Problem Solutions – Chapter 6

Problem 6.1.1 Solution

The random variable X_{33} is a Bernoulli random variable that indicates the result of flip 33. The PMF of X_{33} is

$$P_{X_{33}}(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that each X_i has expected value $E[X] = p$ and variance $\text{Var}[X] = p(1-p)$. The random variable $Y = X_1 + \cdots + X_{100}$ is the number of heads in 100 coin flips. Hence, Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y} p^y (1-p)^{100-y} & y = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since the X_i are independent, by Theorems 6.1 and 6.3, the mean and variance of Y are

$$E[Y] = 100E[X] = 100p \qquad \text{Var}[Y] = 100\,\text{Var}[X] = 100p(1-p) \tag{3}$$

Problem 6.1.2 Solution

Let $Y = X_1 - X_2$.

(a) Since $Y = X_1 + (-X_2)$, Theorem 6.1 says that the expected value of the difference is

$$E[Y] = E[X_1] + E[-X_2] = E[X] - E[X] = 0 \tag{1}$$

(b) By Theorem 6.2, the variance of the difference is

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = 2\,\text{Var}[X] \tag{2}$$

Problem 6.1.3 Solution

(a) The PMF of N_1, the number of phone calls needed to obtain the correct answer, can be determined by observing that if the correct answer is given on the nth call, then the previous $n-1$ calls must have given wrong answers so that

$$P_{N_1}(n) = \begin{cases} (3/4)^{n-1}(1/4) & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) N_1 is a geometric random variable with parameter $p = 1/4$. In Theorem 2.5, the mean of a geometric random variable is found to be $1/p$. For our case, $E[N_1] = 4$.

(c) Using the same logic as in part (a) we recognize that in order for n to be the fourth correct answer, that the previous $n-1$ calls must have contained exactly 3 correct answers and that the fourth correct answer arrived on the n-th call. This is described by a Pascal random variable.

$$P_{N_4}(n_4) = \begin{cases} \binom{n-1}{3}(3/4)^{n-4}(1/4)^4 & n = 4, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(d) Using the hint given in the problem statement we can find the mean of N_4 by summing up the means of the 4 identically distributed geometric random variables each with mean 4. This gives $E[N_4] = 4E[N_1] = 16$.

Problem 6.1.4 Solution

We can solve this problem using Theorem 6.2 which says that

$$\mathrm{Var}[W] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}\,[X, Y] \tag{1}$$

The first two moments of X are

$$E\,[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3 \tag{2}$$

$$E\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6 \tag{3}$$

$$\tag{4}$$

Thus the variance of X is $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 1/18$. By symmetry, it should be apparent that $E[Y] = E[X] = 1/3$ and $\mathrm{Var}[Y] = \mathrm{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$E\,[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12 \tag{5}$$

The covariance is

$$\mathrm{Cov}\,[X, Y] = E\,[XY] - E\,[X]\,E\,[Y] = 1/12 - (1/3)^2 = -1/36 \tag{6}$$

Finally, the variance of the sum $W = X + Y$ is

$$\mathrm{Var}[W] = \mathrm{Var}[X] + \mathrm{Var}[Y] - 2\,\mathrm{Cov}\,[X, Y] = 2/18 - 2/36 = 1/18 \tag{7}$$

For this specific problem, it's arguable whether it would easier to find $\mathrm{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$F_W\,(w) = P\,[X + Y \le w] = \int_0^w \int_0^{w-x} 2\,dy\,dx = \int_0^w 2(w-x)\,dx = w^2 \tag{8}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W\,(w) = \begin{cases} 2w & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

From the PDF, the first and second moments of W are

$$E\,[W] = \int_0^1 2w^2\,dw = 2/3 \qquad E\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2 \tag{10}$$

The variance of W is $\mathrm{Var}[W] = E[W^2] - (E[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 6.1.5 Solution

This problem should be in either Chapter 10 or Chapter 11.

Since each X_i has zero mean, the mean of Y_n is

$$E\,[Y_n] = E\,[X_n + X_{n-1} + X_{n-2}]\,/3 = 0 \tag{1}$$

Since Y_n has zero mean, the variance of Y_n is

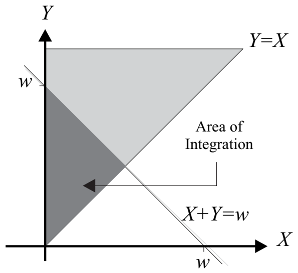
$$
\begin{aligned}
\text{Var}[Y_n] &= E\left[Y_n^2\right] && (2)\\
&= \frac{1}{9}E\left[(X_n + X_{n-1} + X_{n-2})^2\right] && (3)\\
&= \frac{1}{9}E\left[X_n^2 + X_{n-1}^2 + X_{n-2}^2 + 2X_nX_{n-1} + 2X_nX_{n-2} + 2X_{n-1}X_{n-2}\right] && (4)\\
&= \frac{1}{9}(1 + 1 + 1 + 2/4 + 0 + 2/4) = \frac{4}{9} && (5)
\end{aligned}
$$

Problem 6.2.1 Solution

The joint PDF of X and Y is

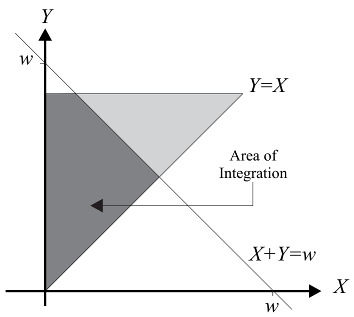
$$
f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}
$$

We wish to find the PDF of W where $W = X + Y$. First we find the CDF of W, $F_W(w)$, but we must realize that the CDF will require different integrations for different values of w.



For values of $0 \le w \le 1$ we look to integrate the shaded area in the figure to the right.

$$
F_W(w) = \int_0^{\frac{w}{2}} \int_x^{w-x} 2\,dy\,dx = \frac{w^2}{2} \tag{2}
$$



For values of w in the region $1 \le w \le 2$ we look to integrate over the shaded region in the graph to the right. From the graph we see that we can integrate with respect to x first, ranging y from 0 to $w/2$, thereby covering the lower right triangle of the shaded region and leaving the upper trapezoid, which is accounted for in the second term of the following expression:

$$
\begin{aligned}
F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\,dx\,dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\,dx\,dy && (3)\\
&= 2w - 1 - \frac{w^2}{2} && (4)
\end{aligned}
$$

Putting all the parts together gives the CDF $F_W(w)$ and (by taking the derivative) the PDF $f_W(w)$.

$$
F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \le w \le 1 \\ 2w - 1 - \frac{w^2}{2} & 1 \le w \le 2 \\ 1 & w > 2 \end{cases} \qquad f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{5}
$$

Problem 6.2.2 Solution

The joint PDF of X and Y is

$$
f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}
$$

Proceeding as in Problem 6.2.1, we must first find $F_W(w)$ by integrating over the square defined by $0 \leq x, y \leq 1$. Again we are forced to find $F_W(w)$ in parts as we did in Problem 6.2.1 resulting in the following integrals for their appropriate regions. For $0 \leq w \leq 1$,

$$F_W(w) = \int_0^w \int_0^{w-x} dx\,dy = w^2/2 \tag{2}$$

For $1 \leq w \leq 2$,

$$F_W(w) = \int_0^{w-1} \int_0^1 dx\,dy + \int_{w-1}^1 \int_0^{w-y} dx\,dy = 2w - 1 - w^2/2 \tag{3}$$

The complete CDF $F_W(w)$ is shown below along with the corresponding PDF $f_W(w) = dF_W(w)/dw$.

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/2 & 0 \leq w \leq 1 \\ 2w - 1 - w^2/2 & 1 \leq w \leq 2 \\ 1 & \text{otherwise} \end{cases} \qquad f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 6.2.3 Solution

By using Theorem 6.5, we can find the PDF of $W = X + Y$ by convolving the two exponential distributions. For $\mu \neq \lambda$,

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w - x)\, dx \tag{1}$$

$$= \int_0^w \lambda e^{-\lambda x} \mu e^{-\mu(w-x)}\, dx \tag{2}$$

$$= \lambda \mu e^{-\mu w} \int_0^w e^{-(\lambda-\mu)x}\, dx \tag{3}$$

$$= \begin{cases} \frac{\lambda\mu}{\lambda-\mu}\left(e^{-\mu w} - e^{-\lambda w}\right) & w \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

When $\mu = \lambda$, the previous derivation is invalid because of the denominator term $\lambda - \mu$. For $\mu = \lambda$, we have

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w - x)\, dx \tag{5}$$

$$= \int_0^w \lambda e^{-\lambda x} \lambda e^{-\lambda(w-x)}\, dx \tag{6}$$

$$= \lambda^2 e^{-\lambda w} \int_0^w dx \tag{7}$$

$$= \begin{cases} \lambda^2 w e^{-\lambda w} & w \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

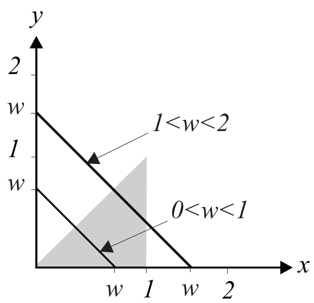
Note that when $\mu = \lambda$, W is the sum of two iid exponential random variables and has a second order Erlang PDF.

