a system with no input derivative is

$$\begin{aligned} h_c(t) &= \left[\frac{1}{p_1 - p_2}\right] [e^{p_1 t} - e^{p_2 t}]\, u(t) \\ &= \left[\frac{e^{j\epsilon t} - e^{-j\epsilon t}}{j2\epsilon}\right] e^{-\alpha t}\, u(t) = \frac{\sin \epsilon t}{\epsilon}\, e^{-\alpha t}\, u(t). \end{aligned} \tag{2.151}$$

Since ϵ is infinitesimally small, $\epsilon \ll \alpha$. Hence, $e^{-\alpha t}$ will decay to approximately zero long before ϵt becomes significant in magnitude, which means that for all practical purposes, the function $\sin \epsilon t$ is relevant only when ϵt is very small, in which case the approximation $\sin(\epsilon t) \approx \epsilon t$ is valid. Accordingly, $h_c(t)$ becomes

$$h_c(t) = te^{-\alpha t}\, u(t), \qquad \text{as } \epsilon \to 0. \tag{2.152}$$

Step 2: Obtain impulse response $h(t)$:

Implementation of Eq. (2.135) to obtain $h(t)$ of the system containing an input derivative dx/dt from that of the same system without the dx/dt term leads to

$$\begin{aligned} h(t) &= b_1 \frac{dh_c}{dt} + b_2 h_c(t) \\ &= (C_1 + C_2 t)e^{-\alpha t}\, u(t) \end{aligned} \tag{2.153}$$

(critically damped impulse response)

with

$$C_1 = b_1 \tag{2.154a}$$

and

$$C_2 = b_2 - \alpha b_1. \tag{2.154b}$$

Example 2-16: Critically Damped Response

Compute and plot the step response of a system described by

$$\frac{d^2 y}{dt^2} + 12\frac{dy}{dt} + 36y(t) = 4\frac{dx}{dt} + 25x(t).$$

Solution: From the LCCDE, $a_1 = 12$, $a_2 = 36$, $b_1 = 4$, and $b_2 = 25$. The damping coefficient is

$$\xi = \frac{a_1}{2\sqrt{a_2}} = \frac{12}{2\sqrt{36}} = 1.$$

Hence, this is a critically damped system. The relevant constants are

$$\alpha = \frac{a_1}{2} = 6 \text{ Np/s}, \ C_1 = 4, \text{ and } C_2 = 1,$$

and the impulse response is

$$h(t) = (4 + t)e^{-6t}\, u(t).$$

The corresponding step response is

$$\begin{aligned} y_{\text{step}}(t) = h(t) * u(t) &= \left[\int_0^t (4 + \tau)e^{-6\tau}\, d\tau\right] u(t) \\ &= \left[\frac{25}{36}(1 - e^{-6t}) - \frac{1}{6}\, te^{-6t}\right] u(t), \end{aligned}$$

and its profile is displayed in Fig. 2-25. The step response starts at zero and approaches a final value of $25/36 = 0.69$ as $t \to \infty$. It exhibits the fastest damping rate possible without oscillation.

The impulse and step responses of the second-order LCCDE, namely Eq. (2.132), are summarized in Table 2-3 for each of the three damping conditions.

Table 2-3: **Impulse and step responses of second-order LCCDE.**

LCCDE $$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = b_1 \frac{dx}{dt} + b_2 x$$

$$\alpha = \frac{a_1}{2}, \qquad \omega_0 = \sqrt{a_2}, \qquad \xi = \frac{\alpha}{\omega_0}, \qquad p_1 = \omega_0[-\xi + \sqrt{\xi^2 - 1}], \qquad p_2 = \omega_0[-\xi - \sqrt{\xi^2 - 1}]$$

Overdamped Case $\xi > 1$

$$h(t) = A_1 e^{p_1 t}\, u(t) + A_2 e^{p_2 t}\, u(t) \qquad y_{\text{step}}(t) = \left[\frac{A_1}{p_1}(e^{p_1 t} - 1) + \frac{A_2}{p_2}(e^{p_2 t} - 1)\right] u(t)$$

$$A_1 = \frac{b_1 p_1 + b_2}{p_1 - p_2}, \qquad A_2 = \frac{-(b_1 p_2 + b_2)}{p_1 - p_2}$$

Underdamped Case $\xi < 1$

$$h(t) = [B_1 \cos\omega_d t + B_2 \sin\omega_d t]e^{-\alpha t}\, u(t)$$

$$y_{\text{step}}(t) = \frac{1}{\alpha^2 + \omega_d^2}\left\{[-(B_1\alpha + B_2\omega_d)\cos\omega_d t + (B_1\omega_d + B_2\alpha)\sin\omega_d t]e^{-\alpha t} + (B_1\alpha + B_2\omega_d)\right\} u(t)$$

$$B_1 = b_1, \qquad B_2 = \frac{b_2 - b_1\alpha}{\omega_d}, \qquad \omega_d = \omega_0\sqrt{1 - \xi^2}$$

Critically Damped Case $\xi = 1$

$$h(t) = (C_1 + C_2 t)e^{-\alpha t}\, u(t) \qquad y_{\text{step}}(t) = \left[\left(\frac{C_1}{\alpha} + \frac{C_2}{\alpha^2}\right)(1 - e^{-\alpha t}) - \frac{C_2}{\alpha}\, t e^{-\alpha t}\right] u(t)$$

$$C_1 = b_1, \qquad C_2 = b_2 - \alpha b_1$$

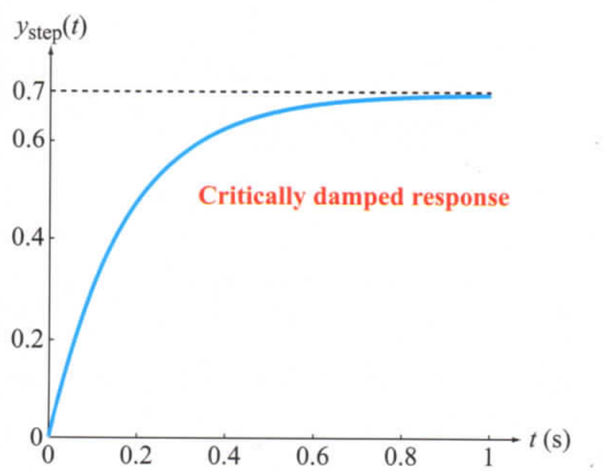


Figure 2-25: Critically damped response of Example 2-16.

Concept Question 2-14: What are the three damping conditions of the impulse response?

Concept Question 2-15: How do input derivatives affect impulse responses?

Exercise 2-15: Which damping condition is exhibited by $h(t)$ of

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y(t) = 2\frac{dx}{dt}\ ?$$

Answer: Overdamped, because $\xi = 1.25 > 1$. (See S^2)

Exercise 2-16: For what constant a_1 is

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + 9y(t) = 2 \frac{dx}{dt}$$

critically damped?

Answer: $a_1 = 6$. (See S^2)

2-9 Car Suspension System

The results of the preceding section will now be used to analyze a car suspension system, which is selected in part because it offers a nice demonstration of how to model both the car suspension system and several examples of input excitations, including driving over a curb, over a pothole, and on a wavy pavement.

2-9.1 Spring-Mass-Damper System

The basic elements of an automobile suspension system are depicted in Fig. 2-26.

- $x(t)$ = input = vertical displacement of the pavement, defined relative to a reference ground level.
- $y(t)$ = output = vertical displacement of the car chassis from its equilibrium position.
- m = *one-fourth* of the car's mass, because the car has four wheels.
- k = ***spring constant*** or ***stiffness*** of the coil.
- b = ***damping coefficient*** of the shock absorber.