Problem 6.2.4 Solution

In this problem, X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the PDF of W using Theorem 6.4: $f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\,dx$. The only tricky part remaining is to determine the limits of the integration. First, for $w < 0$, $f_W(w) = 0$. The two remaining cases are shown in the accompanying figure. The shaded area shows where the joint PDF $f_{X,Y}(x,y)$ is nonzero. The diagonal lines depict $y = w - x$ as a function of x. The intersection of the diagonal line and the shaded area define our limits of integration.



For $0 \le w \le 1$,

$$f_W(w) = \int_{w/2}^{w} 8x(w-x)\,dx \tag{2}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{w} = 2w^3/3 \tag{3}$$

For $1 \le w \le 2$,

$$f_W(w) = \int_{w/2}^{1} 8x(w-x)\,dx \tag{4}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{1} \tag{5}$$

$$= 4w - 8/3 - 2w^3/3 \tag{6}$$

Since $X + Y \le 2$, $f_W(w) = 0$ for $w > 2$. Hence the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 2w^3/3 & 0 \le w \le 1 \\ 4w - 8/3 - 2w^3/3 & 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 6.2.5 Solution

We first find the CDF of W following the same procedure as in the proof of Theorem 6.4.

$$F_W(w) = P[X \le Y + w] = \int_{-\infty}^{\infty} \int_{-\infty}^{y+w} f_{X,Y}(x,y)\,dx\,dy \tag{1}$$

By taking the derivative with respect to w, we obtain

$$f_W(w) = \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} \frac{d}{dw}\left(\int_{-\infty}^{y+w} f_{X,Y}(x,y)\,dx\right) dy \tag{2}$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(w+y, y)\,dy \tag{3}$$

With the variable substitution $y = x - w$, we have $dy = dx$ and

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-w)\,dx \tag{4}$$

Problem 6.2.6 Solution

The random variables K and J have PMFs

$$P_J(j) = \begin{cases} \frac{\alpha^j e^{-\alpha}}{j!} & j = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad P_K(k) = \begin{cases} \frac{\beta^k e^{-\beta}}{k!} & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $n \geq 0$, we can find the PMF of $N = J + K$ via

$$P[N = n] = \sum_{k=-\infty}^{\infty} P[J = n - k, K = k] \tag{2}$$

Since J and K are independent, non-negative random variables,

$$\begin{aligned} P[N = n] &= \sum_{k=0}^{n} P_J(n-k) P_K(k) && (3) \\ &= \sum_{k=0}^{n} \frac{\alpha^{n-k} e^{-\alpha}}{(n-k)!} \frac{\beta^k e^{-\beta}}{k!} && (4) \\ &= \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} \underbrace{\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{\alpha}{\alpha+\beta}\right)^{n-k} \left(\frac{\beta}{\alpha+\beta}\right)^{k}}_{1} && (5) \end{aligned}$$

The marked sum above equals 1 because it is the sum of a binomial PMF over all possible values. The PMF of N is the Poisson PMF

$$P_N(n) = \begin{cases} \frac{(\alpha+\beta)^n e^{-(\alpha+\beta)}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 6.3.1 Solution

For a constant $a > 0$, a zero mean Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \qquad -\infty < x < \infty \tag{1}$$

The moment generating function of X is

$$\begin{aligned} \phi_X(s) = E\left[e^{sX}\right] &= \frac{a}{2} \int_{-\infty}^{0} e^{sx} e^{ax}\, dx + \frac{a}{2} \int_{0}^{\infty} e^{sx} e^{-ax}\, dx && (2) \\ &= \frac{a}{2} \frac{e^{(s+a)x}}{s+a} \bigg|_{-\infty}^{0} + \frac{a}{2} \frac{e^{(s-a)x}}{s-a} \bigg|_{0}^{\infty} && (3) \\ &= \frac{a}{2} \left(\frac{1}{s+a} - \frac{1}{s-a} \right) && (4) \\ &= \frac{a^2}{a^2 - s^2} && (5) \end{aligned}$$

Problem 6.3.2 Solution

(a) By summing across the rows of the table, we see that J has PMF

$$P_J(j) = \begin{cases} 0.6 & j = -2 \\ 0.4 & j = -1 \end{cases} \tag{1}$$

The MGF of J is $\phi_J(s) = E[e^{sJ}] = 0.6e^{-2s} + 0.4e^{-s}$.

(b) Summing down the columns of the table, we see that K has PMF

$$P_K(k) = \begin{cases} 0.7 & k = -1 \\ 0.2 & k = 0 \\ 0.1 & k = 1 \end{cases} \tag{2}$$

The MGF of K is $\phi_K(s) = 0.7e^{-s} + 0.2 + 0.1e^{s}$.

(c) To find the PMF of $M = J + K$, it is easist to annotate each entry in the table with the coresponding value of M:

$P_{J,K}(j,k)$	$k = -1$	$k = 0$	$k = 1$
$j = -2$	$0.42(M = -3)$	$0.12(M = -2)$	$0.06(M = -1)$
$j = -1$	$0.28(M = -2)$	$0.08(M = -1)$	$0.04(M = 0)$

(3)

We obtain $P_M(m)$ by summing over all j, k such that $j + k = m$, yielding

$$P_M(m) = \begin{cases} 0.42 & m = -3 \\ 0.40 & m = -2 \\ 0.14 & m = -1 \\ 0.04 & m = 0 \end{cases} \tag{4}$$

(d) One way to solve this problem, is to find the MGF $\phi_M(s)$ and then take four derivatives. Sometimes its better to just work with definition of $E[M^4]$:

$$E\left[M^4\right] = \sum_m P_M(m)\, m^4 \tag{5}$$

$$= 0.42(-3)^4 + 0.40(-2)^4 + 0.14(-1)^4 + 0.04(0)^4 = 40.434 \tag{6}$$

As best I can tell, the prupose of this problem is to check that you know when not to use the methods in this chapter.

Problem 6.3.3 Solution

We find the MGF by calculating $E[e^{sX}]$ from the PDF $f_X(x)$.

$$\phi_X(s) = E\left[e^{sX}\right] = \int_a^b e^{sX} \frac{1}{b-a}\, dx = \frac{e^{bs} - e^{as}}{s(b-a)} \tag{1}$$

Now to find the first moment, we evaluate the derivative of $\phi_X(s)$ at $s = 0$.

$$E[X] = \left.\frac{d\phi_X(s)}{ds}\right|_{s=0} = \left.\frac{s\left[be^{bs} - ae^{as}\right] - \left[e^{bs} - e^{as}\right]}{(b-a)s^2}\right|_{s=0} \tag{2}$$

Direct evaluation of the above expression at $s = 0$ yields $0/0$ so we must apply l'Hôpital's rule and differentiate the numerator and denominator.

$$E[X] = \lim_{s\to 0} \frac{be^{bs} - ae^{as} + s\left[b^2e^{bs} - a^2e^{as}\right] - \left[be^{bs} - ae^{as}\right]}{2(b-a)s} \tag{3}$$

$$= \lim_{s\to 0} \frac{b^2e^{bs} - a^2e^{as}}{2(b-a)} = \frac{b+a}{2} \tag{4}$$

To find the second moment of X, we first find that the second derivative of $\phi_X(s)$ is

$$\frac{d^2\phi_X(s)}{ds^2} = \frac{s^2\left[b^2e^{bs} - a^2e^{as}\right] - 2s\left[be^{bs} - ae^{as}\right] + 2\left[be^{bs} - ae^{as}\right]}{(b-a)s^3} \tag{5}$$

Substituting $s = 0$ will yield $0/0$ so once again we apply l'Hôpital's rule and differentiate the numerator and denominator.

$$E\left[X^2\right] = \lim_{s\to 0} \frac{d^2\phi_X(s)}{ds^2} = \lim_{s\to 0} \frac{s^2\left[b^3e^{bs} - a^3e^{as}\right]}{3(b-a)s^2} \tag{6}$$

$$= \frac{b^3 - a^3}{3(b-a)} = (b^2 + ab + a^2)/3 \tag{7}$$

In this case, it is probably simpler to find these moments without using the MGF.