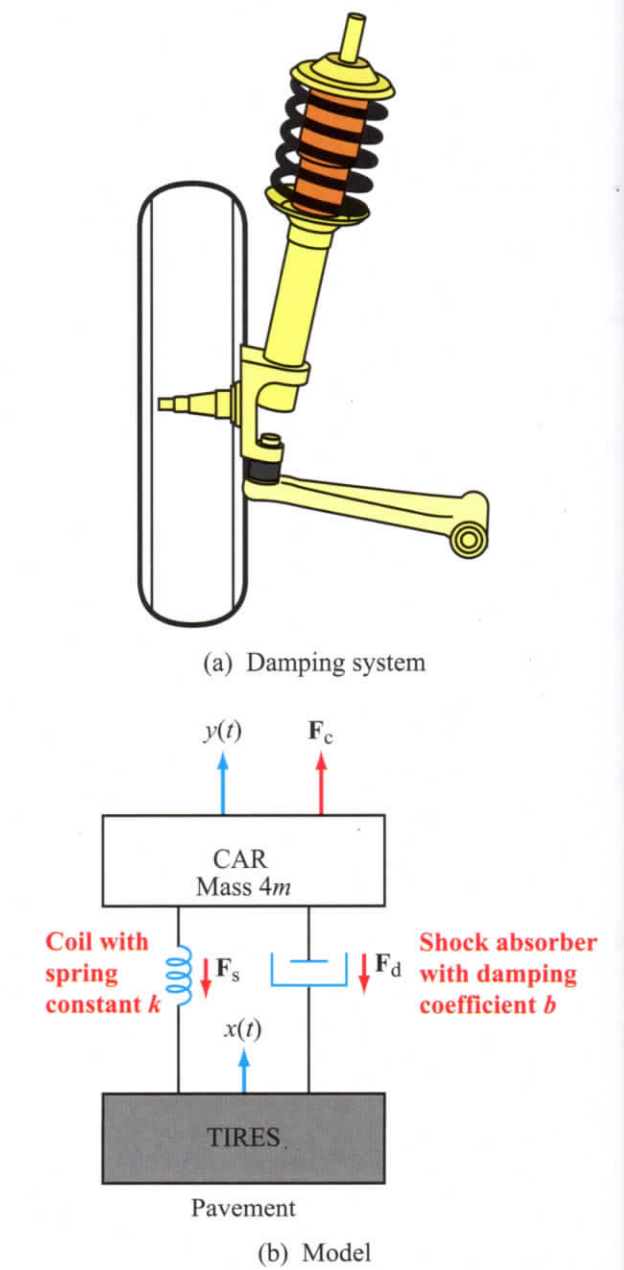


Figure 2-26: Car suspension system model.

The forces exerted by the spring and shock absorber, which act on the car mass in parallel, depend on the relative displacement $(y - x)$ of the car relative to the pavement. They act to oppose the upward inertial force $\mathbf{F}_c$ on the car, which depends on only the car displacement $y(t)$. When $(y - x)$ is positive (car mass moving away from the pavement), the ***spring force*** $\mathbf{F}_s$ is directed downward. Hence, $\mathbf{F}_s$ is given by

$$\mathbf{F}_s = -k(y - x). \tag{2.155}$$

The ***damper force*** $\mathbf{F}_d$ exerted by the shock absorber is governed by viscous compression. It also is pointed downward, but it opposes the *change* in $(y - x)$. Therefore it opposes the derivative of $(y - x)$ rather than $(y - x)$ itself:

$$\mathbf{F}_d = -b \frac{d}{dt}(y - x). \tag{2.156}$$

Using Newton's law, $\mathbf{F}_c = ma = m(d^2y/dt^2)$, the force equation is

$$\mathbf{F}_c = \mathbf{F}_s + \mathbf{F}_d \tag{2.157}$$

or

$$m \frac{d^2y}{dt^2} = -k(y - x) - b \frac{d}{dt}(y - x),$$

which can be recast as

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}\,y = \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}\,x. \tag{2.158}$$

The form of Eq. (2.158) is identical with that of the second-order LCCDE given by Eq. (2.132). Hence, all of the results we derived in the preceding section become directly applicable to the automobile suspension system upon setting $a_1 = b_1 = b/m$ and $a_2 = b_2 = k/m$.

Typical values for a small automobile are:

- $m = 250$ kg for a car with a total mass of one metric ton (1000 kg); each wheel supports one-fourth of the car's mass.
- $k = 10^5$ N/m; it takes a force of 1000 N to compress the spring by 1 cm.
- $b = 10^4$ N·s/m; a vertical motion of 1 m/s incurs a resisting force of 10^4 N.

2-9.2 Pavement Models

Driving on a curb

A car driving over a curb can be modeled as a step in $x(t)$ given by

$$x_1(t) = A_1\, u(t), \tag{2.159}$$

where A_1 is the height of the curb (**Fig. 2-27(a)**).

Driving over a pothole

For a car moving at speed s over a pothole of length d, the pothole represents a depression of duration $T = d/s$. Hence, driving over the pothole can be modeled (**Fig. 2-27(b)**) as

$$x_2(t) = A_2[-u(t) + u(t-T)], \tag{2.160}$$

where A_2 is the depth of the pothole.

Driving over wavy pavement

Figure 2-27(c) depicts a wavy pavement whose elevation is a sinusoid of amplitude A_3 and period T_0. Input $x_3(t)$ is then

$$x_3(t) = A_3 \cos\frac{2\pi t}{T_0}\,. \tag{2.161}$$

Example 2-17: Car Response to a Curb

A car with a mass of 1,000 kg is driven over a 10 cm high curb. Each wheel is supported by a coil with spring constant $k = 10^5$ N/m. Determine the car's response to driving over the curb for each of the following values of b, the damping constant of the shock absorber: (a) 2×10^4 N·s/m, (b) 10^4 N·s/m, and (c) 5000 N·s/m.

Solution: **(a)** The mass per wheel is $m = 1000/4 = 250$ kg. Comparison of the constant coefficients in Eq. (2.158) with those in Eq. (2.132) for the LCCDE of Section 2-8 leads to

LCCDE Suspension System

$$a_1 = \frac{b}{m} = \frac{2\times 10^4}{250} = 80\ \text{s}^{-1},$$

$$a_2 = \frac{k}{m} = \frac{10^5}{250} = 400\ \text{s}^{-2},$$

$$b_1 = \frac{b}{m} = 80\ \text{s}^{-1},$$

$$b_2 = \frac{k}{m} = 400\ \text{s}^{-2},$$

$$\omega_0 = \sqrt{a_2} = 20\ \text{rad/s},$$

and

$$\alpha = \frac{a_1}{2} = 40\ \text{Np/s}.$$

The damping coefficient is

$$\xi = \frac{\alpha}{\omega_0} = \frac{40}{20} = 2.$$

Since $\xi > 1$, the car suspension is an overdamped system. As was noted earlier in Section 2-9.2, the curb is modeled as a step function with an amplitude $A = 10$ cm $= 0.1$ m. From **Table 2-3**, the step response of an overdamped system scaled by a factor of 0.1 is

$$y_1(t) = 0.1\, y_{\text{step}}(t) = 0.1\left[\frac{A_1}{p_1}(e^{p_1 t}-1) + \frac{A_2}{p_2}(e^{p_2 t}-1)\right] u(t). \tag{2.162}$$

From **Table 2-3**, we have

$$p_1 = \omega_0[-\xi + \sqrt{\xi^2-1}] = -5.36\ \text{Np/s},$$

$$p_2 = \omega_0[-\xi - \sqrt{\xi^2-1}] = -74.64\ \text{Np/s},$$

$$A_1 = \frac{b_1 p_1 + b_2}{p_1 - p_2} = \frac{80(-5.36)+400}{-5.36+74.64} = -0.42\ \text{s}^{-1},$$

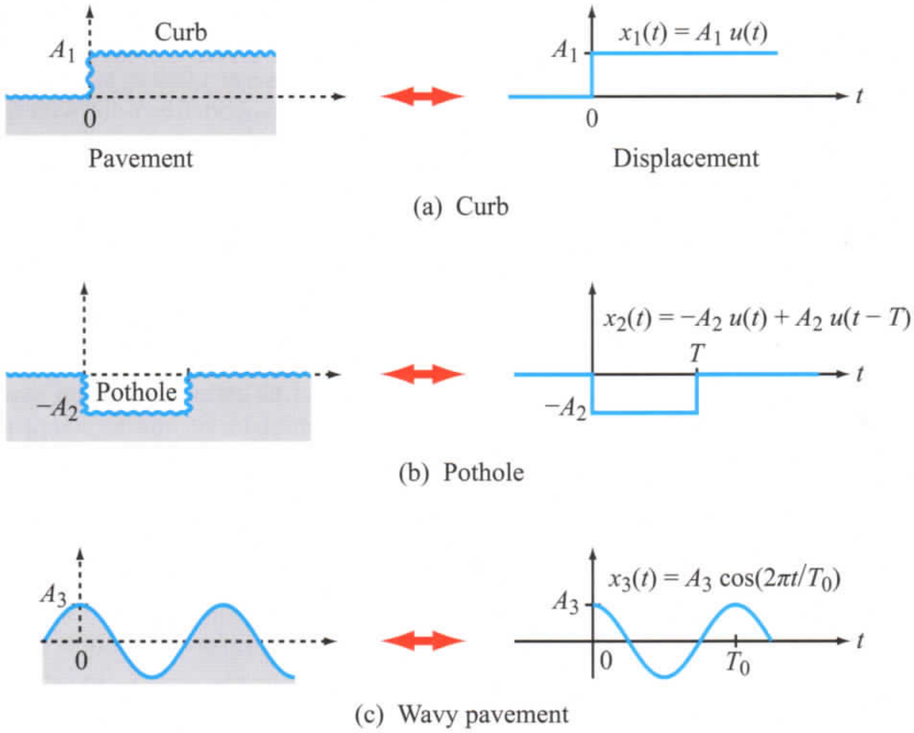


Figure 2-27: Pavement profiles and corresponding models for the vertical displacement $x(t)$.