Problem 6.3.4 Solution

Using the moment generating function of X, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the nth moment of X, $E[X^n]$ by taking the nth derivative of $\phi_X(s)$ and setting $s = 0$.

$$E[X] = \sigma^2 s e^{\sigma^2 s^2/2}\Big|_{s=0} = 0 \tag{1}$$

$$E\left[X^2\right] = \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\Big|_{s=0} = \sigma^2. \tag{2}$$

Continuing in this manner we find that

$$E\left[X^3\right] = \left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0 \tag{3}$$

$$E\left[X^4\right] = \left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4. \tag{4}$$

To calculate the moments of Y, we define $Y = X + \mu$ so that Y is Gaussian (μ, σ). In this case the second moment of Y is

$$E\left[Y^2\right] = E\left[(X+\mu)^2\right] = E\left[X^2 + 2\mu X + \mu^2\right] = \sigma^2 + \mu^2. \tag{5}$$

Similarly, the third moment of Y is

$$E\left[Y^3\right] = E\left[(X+\mu)^3\right] \tag{6}$$

$$= E\left[X^3 + 3\mu X^2 + 3\mu^2 X + \mu^3\right] = 3\mu\sigma^2 + \mu^3. \tag{7}$$

Finally, the fourth moment of Y is

$$E\left[Y^4\right] = E\left[(X+\mu)^4\right] \tag{8}$$

$$= E\left[X^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X + \mu^4\right] \tag{9}$$

$$= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4. \tag{10}$$

Problem 6.3.5 Solution

The PMF of K is

$$P_K(k) = \begin{cases} 1/n & k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The corresponding MGF of K is

$$\begin{aligned} \phi_K(s) = E\left[e^{sK}\right] &= \frac{1}{n}\left(e^s + e^2 s + \cdots + e^{ns}\right) &(2)\\ &= \frac{e^s}{n}\left(1 + e^s + e^{2s} + \cdots + e^{(n-1)s}\right) &(3)\\ &= \frac{e^s(e^{ns}-1)}{n(e^s-1)} &(4) \end{aligned}$$

We can evaluate the moments of K by taking derivatives of the MGF. Some algebra will show that

$$\frac{d\phi_K(s)}{ds} = \frac{ne^{(n+2)s} - (n+1)e^{(n+1)s} + e^s}{n(e^s-1)^2} \tag{5}$$

Evaluating $d\phi_K(s)/ds$ at $s = 0$ yields $0/0$. Hence, we apply l'Hôpital's rule twice (by twice differentiating the numerator and twice differentiating the denominator) when we write

$$\begin{aligned} \left.\frac{d\phi_K(s)}{ds}\right|_{s=0} &= \lim_{s\to 0} \frac{n(n+2)e^{(n+2)s} - (n+1)^2 e^{(n+1)s} + e^s}{2n(e^s-1)} &(6)\\ &= \lim_{s\to 0} \frac{n(n+2)^2 e^{(n+2)s} - (n+1)^3 e^{(n+1)s} + e^s}{2ne^s} = (n+1)/2 &(7) \end{aligned}$$

A significant amount of algebra will show that the second derivative of the MGF is

$$\frac{d^2\phi_K(s)}{ds^2} = \frac{n^2 e^{(n+3)s} - (2n^2+2n-1)e^{(n+2)s} + (n+1)^2 e^{(n+1)s} - e^{2s} - e^s}{n(e^s-1)^3} \tag{8}$$

Evaluating $d^2\phi_K(s)/ds^2$ at $s = 0$ yields $0/0$. Because $(e^s - 1)^3$ appears in the denominator, we need to use l'Hôpital's rule three times to obtain our answer.

$$\begin{aligned} \left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} &= \lim_{s\to 0} \frac{n^2(n+3)^3 e^{(n+3)s} - (2n^2+2n-1)(n+2)^3 e^{(n+2)s} + (n+1)^5 - 8e^{2s} - e^s}{6ne^s} &(9)\\ &= \frac{n^2(n+3)^3 - (2n^2+2n-1)(n+2)^3 + (n+1)^5 - 9}{6n} &(10)\\ &= (2n+1)(n+1)/6 &(11) \end{aligned}$$

We can use these results to derive two well known results. We observe that we can directly use the PMF $P_K(k)$ to calculate the moments

$$E[K] = \frac{1}{n}\sum_{k=1}^{n} k \qquad E\left[K^2\right] = \frac{1}{n}\sum_{k=1}^{n} k^2 \tag{12}$$

Using the answers we found for $E[K]$ and $E[K^2]$, we have the formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{13}$$

Problem 6.4.1 Solution

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Thus N is the sum of 100 independent Bernoulli $(p = 0.4)$ and M is the sum of 50 independent Bernoulli $(p = 0.4)$ random variables. Since M and N are independent, $L = M + N$ is the sum of 150 independent Bernoulli $(p = 0.4)$ random variables. Hence L is a binomial $n = 150, p = 0.4)$ and has PMF

$$P_L(l) = \binom{150}{l}(0.4)^l(0.6)^{150-l}. \tag{1}$$

Problem 6.4.2 Solution

Random variable Y has the moment generating function $\phi_Y(s) = 1/(1-s)$. Random variable V has the moment generating function $\phi_V(s) = 1/(1-s)^4$. Y and V are independent. $W = Y + V$.

(a) From Table 6.1, Y is an exponential $(\lambda = 1)$ random variable. For an exponential (λ) random variable, Example 6.5 derives the moments of the exponential random variable. For $\lambda = 1$, the moments of Y are

$$E[Y] = 1, \qquad E\left[Y^2\right] = 2, \qquad E\left[Y^3\right] = 3! = 6. \tag{1}$$

(b) Since Y and V are independent, $W = Y + V$ has MGF

$$\phi_W(s) = \phi_Y(s)\phi_V(s) = \left(\frac{1}{1-s}\right)\left(\frac{1}{1-s}\right)^4 = \left(\frac{1}{1-s}\right)^5. \tag{2}$$

W is the sum of five independent exponential $(\lambda = 1)$ random variables $X_1, \ldots, X_5$. (That is, W is an Erlang $(n = 5, \lambda = 1)$ random variable.) Each X_i has expected value $E[X] = 1$ and variance $\text{Var}[X] = 1$. From Theorem 6.1 and Theorem 6.3,

$$E[W] = 5E[X] = 5, \qquad \text{Var}[W] = 5\,\text{Var}[X] = 5. \tag{3}$$

It follows that

$$E\left[W^2\right] = \text{Var}[W] + (E[W])^2 = 5 + 25 = 30. \tag{4}$$

Problem 6.4.3 Solution

In the iid random sequence $K_1, K_2, \ldots$, each K_i has PMF

$$P_K(k) = \begin{cases} 1-p & k = 0, \\ p & k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The MGF of K is $\phi_K(s) = E[e^{sK}] = 1 - p + pe^s$.

(b) By Theorem 6.8, $M = K_1 + K_2 + \ldots + K_n$ has MGF

$$\phi_M(s) = [\phi_K(s)]^n = [1 - p + pe^s]^n \tag{2}$$

(c) Although we could just use the fact that the expectation of the sum equals the sum of the expectations, the problem asks us to find the moments using $\phi_M(s)$. In this case,

$$E[M] = \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} = n(1-p+pe^s)^{n-1}pe^s\big|_{s=0} = np \tag{3}$$

The second moment of M can be found via

$$\begin{aligned} E\left[M^2\right] &= \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} && (4)\\ &= np\left((n-1)(1-p+pe^s)pe^{2s} + (1-p+pe^s)^{n-1}e^s\right)\big|_{s=0} && (5)\\ &= np[(n-1)p+1] && (6) \end{aligned}$$

The variance of M is

$$\text{Var}[M] = E\left[M^2\right] - (E[M])^2 = np(1-p) = n\,\text{Var}[K] \tag{7}$$

Problem 6.4.4 Solution

Based on the problem statement, the number of points X_i that you earn for game i has PMF

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0,1,2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The MGF of X_i is

$$\phi_{X_i}(s) = E\left[e^{sX_i}\right] = 1/3 + e^s/3 + e^{2s}/3 \tag{2}$$

Since $Y = X_1 + \cdots + X_n$, Theorem 6.8 implies

$$\phi_Y(s) = [\phi_{X_i}(s)]^n = [1 + e^s + e^{2s}]^n/3^n \tag{3}$$

(b) First we observe that first and second moments of X_i are

$$\begin{aligned} E[X_i] &= \sum_x x P_{X_i}(x) = 1/3 + 2/3 = 1 && (4)\\ E\left[X_i^2\right] &= \sum_x x^2 P_{X_i}(x) = 1^2/3 + 2^2/3 = 5/3 && (5) \end{aligned}$$

Hence,

$$\text{Var}[X_i] = E\left[X_i^2\right] - (E[X_i])^2 = 2/3. \tag{6}$$

By Theorems 6.1 and 6.3, the mean and variance of Y are

$$\begin{aligned} E[Y] &= nE[X] = n && (7)\\ \text{Var}[Y] &= n\,\text{Var}[X] = 2n/3 && (8) \end{aligned}$$

Another more complicated way to find the mean and variance is to evaluate derivatives of $\phi_Y(s)$ as $s = 0$.

Problem 6.4.5 Solution

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

And let $R_i = K_1 + K_2 + \ldots + K_i$

(a) From Table 6.1, we find that the Poisson ($\alpha = 2$) random variable K has MGF $\phi_K(s) = e^{2(e^s-1)}$.

(b) The MGF of R_i is the product of the MGFs of the K_i's.

$$\phi_{R_i}(s) = \prod_{n=1}^{i} \phi_K(s) = e^{2i(e^s-1)} \tag{2}$$

(c) Since the MGF of R_i is of the same form as that of the Poisson with parameter, $\alpha = 2i$. Therefore we can conclude that R_i is in fact a Poisson random variable with parameter $\alpha = 2i$. That is,

$$P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i}/r! & r = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(d) Because R_i is a Poisson random variable with parameter $\alpha = 2i$, the mean and variance of R_i are then both $2i$.