and

$$A_2 = \frac{-(b_1 p_2 + b_2)}{p_1 - p_2} = \frac{-[80(-74.64) + 400]}{-5.36 + 74.64} = 80.42 \text{ s}^{-1}.$$

Hence, $y_1(t)$ in meters becomes

$$y_1(t) = [0.108(1 - e^{-74.64t}) - 0.008(1 - e^{-5.36t})]\, u(t) \text{ m}, \tag{2.163}$$

A plot of $y_1(t)$ is displayed in **Fig. 2-28**.

(b) Changing the value of b to 10^4 N·s/m leads to

$$a_1 = b_1 = \frac{10^4}{250} = 40 \text{ s}^{-1},$$

$$a_2 = b_2 = \frac{k}{m} = 400 \text{ s}^{-2} \text{ (unchanged)},$$

$$\omega_0 = \sqrt{a_2} = 20 \text{ rad/s (unchanged)},$$

$$\alpha = \frac{a_1}{2} = 20 \text{ Np/s},$$

and

$$\xi = \frac{\alpha}{\omega_0} = \frac{20}{20} = 1.$$

For this critically damped case, the expressions given in **Table 2-3** lead to

$$y_2(t) = 0.1[(1 - e^{-20t}) + 20te^{-20t}]\, u(t) \text{ m}. \tag{2.164}$$

From the plot of $y_2(t)$ in **Fig. 2-28**, it is easy to see that it approaches its final destination of 0.1 m (height of the curb) much sooner than the overdamped response exhibited by $y_1(t)$.

(c) For an old shock absorber with $b = 5000$ N·s/m, the parameter values are

$$a_1 = b_1 = \frac{5000}{250} = 20 \text{ s}^{-1},$$

$$a_2 = b_2 = \frac{k}{m} = 400 \text{ s}^{-2} \text{ (unchanged)},$$

$$\omega_0 = \sqrt{a_2} = 20 \text{ rad/s (unchanged)},$$

$$\alpha = \frac{a_1}{2} = \frac{20}{2} = 10 \text{ Np/s},$$

and

$$\xi = \frac{\alpha}{\omega_0} = \frac{10}{20} = 0.5.$$

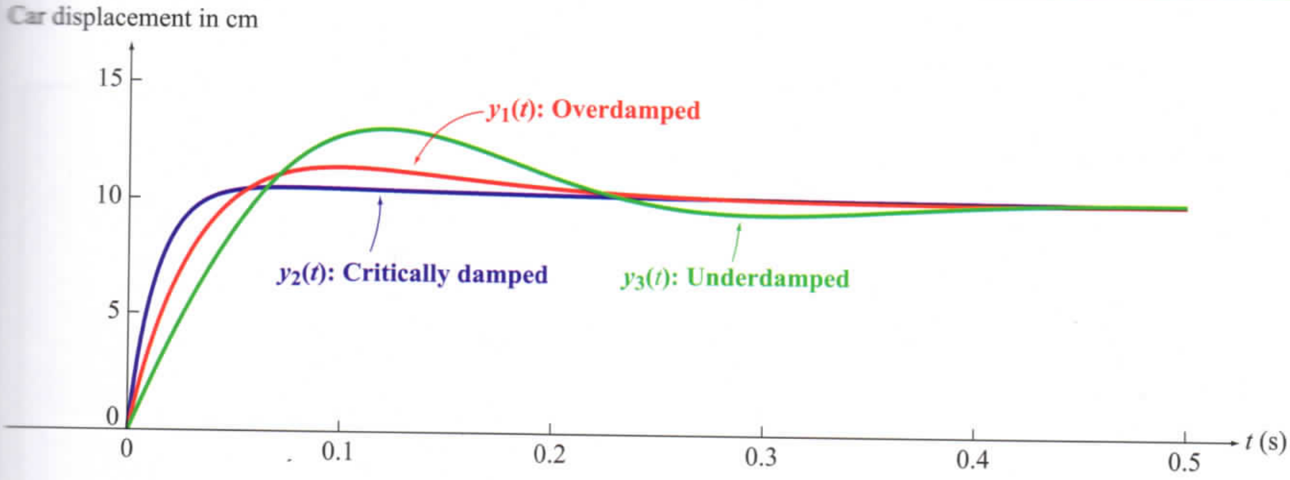


Figure 2-28: Car's response to driving over a 10 cm curb.

Since $\xi < 1$, the system is underdamped, which means that the car response to the curb will include some oscillations. Using Table 2-3, we have

$$\omega_d = \omega_0\sqrt{1-\xi^2} = 20\sqrt{1-0.25} = 17.32 \text{ rad/s}$$

and

$$y_3(t) = 0.1\{[-\cos 17.32t + 1.15\sin 17.32t]e^{-10t} + 1\}\, u(t) \text{ m}.$$

The oscillatory behavior of $y_3(t)$ is clearly evident in the plot of its profile in Fig. 2-28 (see S² for details).

Example 2-18: Car Response to a Pothole

Simulate the response of a car driven at 5 m/s over a 1 m long, 10 cm deep pothole if the damping constant of its shock absorber is (a) 10^4 N·s/m or (b) 2000 N·s/m. All other attributes are the same as those in Example 2-17, namely, $m = 250$ kg and $k = 10^5$ N/m.

Solution: (a) The travel time across the pothole is $T = \frac{1}{5} = 0.2$ s. According to part (b) of the solution of Example 2-17, $\xi = 1$ when $b = 10^4$ N·s/m, representing a critically damped system with the response given by Eq. (2.164) as

$$y_2(t) = 0.1[(1-e^{-20t}) + 20te^{-20t}]\, u(t) \text{ m}. \qquad (2.165)$$

The car's vertical displacement $y_2(t)$ is in response to a 0.1 m vertical step (curb). For the pothole model shown in Fig. 2-27, the response $y_4(t)$ can be synthesized as

$$\begin{aligned} y_4(t) &= -y_2(t) + y_2(t-0.2) \\ &= -0.1[(1-e^{-20t}) + 20te^{-20t}]\, u(t) \\ &\quad + 0.1(1-e^{-20(t-0.2)})\, u(t-0.2) \\ &\quad + 2(t-0.2)e^{-20(t-0.2)}\, u(t-0.2) \text{ m}. \end{aligned} \qquad (2.166)$$

Because the height of the curb and the depth of the pothole are both 0.1 m, no scaling was necessary in this case. For a pothole of depth A, the multiplying coefficient (0.1) should be replaced with A.