Problem 6.4.6 Solution

The total energy stored over the 31 days is

$$Y = X_1 + X_2 + \cdots + X_{31} \tag{1}$$

The random variables $X_1, \ldots, X_{31}$ are Gaussian and independent but not identically distributed. However, since the sum of independent Gaussian random variables is Gaussian, we know that Y is Gaussian. Hence, all we need to do is find the mean and variance of Y in order to specify the PDF of Y. The mean of Y is

$$E[Y] = \sum_{i=1}^{31} E[X_i] = \sum_{i=1}^{31}(32 - i/4) = 32(31) - \frac{31(32)}{8} = 868 \text{ kW-hr} \tag{2}$$

Since each X_i has variance of $100(\text{kW-hr})^2$, the variance of Y is

$$\text{Var}[Y] = \text{Var}[X_1] + \cdots + \text{Var}[X_{31}] = 31\,\text{Var}[X_i] = 3100 \tag{3}$$

Since $E[Y] = 868$ and $\text{Var}[Y] = 3100$, the Gaussian PDF of Y is

$$f_Y(y) = \frac{1}{\sqrt{6200\pi}} e^{-(y-868)^2/6200} \tag{4}$$

Problem 6.4.7 Solution

By Theorem 6.8, we know that $\phi_M(s) = [\phi_K(s)]^n$.

(a) The first derivative of $\phi_M(s)$ is

$$\frac{d\phi_M(s)}{ds} = n\left[\phi_K(s)\right]^{n-1}\frac{d\phi_K(s)}{ds} \tag{1}$$

We can evaluate $d\phi_M(s)/ds$ at $s = 0$ to find $E[M]$.

$$E\left[M\right] = \left.\frac{d\phi_M(s)}{ds}\right|_{s=0} = n\left[\phi_K(s)\right]^{n-1}\left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = nE\left[K\right] \tag{2}$$

(b) The second derivative of $\phi_M(s)$ is

$$\frac{d^2\phi_M(s)}{ds^2} = n(n-1)\left[\phi_K(s)\right]^{n-2}\left(\frac{d\phi_K(s)}{ds}\right)^2 + n\left[\phi_K(s)\right]^{n-1}\frac{d^2\phi_K(s)}{ds^2} \tag{3}$$

Evaluating the second derivative at $s = 0$ yields

$$E\left[M^2\right] = \left.\frac{d^2\phi_M(s)}{ds^2}\right|_{s=0} = n(n-1)\left(E\left[K\right]\right)^2 + nE\left[K^2\right] \tag{4}$$

Problem 6.5.1 Solution

(a) From Table 6.1, we see that the exponential random variable X has MGF

$$\phi_X(s) = \frac{\lambda}{\lambda - s} \tag{1}$$

(b) Note that K is a geometric random variable identical to the geometric random variable X in Table 6.1 with parameter $p = 1 - q$. From Table 6.1, we know that random variable K has MGF

$$\phi_K(s) = \frac{(1-q)e^s}{1-qe^s} \tag{2}$$

Since K is independent of each X_i, $V = X_1 + \cdots + X_K$ is a random sum of random variables. From Theorem 6.12,

$$\phi_V(s) = \phi_K(\ln\phi_X(s)) = \frac{(1-q)\frac{\lambda}{\lambda-s}}{1-q\frac{\lambda}{\lambda-s}} = \frac{(1-q)\lambda}{(1-q)\lambda - s} \tag{3}$$

We see that the MGF of V is that of an exponential random variable with parameter $(1-q)\lambda$. The PDF of V is

$$f_V\left(v\right) = \begin{cases} (1-q)\lambda e^{-(1-q)\lambda v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 6.5.2 Solution

The number N of passes thrown has the Poisson PMF and MGF

$$P_N(n) = \begin{cases} (30)^n e^{-30}/n! & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N(s) = e^{30(e^s - 1)} \tag{1}$$

Let $X_i = 1$ if pass i is thrown and completed and otherwise $X_i = 0$. The PMF and MGF of each X_i is

$$P_{X_i}(x) = \begin{cases} 1/3 & x = 0 \\ 2/3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_{X_i}(s) = 1/3 + (2/3)e^s \tag{2}$$

The number of completed passes can be written as the random sum of random variables

$$K = X_1 + \cdots + X_N \tag{3}$$

Since each X_i is independent of N, we can use Theorem 6.12 to write

$$\phi_K(s) = \phi_N(\ln \phi_X(s)) = e^{30(\phi_X(s) - 1)} = e^{30(2/3)(e^s - 1)} \tag{4}$$

We see that K has the MGF of a Poisson random variable with mean $E[K] = 30(2/3) = 20$, variance $\text{Var}[K] = 20$, and PMF

$$P_K(k) = \begin{cases} (20)^k e^{-20}/k! & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Problem 6.5.3 Solution

In this problem, $Y = X_1 + \cdots + X_N$ is not a straightforward random sum of random variables because N and the X_i's are dependent. In particular, given $N = n$, then we know that there were exactly 100 heads in N flips. Hence, given N, $X_1 + \cdots + X_N = 100$ *no matter what is the actual value of N*. Hence $Y = 100$ every time and the PMF of Y is

$$P_Y(y) = \begin{cases} 1 & y = 100 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 6.5.4 Solution

Donovan McNabb's passing yardage is the random sum of random variables

$$V + Y_1 + \cdots + Y_K \tag{1}$$

where Y_i has the exponential PDF

$$f_{Y_i}(y) = \begin{cases} \frac{1}{15} e^{-y/15} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

From Table 6.1, the MGFs of Y and K are

$$\phi_Y(s) = \frac{1/15}{1/15 - s} = \frac{1}{1 - 15s} \qquad \phi_K(s) = e^{20(e^s - 1)} \tag{3}$$

From Theorem 6.12, V has MGF

$$\phi_V(s) = \phi_K(\ln \phi_Y(s)) = e^{20(\phi_Y(s) - s)} = e^{300s/(1-15s)} \tag{4}$$

The PDF of V cannot be found in a simple form. However, we can use the MGF to calculate the mean and variance. In particular,

$$E[V] = \left.\frac{d\phi_V(s)}{ds}\right|_{s=0} = \left.e^{300s/(1-15s)}\frac{300}{(1-15s)^2}\right|_{s=0} = 300 \tag{5}$$

$$E[V^2] = \left.\frac{d^2\phi_V(s)}{ds^2}\right|_{s=0} \tag{6}$$

$$= \left.e^{300s/(1-15s)}\left(\frac{300}{(1-15s)^2}\right)^2 + e^{300s/(1-15s)}\frac{9000}{(1-15s)^3}\right|_{s=0} = 99,000 \tag{7}$$

Thus, V has variance $\text{Var}[V] = E[V^2] - (E[V])^2 = 9,000$ and standard deviation $\sigma_V \approx 94.9$.

A second way to calculate the mean and variance of V is to use Theorem 6.13 which says

$$E[V] = E[K]\,E[Y] = 20(15) = 200 \tag{8}$$

$$\text{Var}[V] = E[K]\,\text{Var}[Y] + \text{Var}[K](E[Y])^2 = (20)15^2 + (20)15^2 = 9000 \tag{9}$$

Problem 6.5.5 Solution

Since each ticket is equally likely to have one of $\binom{46}{6}$ combinations, the probability a ticket is a winner is

$$q = \frac{1}{\binom{46}{6}} \tag{1}$$

Let $X_i = 1$ if the ith ticket sold is a winner; otherwise $X_i = 0$. Since the number K of tickets sold has a Poisson PMF with $E[K] = r$, the number of winning tickets is the random sum

$$V = X_1 + \cdots + X_K \tag{2}$$

From Appendix A,

$$\phi_X(s) = (1-q) + qe^s \qquad \phi_K(s) = e^{r[e^s-1]} \tag{3}$$

By Theorem 6.12,

$$\phi_V(s) = \phi_K(\ln\phi_X(s)) = e^{r[\phi_X(s)-1]} = e^{rq(e^s-1)} \tag{4}$$

Hence, we see that V has the MGF of a Poisson random variable with mean $E[V] = rq$. The PMF of V is

$$P_V(v) = \begin{cases} (rq)^v e^{-rq}/v! & v = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Problem 6.5.6 Solution

(a) We can view K as a shifted geometric random variable. To find the MGF, we start from first principles with Definition 6.1:

$$\phi_K(s) = \sum_{k=0}^{\infty} e^{sk}p(1-p)^k = p\sum_{n=0}^{\infty}[(1-p)e^s]^k = \frac{p}{1-(1-p)e^s} \tag{1}$$

(b) First, we need to recall that each X_i has MGF $\phi_X(s) = e^{s+s^2/2}$. From Theorem 6.12, the MGF of R is

$$\phi_R(s) = \phi_K(\ln \phi_X(s)) = \phi_K(s + s^2/2) = \frac{p}{1-(1-p)e^{s+s^2/2}} \tag{2}$$

(c) To use Theorem 6.13, we first need to calculate the mean and variance of K:

$$E[K] = \left.\frac{d\phi_K(s)}{ds}\right|_{s=0} = \left.\frac{p(1-p)e^s}{1-(1-p)e^s}^{2}\right|_{s=0} = \frac{1-p}{p} \tag{3}$$

$$E\left[K^2\right] = \left.\frac{d^2\phi_K(s)}{ds^2}\right|_{s=0} = p(1-p)\left.\frac{[1-(1-p)e^s]e^s + 2(1-p)e^{2s}}{[1-(1-p)e^s]^3}\right|_{s=0} \tag{4}$$

$$= \frac{(1-p)(2-p)}{p^2} \tag{5}$$

Hence, $\text{Var}[K] = E[K^2] - (E[K])^2 = (1-p)/p^2$. Finally. we can use Theorem 6.13 to write

$$\text{Var}[R] = E[K]\,\text{Var}[X] + (E[X])^2\,\text{Var}[K] = \frac{1-p}{p} + \frac{1-p}{p^2} = \frac{1-p^2}{p^2} \tag{6}$$

Problem 6.5.7 Solution

The way to solve for the mean and variance of U is to use conditional expectations. Given $K = k$, $U = X_1 + \cdots + X_k$ and

$$E[U|K=k] = E[X_1 + \cdots + X_k | X_1 + \cdots + X_n = k] \tag{1}$$

$$= \sum_{i=1}^{k} E[X_i | X_1 + \cdots + X_n = k] \tag{2}$$

Since X_i is a Bernoulli random variable,

$$E[X_i|X_1 + \cdots + X_n = k] = P\left[X_i = 1 | \sum_{j=1}^{n} X_j = k\right] \tag{3}$$

$$= \frac{P\left[X_i = 1, \sum_{j\neq i} X_j = k-1\right]}{P\left[\sum_{j=1}^{n} X_j = k\right]} \tag{4}$$

Note that $\sum_{j=1}^{n} X_j$ is just a binomial random variable for n trials while $\sum_{j\neq i} X_j$ is a binomial random variable for $n-1$ trials. In addition, X_i and $\sum_{j\neq i} X_j$ are independent random variables. This implies

$$E[X_i|X_1 + \cdots + X_n = k] = \frac{P[X_i = 1]\,P\left[\sum_{j\neq i} X_j = k-1\right]}{P\left[\sum_{j=1}^{n} X_j = k\right]} \tag{5}$$

$$= \frac{p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-1-(k-1)}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{k}{n} \tag{6}$$

A second way is to argue that symmetry implies $E[X_i|X_1 + \cdots + X_n = k] = \gamma$, the same for each i. In this case,

$$n\gamma = \sum_{i=1}^{n} E\left[X_i|X_1 + \cdots + X_n = k\right] = E\left[X_1 + \cdots + X_n|X_1 + \cdots + X_n = k\right] = k \tag{7}$$

Thus $\gamma = k/n$. At any rate, the conditional mean of U is

$$E\left[U|K = k\right] = \sum_{i=1}^{k} E\left[X_i|X_1 + \cdots + X_n = k\right] = \sum_{i=1}^{k} \frac{k}{n} = \frac{k^2}{n} \tag{8}$$

This says that the random variable $E[U|K] = K^2/n$. Using iterated expectations, we have

$$E\left[U\right] = E\left[E\left[U|K\right]\right] = E\left[K^2/n\right] \tag{9}$$

Since K is a binomial random variable, we know that $E[K] = np$ and $\text{Var}[K] = np(1-p)$. Thus,

$$E\left[U\right] = \frac{1}{n}E\left[K^2\right] = \frac{1}{n}\left(\text{Var}[K] + \left(E\left[K\right]\right)^2\right) = p(1-p) + np^2 \tag{10}$$

On the other hand, V is just and ordinary random sum of independent random variables and the mean of $E[V] = E[X]E[M] = np^2$.

Problem 6.5.8 Solution

Using N to denote the number of games played, we can write the total number of points earned as the random sum

$$Y = X_1 + X_2 + \cdots + X_N \tag{1}$$

(a) It is tempting to use Theorem 6.12 to find $\phi_Y(s)$; however, this would be wrong since each X_i is not independent of N. In this problem, we must start from first principles using iterated expectations.

$$\phi_Y(s) = E\left[E\left[e^{s(X_1+\cdots+X_N)}|N\right]\right] = \sum_{n=1}^{\infty} P_N\left(n\right) E\left[e^{s(X_1+\cdots+X_n)}|N = n\right] \tag{2}$$

Given $N = n$, $X_1, \ldots, X_n$ are independent so that

$$E\left[e^{s(X_1+\cdots+X_n)}|N = n\right] = E\left[e^{sX_1}|N = n\right] E\left[e^{sX_2}|N = n\right] \cdots E\left[e^{sX_n}|N = n\right] \tag{3}$$

Given $N = n$, we know that games 1 through $n-1$ were either wins or ties and that game n was a loss. That is, given $N = n$, $X_n = 0$ and for $i < n$, $X_i \neq 0$. Moreover, for $i < n$, X_i has the conditional PMF

$$P_{X_i|N=n}\left(x\right) = P_{X_i|X_i\neq 0}\left(x\right) = \begin{cases} 1/2 & x = 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

These facts imply

$$E\left[e^{sX_n}|N = n\right] = e^0 = 1 \tag{5}$$

and that for $i < n$,

$$E\left[e^{sX_i}|N = n\right] = (1/2)e^s + (1/2)e^{2s} = e^s/2 + e^{2s}/2 \tag{6}$$

Now we can find the MGF of Y.

$$\phi_Y(s) = \sum_{n=1}^{\infty} P_N(n)\, E\left[e^{sX_1}|N=n\right] E\left[e^{sX_2}|N=n\right] \cdots E\left[e^{sX_n}|N=n\right] \tag{7}$$

$$= \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^{n-1} = \frac{1}{e^s/2 + e^{2s}/2} \sum_{n=1}^{\infty} P_N(n) \left[e^s/2 + e^{2s}/2\right]^n \tag{8}$$

It follows that

$$\phi_Y(s) = \frac{1}{e^s/2 + e^{2s}/2} \sum_{n=1}^{\infty} P_N(n)\, e^{n \ln[(e^s + e^{2s})/2]} = \frac{\phi_N(\ln[e^s/2 + e^{2s}/2])}{e^s/2 + e^{2s}/2} \tag{9}$$

The tournament ends as soon as you lose a game. Since each game is a loss with probability 1/3 independent of any previous game, the number of games played has the geometric PMF and corresponding MGF

$$P_N(n) = \begin{cases} (2/3)^{n-1}(1/3) & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad \phi_N(s) = \frac{(1/3)e^s}{1 - (2/3)e^s} \tag{10}$$

Thus, the MGF of Y is

$$\phi_Y(s) = \frac{1/3}{1 - (e^s + e^{2s})/3} \tag{11}$$

(b) To find the moments of Y, we evaluate the derivatives of the MGF $\phi_Y(s)$. Since

$$\frac{d\phi_Y(s)}{ds} = \frac{e^s + 2e^{2s}}{9\left[1 - e^s/3 - e^{2s}/3\right]^2} \tag{12}$$

we see that

$$E[Y] = \left.\frac{d\phi_Y(s)}{ds}\right|_{s=0} = \frac{3}{9(1/3)^2} = 3 \tag{13}$$

If you're curious, you may notice that $E[Y] = 3$ precisely equals $E[N]E[X_i]$, the answer you would get if you mistakenly assumed that N and each X_i were independent. Although this may seem like a coincidence, its actually the result of theorem known as Wald's equality.

The second derivative of the MGF is

$$\frac{d^2\phi_Y(s)}{ds^2} = \frac{(1 - e^s/3 - e^{2s}/3)(e^s + 4e^{2s}) + 2(e^s + 2e^{2s})^2/3}{9(1 - e^s/3 - e^{2s}/3)^3} \tag{14}$$

The second moment of Y is

$$E\left[Y^2\right] = \left.\frac{d^2\phi_Y(s)}{ds^2}\right|_{s=0} = \frac{5/3 + 6}{1/3} = 23 \tag{15}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23 - 9 = 14$.

Problem 6.6.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \le w \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$.

(a) The expected value of X is $E[X] = E[W + 3] = E[W] + 3 = 5 + 3 = 8$.

(b) The variance of X is $\text{Var}[X] = \text{Var}[W + 3] = \text{Var}[W] = 25/3$.

(c) The expected value of A is $E[A] = 12E[X] = 96$.

(d) The standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$

Problem 6.6.2 Solution

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed withe the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16 \tag{2}$$

With these facts, we can answer the questions posed by the problem.

(a) $E[K_{100}] = 100E[D] = 20$

(b) $\text{Var}[K_{100}] = \sqrt{100\,\text{Var}[D]} = \sqrt{16} = 4$

(c) $P[K_{100} \ge 18] = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915$

(d) $P[16 \le K_{100} \le 24] = \Phi(\frac{24-20}{4}) - \Phi(\frac{16-20}{4}) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

Problem 6.6.3 Solution

(a) Let $X_1, \ldots, X_{120}$ denote the set of call durations (measured in minutes) during the month. From the problem statement, each $X - I$ is an exponential (λ) random variable with $E[X_i] = 1/\lambda = 2.5$ min and $\text{Var}[X_i] = 1/\lambda^2 = 6.25$ min^2. The total number of minutes used during the month is $Y = X_1 + \cdots + X_{120}$. By Theorem 6.1 and Theorem 6.3,

$$E[Y] = 120E[X_i] = 300 \qquad \text{Var}[Y] = 120\,\text{Var}[X_i] = 750. \tag{1}$$

The subscriber's bill is $30 + 0.4(y - 300)^+$ where $x^+ = x$ if $x \geq 0$ or $x^+ = 0$ if $x < 0$. the subscribers bill is exactly \$36 if $Y = 315$. The probability the subscribers bill exceeds \$36 equals

$$P[Y > 315] = P\left[\frac{Y - 300}{\sigma_Y} > \frac{315 - 300}{\sigma_Y}\right] = Q\left(\frac{15}{\sqrt{750}}\right) = 0.2919. \tag{2}$$

(b) If the actual call duration is X_i, the subscriber is billed for $M_i = \lceil X_i \rceil$ minutes. Because each X_i is an exponential (λ) random variable, Theorem 3.9 says that M_i is a geometric (p) random variable with $p = 1 - e^{-\lambda} = 0.3297$. Since M_i is geometric,

$$E[M_i] = \frac{1}{p} = 3.033, \qquad \text{Var}[M_i] = \frac{1-p}{p^2} = 6.167. \tag{3}$$

The number of billed minutes in the month is $B = M_1 + \cdots + M_{120}$. Since $M_1, \ldots, M_{120}$ are iid random variables,

$$E[B] = 120E[M_i] = 364.0, \qquad \text{Var}[B] = 120\,\text{Var}[M_i] = 740.08. \tag{4}$$

Similar to part (a), the subscriber is billed \$36 if $B = 315$ minutes. The probability the subscriber is billed more than \$36 is

$$P[B > 315] = P\left[\frac{B - 364}{\sqrt{740.08}} > \frac{315 - 365}{\sqrt{740.08}}\right] = Q(-1.8) = \Phi(1.8) = 0.964. \tag{5}$$

Problem 6.7.1 Solution

In Problem 6.2.6, we learned that a sum of iid Poisson random variables is a Poisson random variable. Hence W_n is a Poisson random variable with mean $E[W_n] = nE[K] = n$. Thus W_n has variance $\text{Var}[W_n] = n$ and PMF

$$P_{W_n}(w) = \begin{cases} n^w e^{-n}/w! & w = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

All of this implies that we can exactly calculate

$$P[W_n = n] = P_{W_n}(n) = n^n e^{-n}/n! \tag{2}$$

Since we can perform the exact calculation, using a central limit theorem may seem silly; however for large n, calculating n^n or $n!$ is difficult for large n. Moreover, it's interesting to see how good the approximation is. In this case, the approximation is

$$P[W_n = n] = P[n \leq W_n \leq n] \approx \Phi\left(\frac{n + 0.5 - n}{\sqrt{n}}\right) - \Phi\left(\frac{n - 0.5 - n}{\sqrt{n}}\right) = 2\Phi\left(\frac{1}{2\sqrt{n}}\right) - 1 \tag{3}$$

The comparison of the exact calculation and the approximation are given in the following table.

$P[W_n = n]$	$n = 1$	$n = 4$	$n = 16$	$n = 64$
exact	0.3679	0.1954	0.0992	0.0498
approximate	0.3829	0.1974	0.0995	0.0498

(4)

Problem 6.7.2 Solution

(a) Since the number of requests N has expected value $E[N] = 300$ and variance $\text{Var}[N] = 300$, we need C to satisfy

$$P[N > C] = P\left[\frac{N-300}{\sqrt{300}} > \frac{C-300}{\sqrt{300}}\right] \tag{1}$$

$$= 1 - \Phi\left(\frac{C-300}{\sqrt{300}}\right) = 0.05. \tag{2}$$

From Table 3.1, we note that $\Phi(1.65) = 0.9505$. Thus,

$$C = 300 + 1.65\sqrt{300} = 328.6. \tag{3}$$

(b) For $C = 328.6$, the exact probability of overload is

$$P[N > C] = 1 - P[N \leq 328] = 1 - \texttt{poissoncdf(300,328)} = 0.0516, \tag{4}$$

which shows the central limit theorem approximation is reasonable.

(c) This part of the problem could be stated more carefully. Re-examining Definition 2.10 for the Poisson random variable and the accompanying discussion in Chapter 2, we observe that the webserver has an arrival rate of $\lambda = 300$ hits/min, or equivalently $\lambda = 5$ hits/sec. Thus in a one second interval, the number of requests N' is a Poisson ($\alpha = 5$) random variable.

However, since the server "capacity" in a one second interval is not precisely defined, we will make the somewhat arbitrary definition that the server capacity is $C' = 328.6/60 = 5.477$ packets/sec. With this somewhat arbitrary definition, the probability of overload in a one second interval is

$$P\left[N' > C'\right] = 1 - P\left[N' \leq 5.477\right] = 1 - P\left[N' \leq 5\right]. \tag{5}$$

Because the number of arrivals in the interval is small, it would be a mistake to use the Central Limit Theorem to estimate this overload probability. However, the direct calculation of the overload probability is not hard. For $E[N'] = \alpha = 5$,

$$1 - P\left[N' \leq 5\right] = 1 - \sum_{n=0}^{5} P_N(n) = 1 - e^{-\alpha} \sum_{n=0}^{5} \frac{\alpha^n}{n!} = 0.3840. \tag{6}$$

(d) Here we find the smallest C such that $P[N' \leq C] \geq 0.95$. From the previous step, we know that $C > 5$. Since N' is a Poisson ($\alpha = 5$) random variable, we need to find the smallest C such that

$$P[N \leq C] = \sum_{n=0}^{C} \alpha^n e^{-\alpha}/n! \geq 0.95. \tag{7}$$

Some experiments with `poissoncdf(alpha,c)` will show that $P[N \leq 8] = 0.9319$ while $P[N \leq 9] = 0.9682$. Hence $C = 9$.

(e) If we use the Central Limit theorem to estimate the overload probability in a one second interval, we would use the facts that $E[N'] = 5$ and $\text{Var}[N'] = 5$ to estimate the the overload probability as

$$1 - P\left[N' \leq 5\right] = 1 - \Phi\left(\frac{5-5}{\sqrt{5}}\right) = 0.5 \tag{8}$$

which overestimates the overload probability by roughly 30 percent. We recall from Chapter 2 that a Poisson random is the limiting case of the (n, p) binomial random variable when n is large and $np = \alpha$.In general, for fixed p, the Poisson and binomial PMFs become closer as n increases. Since large n is also the case for which the central limit theorem applies, it is not surprising that the the CLT approximation for the Poisson (α) CDF is better when $\alpha = np$ is large.

Comment: Perhaps a more interesting question is why the overload probability in a one-second interval is so much higher than that in a one-minute interval? To answer this, consider a T-second interval in which the number of requests N_T is a Poisson (λT) random variable while the server capacity is cT hits. In the earlier problem parts, $c = 5.477$ hits/sec. We make the assumption that the server system is reasonably well-engineered in that $c > \lambda$. (We will learn in Chapter 12 that to assume otherwise means that the backlog of requests will grow without bound.) Further, assuming T is fairly large, we use the CLT to estimate the probability of overload in a T-second interval as

$$P\left[N_T \geq cT\right] = P\left[\frac{N_T - \lambda T}{\sqrt{\lambda T}} \geq \frac{cT - \lambda T}{\sqrt{\lambda T}}\right] = Q\left(k\sqrt{T}\right), \tag{9}$$

where $k = (c - \lambda)/\sqrt{\lambda}$. As long as $c > \lambda$, the overload probability decreases with increasing T. In fact, the overload probability goes rapidly to zero as T becomes large. The reason is that the gap $cT - \lambda T$ between server capacity cT and the expected number of requests λT grows linearly in T while the standard deviation of the number of requests grows proportional to $\sqrt{T}$. However, one should add that the definition of a T-second overload is somewhat arbitrary. In fact, one can argue that as T becomes large, the requirement for no overloads simply becomes less stringent. In Chapter 12, we will learn techniques to analyze a system such as this webserver in terms of the average backlog of requests and the average delay in serving in serving a request. These statistics won't depend on a particular time period T and perhaps better describe the system performance.

Problem 6.7.3 Solution

(a) The number of tests L needed to identify 500 acceptable circuits is a Pascal $(k = 500, p = 0.8)$ random variable, which has expected value $E[L] = k/p = 625$ tests.

(b) Let K denote the number of acceptable circuits in $n = 600$ tests. Since K is binomial $(n = 600, p = 0.8)$, $E[K] = np = 480$ and $\text{Var}[K] = np(1-p) = 96$. Using the CLT, we estimate the probability of finding at least 500 acceptable circuits as

$$P\left[K \geq 500\right] = P\left[\frac{K - 480}{\sqrt{96}} \geq \frac{20}{\sqrt{96}}\right] \approx Q\left(\frac{20}{\sqrt{96}}\right) = 0.0206. \tag{1}$$

(c) Using MATLAB, we observe that

```
1.0-binomialcdf(600,0.8,499)
ans =
    0.0215
```

(d) We need to find the smallest value of n such that the binomial (n, p) random variable K satisfies $P[K \geq 500] \geq 0.9$. Since $E[K] = np$ and $\text{Var}[K] = np(1-p)$, the CLT approximation yields

$$P[K \geq 500] = P\left[\frac{K - np}{\sqrt{np(1-p)}} \geq \frac{500 - np}{\sqrt{np(1-p)}}\right] \approx 1 - \Phi(z) = 0.90. \qquad (2)$$

where $z = (500 - np)/\sqrt{np(1-p)}$. It follows that $1 - \Phi(z) = \Phi(-z) \geq 0.9$, implying $z = -1.29$. Since $p = 0.8$, we have that

$$np - 500 = 1.29\sqrt{np(1-p)}. \qquad (3)$$

Equivalently, for $p = 0.8$, solving the quadratic equation

$$\left(n - \frac{500}{p}\right)^2 = (1.29)^2 \frac{1-p}{p} n \qquad (4)$$

we obtain $n = 641.3$. Thus we should test $n = 642$ circuits.

Problem 6.8.1 Solution

The $N[0, 1]$ random variable Z has MGF $\phi_Z(s) = e^{s^2/2}$. Hence the Chernoff bound for Z is

$$P[Z \geq c] \leq \min_{s \geq 0} e^{-sc} e^{s^2/2} = \min_{s \geq 0} e^{s^2/2 - sc} \qquad (1)$$

We can minimize $e^{s^2/2-sc}$ by minimizing the exponent $s^2/2 - sc$. By setting

$$\frac{d}{ds}\left(s^2/2 - sc\right) = 2s - c = 0 \qquad (2)$$

we obtain $s = c$. At $s = c$, the upper bound is $P[Z \geq c] \leq e^{-c^2/2}$. The table below compares this upper bound to the true probability. Note that for $c = 1, 2$ we use Table 3.1 and the fact that $Q(c) = 1 - \Phi(c)$.

	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
Chernoff bound	0.606	0.135	0.011	3.35×10^{-4}	3.73×10^{-6}
$Q(c)$	0.1587	0.0228	0.0013	3.17×10^{-5}	2.87×10^{-7}

(3)

We see that in this case, the Chernoff bound typically overestimates the true probability by roughly a factor of 10.

Problem 6.8.2 Solution

For an $N[\mu, \sigma^2]$ random variable X, we can write

$$P[X \geq c] = P[(X - \mu)/\sigma \geq (c - \mu)/\sigma] = P[Z \geq (c - \mu)/\sigma] \qquad (1)$$

Since Z is $N[0, 1]$, we can apply the result of Problem 6.8.1 with c replaced by $(c - \mu)/\sigma$. This yields

$$P[X \geq c] = P[Z \geq (c - \mu)/\sigma] \leq e^{-(c-\mu)^2/2\sigma^2} \qquad (2)$$

Problem 6.8.3 Solution

From Appendix A, we know that the MGF of K is

$$\phi_K(s) = e^{\alpha(e^s-1)} \tag{1}$$

The Chernoff bound becomes

$$P[K \geq c] \leq \min_{s\geq 0} e^{-sc} e^{\alpha(e^s-1)} = \min_{s\geq 0} e^{\alpha(e^s-1)-sc} \tag{2}$$

Since e^y is an increasing function, it is sufficient to choose s to minimize $h(s) = \alpha(e^s - 1) - sc$. Setting $dh(s)/ds = \alpha e^s - c = 0$ yields $e^s = c/\alpha$ or $s = \ln(c/\alpha)$. Note that for $c < \alpha$, the minimizing s is negative. In this case, we choose $s = 0$ and the Chernoff bound is $P[K \geq c] \leq 1$. For $c \geq \alpha$, applying $s = \ln(c/\alpha)$ yields $P[K \geq c] \leq e^{-\alpha}(\alpha e/c)^c$. A complete expression for the Chernoff bound is

$$P[K \geq c] \leq \begin{cases} 1 & c < \alpha \\ \alpha^c e^c e^{-\alpha}/c^c & c \geq \alpha \end{cases} \tag{3}$$

Problem 6.8.4 Solution

This problem is solved completely in the solution to Quiz 6.8! We repeat that solution here. Since $W = X_1 + X_2 + X_3$ is an Erlang $(n = 3, \lambda = 1/2)$ random variable, Theorem 3.11 says that for any $w > 0$, the CDF of W satisfies

$$F_W(w) = 1 - \sum_{k=0}^{2} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \tag{1}$$

Equivalently, for $\lambda = 1/2$ and $w = 20$,

$$P[W > 20] = 1 - F_W(20) \tag{2}$$

$$= e^{-10}\left(1 + \frac{10}{1!} + \frac{10^2}{2!}\right) = 61e^{-10} = 0.0028 \tag{3}$$

Problem 6.8.5 Solution

Let $W_n = X_1 + \cdots + X_n$. Since $M_n(X) = W_n/n$, we can write

$$P[M_n(X) \geq c] = P[W_n \geq nc] \tag{1}$$

Since $\phi_{W_n}(s) = (\phi_X(s))^n$, applying the Chernoff bound to W_n yields

$$P[W_n \geq nc] \leq \min_{s\geq 0} e^{-snc}\phi_{W_n}(s) = \min_{s\geq 0} \left(e^{-sc}\phi_X(s)\right)^n \tag{2}$$

For $y \geq 0$, y^n is a nondecreasing function of y. This implies that the value of s that minimizes $e^{-sc}\phi_X(s)$ also minimizes $(e^{-sc}\phi_X(s))^n$. Hence

$$P[M_n(X) \geq c] = P[W_n \geq nc] \leq \left(\min_{s\geq 0} e^{-sc}\phi_X(s)\right)^n \tag{3}$$

Problem 6.9.1 Solution

Note that W_n is a binomial $(10^n, 0.5)$ random variable. We need to calculate

$$P[B_n] = P\left[0.499 \times 10^n \le W_n \le 0.501 \times 10^n\right] \quad (1)$$
$$= P\left[W_n \le 0.501 \times 10^n\right] - P\left[W_n < 0.499 \times 10^n\right]. \quad (2)$$

A complication is that the event $W_n < w$ is not the same as $W_n \le w$ when w is an integer. In this case, we observe that

$$P[W_n < w] = P[W_n \le \lceil w \rceil - 1] = F_{W_n}(\lceil w \rceil - 1) \quad (3)$$

Thus

$$P[B_n] = F_{W_n}(0.501 \times 10^n) - F_{W_n}\left(\lceil 0.499 \times 10^9 \rceil - 1\right) \quad (4)$$

For $n = 1, \ldots, N$, we can calculate $P[B_n]$ in this MATLAB program:

```
function pb=binomialcdftest(N);
pb=zeros(1,N);
for n=1:N,
    w=[0.499 0.501]*10^n;
    w(1)=ceil(w(1))-1;
    pb(n)=diff(binomialcdf(10^n,0.5,w));
end
```

Unfortunately, on this user's machine (a Windows XP laptop), the program fails for $N = 4$. The problem, as noted earlier is that `binomialcdf.m` uses `binomialpmf.m`, which fails for a binomial $(10000, p)$ random variable. Of course, your mileage may vary. A slightly better solution is to use the `bignomialcdf.m` function, which is identical to `binomialcdf.m` except it calls `bignomialpmf.m` rather than `binomialpmf.m`. This enables calculations for larger values of n, although at some cost in numerical accuracy. Here is the code:

```
function pb=bignomialcdftest(N);
pb=zeros(1,N);
for n=1:N,
    w=[0.499 0.501]*10^n;
    w(1)=ceil(w(1))-1;
    pb(n)=diff(bignomialcdf(10^n,0.5,w));
end
```

For comparison, here are the outputs of the two programs:

```
>> binomialcdftest(4)
ans =
  0.2461  0.0796  0.0756     NaN
>> bignomialcdftest(6)
ans =
  0.2461  0.0796  0.0756  0.1663  0.4750  0.9546
```

The result 0.9546 for $n = 6$ corresponds to the exact probability in Example 6.15 which used the CLT to estimate the probability as 0.9544. Unfortunately for this user, for $n = 7$, `bignomialcdftest(7)` failed.

Problem 6.9.2 Solution

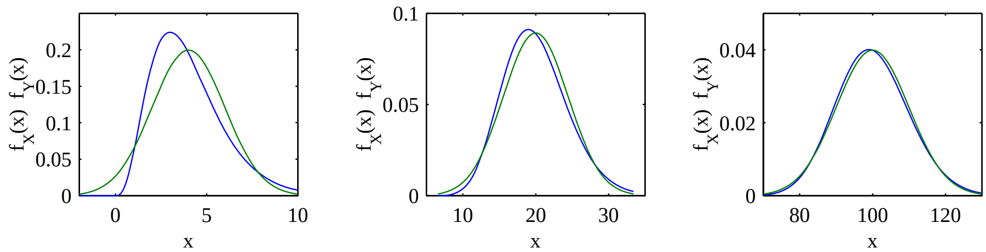
The Erlang $(n, \lambda = 1)$ random variable X has expected value $E[X] = n/\lambda = n$ and variance $\text{Var}[X] = n/\lambda^2 = n$. The PDF of X as well as the PDF of a Gaussian random variable Y with the same expected value and variance are

$$f_X(x) = \begin{cases} \frac{x^{n-1}e^{-x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(x) = \frac{1}{\sqrt{2\pi n}} e^{-x^2/2n} \tag{1}$$

```
function df=erlangclt(n);
r=3*sqrt(n);
x=(n-r):(2*r)/100:n+r;
fx=erlangpdf(n,1,x);
fy=gausspdf(n,sqrt(n),x);
plot(x,fx,x,fy);
df=fx-fy;
```

From the forms of the functions, it not likely to be apparent that $f_X(x)$ and $f_Y(x)$ are similar. The following program plots $f_X(x)$ and $f_Y(x)$ for values of x within three standard deviations of the expected value n. Below are sample outputs of `erlangclt(n)` for $n = 4, 20, 100$.

In the graphs we will see that as n increases, the Erlang PDF becomes increasingly similar to the Gaussian PDF of the same expected value and variance. This is not surprising since the Erlang (n, λ) random variable is the sum of n of exponential random variables and the CLT says that the Erlang CDF should converge to a Gaussian CDF as n gets large.



`erlangclt(4)` `erlangclt(20)` `erlangclt(100)`

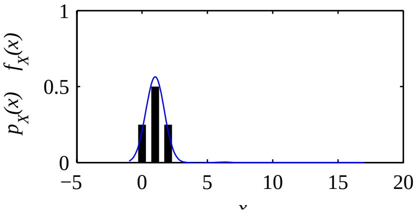
On the other hand, the convergence should be viewed with some caution. For example, the mode (the peak value) of the Erlang PDF occurs at $x = n-1$ while the mode of the Gaussian PDF is at $x = n$. This difference only appears to go away for $n = 100$ because the graph x-axis range is expanding. More important, the two PDFs are quite different far away from the center of the distribution. The Erlang PDF is always zero for $x < 0$ while the Gaussian PDF is always positive. For large postive x, the two distributions do not have the same exponential decay. Thus it's not a good idea to use the CLT to estimate probabilities of rare events such as $\{X > x\}$ for extremely large values of x.

Problem 6.9.3 Solution

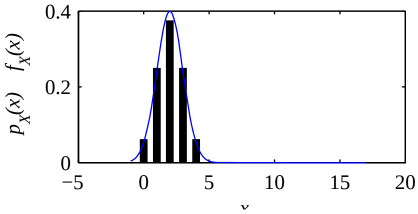
In this problem, we re-create the plots of Figure 6.3 except we use the binomial PMF and corresponding Gaussian PDF. Here is a MATLAB program that compares the binomial (n, p) PMF and the Gaussian PDF with the same expected value and variance.

```
function y=binomcltpmf(n,p)
x=-1:17;
xx=-1:0.05:17;
y=binomialpmf(n,p,x);
std=sqrt(n*p*(1-p));
clt=gausspdf(n*p,std,xx);
hold off;
pmfplot(x,y,'\it x','\it p_X(x)    f_X(x)');
hold on; plot(xx,clt); hold off;
```

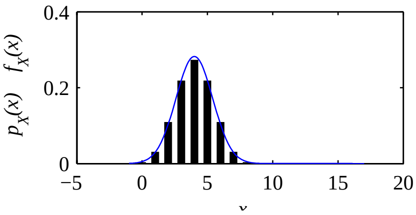
Here are the output plots for $p = 1/2$ and $n = 2, 4, 8, 16$.



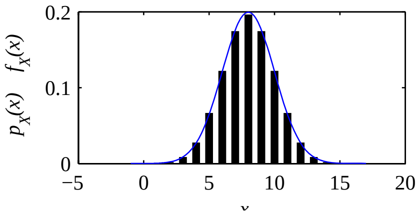
`binomcltpmf(2,0.5)`



`binomcltpmf(4,0.5)`



`binomcltpmf(8,0.5)`



`binomcltpmf(16,0.5)`

To see why the values of the PDF and PMF are roughly the same, consider the Gaussian random variable Y. For small Δ,

$$f_Y(x)\,\Delta \approx \frac{F_Y(x+\Delta/2) - F_Y(x-\Delta/2)}{\Delta}. \tag{1}$$

For $\Delta = 1$, we obtain

$$f_Y(x) \approx F_Y(x+1/2) - F_Y(x-1/2). \tag{2}$$

Since the Gaussian CDF is approximately the same as the CDF of the binomial (n, p) random variable X, we observe for an integer x that

$$f_Y(x) \approx F_X(x+1/2) - F_X(x-1/2) = P_X(x). \tag{3}$$

Although the equivalence in heights of the PMF and PDF is only an approximation, it can be useful for checking the correctness of a result.

Problem 6.9.4 Solution

Since the `conv` function is for convolving signals in time, we treat $P_{X_1}(x)$ and $P_{X_2}(x_2)x$, or as though they were signals in time starting at time $x = 0$. That is,

$$\texttt{px1} = \begin{bmatrix} P_{X_1}(0) & P_{X_1}(1) & \cdots & P_{X_1}(25) \end{bmatrix} \tag{1}$$

$$\texttt{px2} = \begin{bmatrix} P_{X_2}(0) & P_{X_2}(1) & \cdots & P_{X_2}(100) \end{bmatrix} \tag{2}$$

```
%convx1x2.m
sw=(0:125);
px1=[0,0.04*ones(1,25)];
px2=zeros(1,101);
px2(10*(1:10))=10*(1:10)/550;
pw=conv(px1,px2);
h=pmfplot(sw,pw,...
    '\itw','\itP_W(w)');
set(h,'LineWidth',0.25);
```

In particular, between its minimum and maximum values, the vector `px2` must enumerate all integer values, including those which have zero probability. In addition, we write down `sw=0:125` directly based on knowledge that the range enumerated by `px1` and `px2` corresponds to $X_1 + X_2$ having a minimum value of 0 and a maximum value of 125.

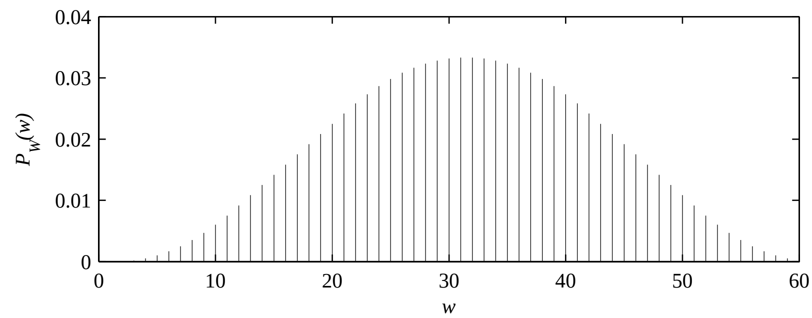
The resulting plot will be essentially identical to Figure 6.4. One final note, the command `set(h,'LineWidth',0.25)` is used to make the bars of the PMF thin enough to be resolved individually.

Problem 6.9.5 Solution

```
sx1=(1:10);px1=0.1*ones(1,10);
sx2=(1:20);px2=0.05*ones(1,20);
sx3=(1:30);px3=ones(1,30)/30;
[SX1,SX2,SX3]=ndgrid(sx1,sx2,sx3);
[PX1,PX2,PX3]=ndgrid(px1,px2,px3);
SW=SX1+SX2+SX3;
PW=PX1.*PX2.*PX3;
sw=unique(SW);
pw=finitepmf(SW,PW,sw);
h=pmfplot(sw,pw,'\itw','\itP_W(w)');
set(h,'LineWidth',0.25);
```

Since the `mdgrid` function extends naturally to higher dimensions, this solution follows the logic of `sumx1x2` in Example 6.19.

The output of `sumx1x2x3` is the plot of the PMF of W shown below. We use the command `set(h,'LineWidth',0.25)` to ensure that the bars of the PMF are thin enough to be resolved individually.



Problem 6.9.6 Solution

```
function [pw,sw]=sumfinitepmf(px,sx,py,sy);
[SX,SY]=ndgrid(sx,sy);
[PX,PY]=ndgrid(px,py);
SW=SX+SY;PW=PX.*PY;
sw=unique(SW);
pw=finitepmf(SW,PW,sw);
```

`sumfinitepmf` generalizes the method of Example 6.19. The only difference is that the PMFs `px` and `py` and ranges `sx` and `sy` are not hard coded, but instead are function inputs.

As an example, suppose X is a discrete uniform $(0, 20)$ random variable and Y is an independent discrete uniform $(0, 80)$ random variable. The following program `sum2unif` will generate and plot the PMF of $W = X + Y$.

```
%sum2unif.m
sx=0:20;px=ones(1,21)/21;
sy=0:80;py=ones(1,81)/81;
[pw,sw]=sumfinitepmf(px,sx,py,sy);
h=pmfplot(sw,pw,'\it w','\it P_W(w)');
set(h,'LineWidth',0.25);
```

Here is the graph generated by `sum2unif`.