(b) For $b = 2000$ N·s/m, we have

$$a_1 = b_1 = \frac{2000}{250} = 8 \text{ s}^{-1},$$

$$a_2 = b_2 = \frac{k}{m} = 400 \text{ s}^{-2},$$

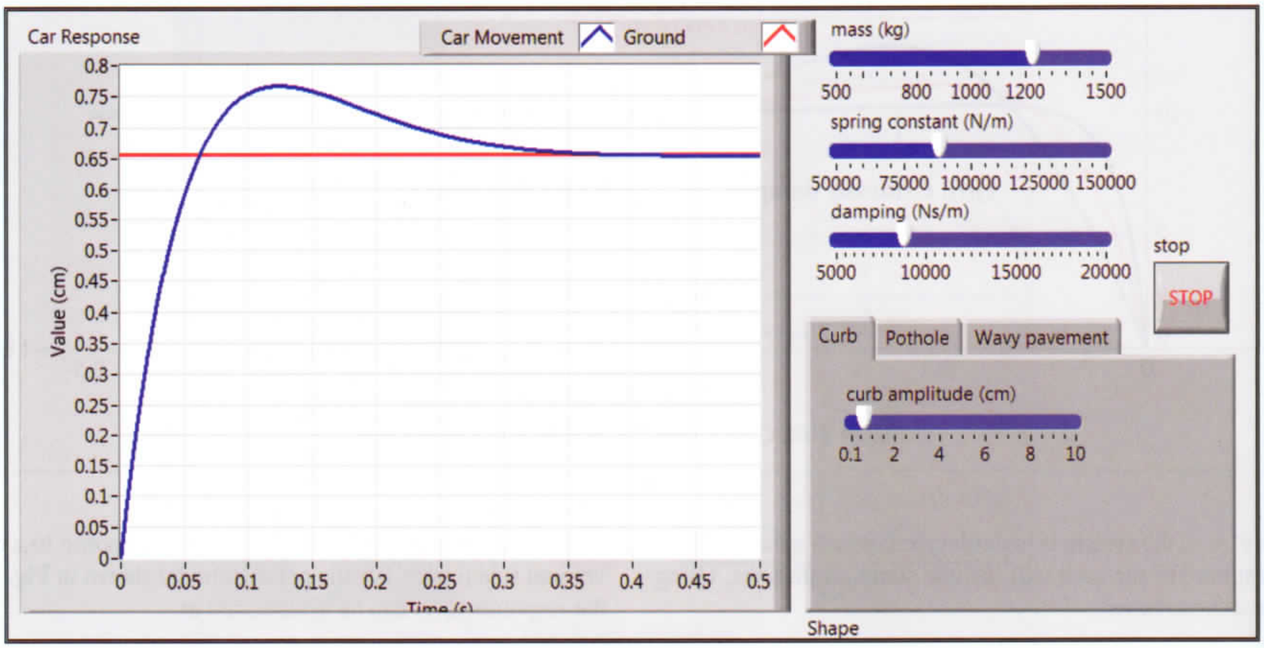
$$\omega_0 = \sqrt{a_2} = 20 \text{ rad/s},$$

$$\alpha = \frac{a_1}{2} = \frac{8}{2} = 4 \text{ Np/s},$$

and

$$\xi = \frac{\alpha}{\omega_0} = \frac{4}{20} = 0.2.$$

Module 2.2 Automobile Suspension Response Select curb, pothole, or wavy pavement. Then, select the pavement characteristics, the automobile's mass, and its suspension's spring constant and damping coefficient.



For this underdamped case, the expressions in **Table 2-3** lead to

$$\omega_d = \omega_0\sqrt{1-\xi^2} = 20\sqrt{1-0.2^2} = 19.6 \text{ rad/s}$$

and a unit-step response given by

$$y(t) = \{[-\cos 19.6t + 0.58 \sin 19.6t]e^{-4t} + 1\}\, u(t) \text{ m.} \tag{2.167}$$

For the pothole response, we have

$$\begin{aligned} y_5(t) &= -0.1\, y(t) + 0.1\, y(t-0.2) \\ &= 0.1\{[\cos 19.6t - 0.58 \sin 19.6t]e^{-4t} - 1\}\, u(t) \\ &\quad - 0.1\{[\cos(19.6(t-0.2)) \\ &\quad - 0.58\sin(19.6(t-0.2))]e^{-4(t-0.2)} \\ &\quad - 1\}\, u(t-0.2) \text{ m.} \end{aligned} \tag{2.168}$$

Plots for $y_4(t)$ and $y_5(t)$ are displayed in **Fig. 2-29** [see S² for details].

Example 2-19: Driving over Wavy Pavement

A 1,000 kg car is driven over a wavy pavement (**Fig. 2-27(c)**) of amplitude $A_3 = 5$ cm and a period $T_0 = 0.314$ s. The suspension system has a spring constant $k = 10^5$ N/m and a damping constant $b = 10^4$ N·s/m. Simulate the car displacement as a function of time.

Solution: The car suspension parameter values are

$$\frac{b}{m} = \frac{10^4}{250} = 40 \text{ s}^{-1}$$

and

$$\frac{k}{m} = \frac{10^5}{250} = 400 \text{ s}^{-2}.$$

Using these values in Eq. (2.158) gives

$$\frac{d^2y}{dt^2} + 40\,\frac{dy}{dt} + 400y = 40\,\frac{dx}{dt} + 400x. \tag{2.169}$$

Following the recipe outlined in Section 2-7.3, wherein we set $x(t) = e^{j\omega t}$ and $y(t) = \mathbf{H}(\omega)\, e^{j\omega t}$, Eq. (2.169) leads to the following expression for the frequency response function $\mathbf{H}(\omega)$:

$$\mathbf{H}(\omega) = \frac{400 + j40\omega}{(j\omega)^2 + j40\omega + 400} = \frac{400 + j40\omega}{(400-\omega^2) + j40\omega}. \tag{2.170}$$

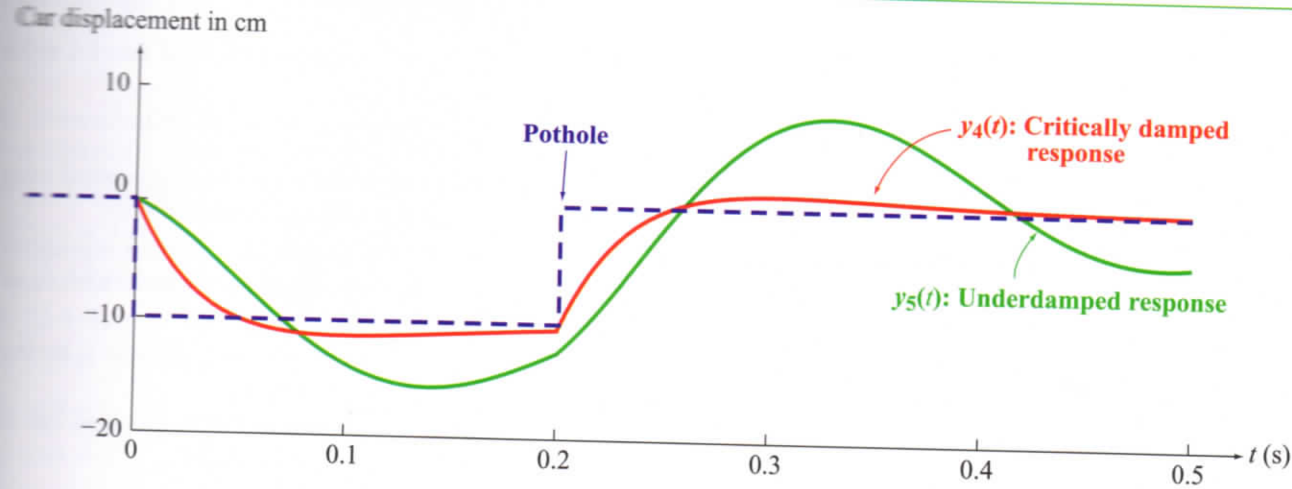


Figure 2-29: Car's response to driving over a 10 cm deep pothole.

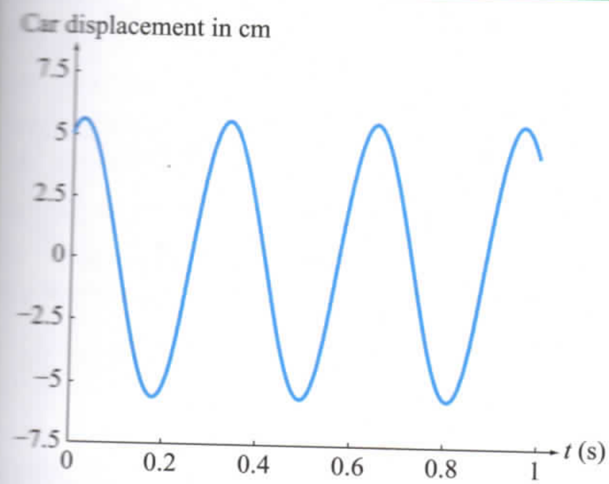


Figure 2-30: Car's response to driving over a wavy pavement with a 5 cm amplitude.

The angular frequency ω of the wavy pavement is

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{0.314} = 20 \text{ rad/s}. \tag{2.171}$$

Evaluating $\mathbf{H}(\omega)$ at $\omega_0 = 20$ rad/s gives

$$\mathbf{H}(20) = \frac{400 + j800}{(400 - 400) + j800} = 1 - j0.5 = 1.12e^{-j26.6^\circ}. \tag{2.172}$$

Application of Eq. (2.118) with a scaling amplitude of 0.05 m yields

$$\begin{aligned} y_6(t) &= 0.05|\mathbf{H}(20)| \cos(\omega_0 t + \theta) \\ &= 5.6 \cos(20t - 26.6^\circ) \times 10^{-2} \text{ m}. \end{aligned} \tag{2.173}$$

Note that the amplitude of $y_6(t)$ in **Fig. 2-30** is slightly greater than the amplitude of the pavement displacement [5.6 cm compared with 5 cm for $x(t)$]. This is an example of ***resonance***.

Exercise 2-17: Use LabVIEW Module 2.2 to compute the wavy pavement response in Example 2-19 and shown in **Fig. 2-30**.

Answer: