Problem Solutions – Chapter 10

Problem 10.2.1 Solution

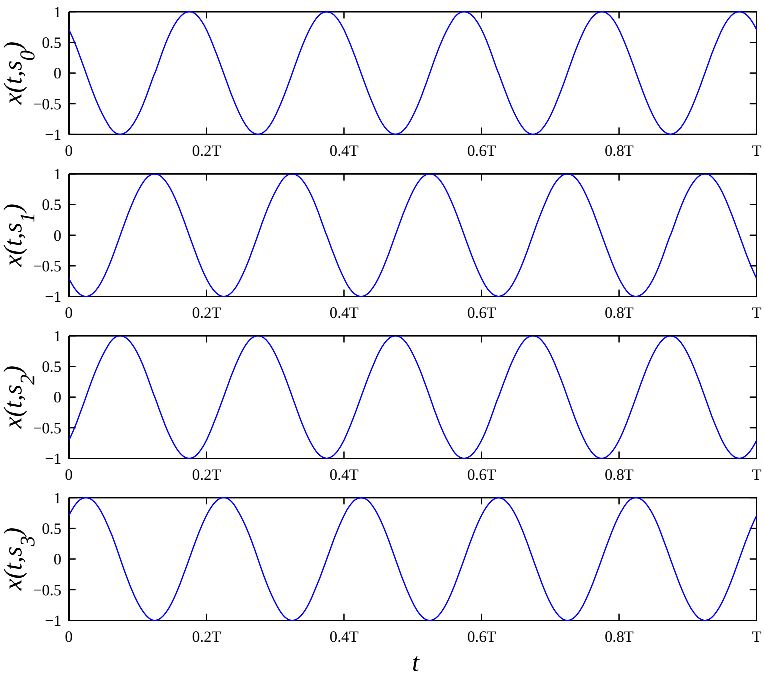
- In Example 10.3, the daily noontime temperature at Newark Airport is a discrete time, continuous value random process. However, if the temperature is recorded only in units of one degree, then the process was would be discrete value.
- In Example 10.4, the number of active telephone calls is discrete time and discrete value.
- The dice rolling experiment of Example 10.5 yields a discrete time, discrete value random process.
- The QPSK system of Example 10.6 is a continuous time and continuous value random process.

Problem 10.2.2 Solution

The sample space of the underlying experiment is $S = \{s_0, s_1, s_2, s_3\}$. The four elements in the sample space are equally likely. The ensemble of sample functions is $\{x(t, s_i)|i = 0, 1, 2, 3\}$ where

$$x(t, s_i) = \cos(2\pi f_0 t + \pi/4 + i\pi/2) \qquad (0 \leq t \leq T) \tag{1}$$

For $f_0 = 5/T$, this ensemble is shown below.

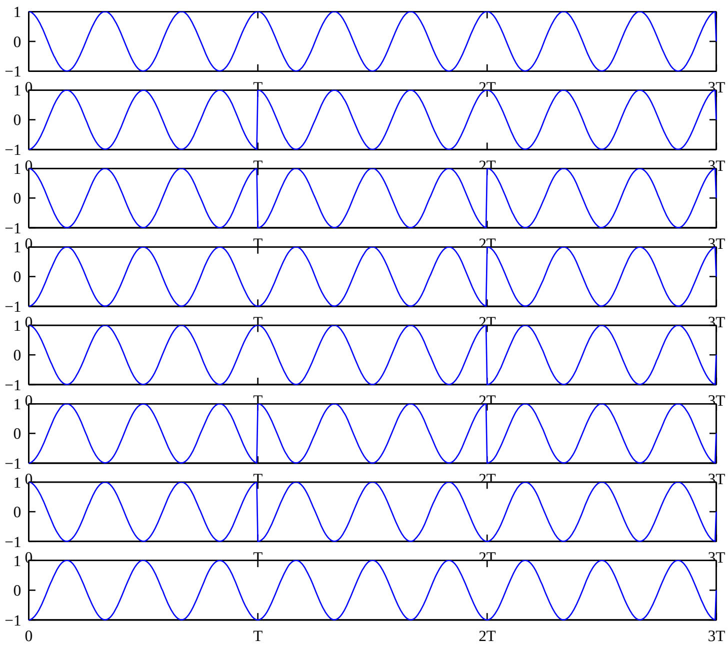


Problem 10.2.3 Solution

The eight possible waveforms correspond to the bit sequences

$$\{(0, 0, 0), (1, 0, 0), (1, 1, 0), \ldots, (1, 1, 1)\} \tag{1}$$

The corresponding eight waveforms are:



Problem 10.2.4 Solution

The statement is *false*. As a counterexample, consider the rectified cosine waveform $X(t) = R|\cos 2\pi f t|$ of Example 10.9. When $t = \pi/2$, then $\cos 2\pi f t = 0$ so that $X(\pi/2) = 0$. Hence $X(\pi/2)$ has PDF

$$f_{X(\pi/2)}(x) = \delta(x) \tag{1}$$

That is, $X(\pi/2)$ is a discrete random variable.

Problem 10.3.1 Solution

In this problem, we start from first principles. What makes this problem fairly straightforward is that the ramp is defined for all time. That is, the ramp doesn't start at time $t = W$.

$$P[X(t) \leq x] = P[t - W \leq x] = P[W \geq t - x] \tag{1}$$

Since $W \geq 0$, if $x \geq t$ then $P[W \geq t - x] = 1$. When $x < t$,

$$P[W \geq t - x] = \int_{t-x}^{\infty} f_W(w)\, dw = e^{-(t-x)} \tag{2}$$

Combining these facts, we have

$$F_{X(t)}(x) = P[W \geq t - x] = \begin{cases} e^{-(t-x)} & x < t \\ 1 & t \leq x \end{cases} \tag{3}$$

We note that the CDF contain no discontinuities. Taking the derivative of the CDF $F_{X(t)}(x)$ with respect to x, we obtain the PDF

$$f_{X(t)}(x) = \begin{cases} e^{x-t} & x < t \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 10.3.2 Solution

(a) Each resistor has frequency W in Hertz with uniform PDF

$$f_R(r) = \begin{cases} 0.025 & 9980 \leq r \leq 1020 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The probability that a test yields a one part in 10^4 oscillator is

$$p = P[9999 \leq W \leq 10001] = \int_{9999}^{10001} (0.025)\, dr = 0.05 \tag{2}$$

(b) To find the PMF of T_1, we view each oscillator test as an independent trial. A success occurs on a trial with probability p if we find a one part in 10^4 oscillator. The first one part in 10^4 oscillator is found at time $T_1 = t$ if we observe failures on trials $1, \ldots, t-1$ followed by a success on trial t. Hence, just as in Example 2.11, T_1 has the geometric PMF

$$P_{T_1}(t) = \begin{cases} (1-p)^{t-1} p & t = 1, 2, \ldots \\ 9 & \text{otherwise} \end{cases} \tag{3}$$

A geometric random variable with success probability p has mean $1/p$. This is derived in Theorem 2.5. The expected time to find the first good oscillator is $E[T_1] = 1/p = 20$ minutes.

(c) Since $p = 0.05$, the probability the first one part in 10^4 oscillator is found in exactly 20 minutes is $P_{T_1}(20) = (0.95)^{19}(0.05) = 0.0189$.

(d) The time T_5 required to find the 5th one part in 10^4 oscillator is the number of trials needed for 5 successes. T_5 is a Pascal random variable. If this is not clear, see Example 2.15 where the Pascal PMF is derived. When we are looking for 5 successes, the Pascal PMF is

$$P_{T_5}(t) = \begin{cases} \binom{t-1}{4} p^5 (1-p)^{t-5} & t = 5, 6, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Looking up the Pascal PMF in Appendix A, we find that $E[T_5] = 5/p = 100$ minutes. The following argument is a second derivation of the mean of T_5. Once we find the first one part in 10^4 oscillator, the number of additional trials needed to find the next one part in 10^4 oscillator once again has a geometric PMF with mean $1/p$ since each independent trial is a success with probability p. Similarly, the time required to find 5 one part in 10^4 oscillators is the sum of five independent geometric random variables. That is,

$$T_5 = K_1 + K_2 + K_3 + K_4 + K_5 \tag{5}$$

where each K_i is identically distributed to T_1. Since the expectation of the sum equals the sum of the expectations,

$$E[T_5] = E[K_1 + K_2 + K_3 + K_4 + K_5] = 5E[K_i] = 5/p = 100 \text{ minutes} \tag{6}$$

Problem 10.3.3 Solution

Once we find the first one part in 10^4 oscillator, the number of additional tests needed to find the next one part in 10^4 oscillator once again has a geometric PMF with mean $1/p$ since each independent trial is a success with probability p. That is $T_2 = T_1 + T'$ where T' is independent and identically distributed to T_1. Thus,

$$\begin{aligned} E[T_2|T_1 = 3] &= E[T_1|T_1 = 3] + E\left[T'|T_1 = 3\right] && (1)\\ &= 3 + E\left[T'\right] = 23 \text{ minutes.} && (2) \end{aligned}$$

Problem 10.3.4 Solution

Since the problem states that the pulse is delayed, we will assume $T \geq 0$. This problem is difficult because the answer will depend on t. In particular, for $t < 0$, $X(t) = 0$ and $f_{X(t)}(x) = \delta(x)$. Things are more complicated when $t > 0$. For $x < 0$, $P[X(t) > x] = 1$. For $x \geq 1$, $P[X(t) > x] = 0$. Lastly, for $0 \leq x < 1$,

$$\begin{aligned} P[X(t) > x] &= P\left[e^{-(t-T)}u(t-T) > x\right] && (1)\\ &= P[t + \ln x < T \leq t] && (2)\\ &= F_T(t) - F_T(t + \ln x) && (3) \end{aligned}$$

Note that condition $T \leq t$ is needed to make sure that the pulse doesn't arrive after time t. The other condition $T > t + \ln x$ ensures that the pulse didn't arrrive too early and already decay too much. We can express these facts in terms of the CDF of $X(t)$.

$$F_{X(t)}(x) = 1 - P[X(t) > x] = \begin{cases} 0 & x < 0 \\ 1 + F_T(t + \ln x) - F_T(t) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (4)$$

We can take the derivative of the CDF to find the PDF. However, we need to keep in mind that the CDF has a jump discontinuity at $x = 0$. In particular, since $\ln 0 = -\infty$,

$$F_{X(t)}(0) = 1 + F_T(-\infty) - F_T(t) = 1 - F_T(t) \quad (5)$$

Hence, when we take a derivative, we will see an impulse at $x = 0$. The PDF of $X(t)$ is

$$f_{X(t)}(x) = \begin{cases} (1 - F_T(t))\delta(x) + f_T(t + \ln x)/x & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Problem 10.4.1 Solution

Each Y_k is the sum of two identical independent Gaussian random variables. Hence, each Y_k must have the same PDF. That is, the Y_k are identically distributed. Next, we observe that the sequence of Y_k is independent. To see this, we observe that each Y_k is composed of two samples of X_k that are unused by any other Y_j for $j \neq k$.

Problem 10.4.2 Solution

Each W_n is the sum of two identical independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. However, since W_{n-1} and

W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We can verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30 \tag{1}$$

Next, we observe that W_{n-1} and W_n have covariance

$$\begin{aligned}\text{Cov}[W_{n-1}, W_n] &= E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] &(2)\\ &= \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 &(3)\end{aligned}$$

We observe that for $n \neq m$, $E[X_nX_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \text{Var}[X_n] + (E[X_n])^2 = 916 \tag{4}$$

Thus,

$$\text{Cov}[W_{n-1}, W_n] = \frac{900 + 916 + 900 + 900}{4} - 900 = 4 \tag{5}$$

Since $\text{Cov}[W_{n-1}, W_n] \neq 0$, W_n and W_{n-1} must be dependent.

Problem 10.4.3 Solution

The number Y_k of failures between successes $k-1$ and k is exactly $y \geq 0$ iff after success $k-1$, there are y failures followed by a success. Since the Bernoulli trials are independent, the probability of this event is $(1-p)^y p$. The complete PMF of Y_k is

$$P_{Y_k}(y) = \begin{cases} (1-p)^y p & y = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since this argument is valid for all k including $k = 1$, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. Moreover, since the trials are independent, the failures between successes $k-1$ and k and the number of failures between successes $k'-1$ and k' are independent. Hence, $Y_1, Y_2, \ldots$ is an iid sequence.

Problem 10.5.1 Solution

This is a very straightforward problem. The Poisson process has rate $\lambda = 4$ calls per second. When t is measured in seconds, each $N(t)$ is a Poisson random variable with mean $4t$ and thus has PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(4t)^n}{n!} e^{-4t} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Using the general expression for the PMF, we can write down the answer for each part.

(a) $P_{N(1)}(0) = 4^0 e^{-4}/0! = e^{-4} \approx 0.0183$.

(b) $P_{N(1)}(4) = 4^4 e^{-4}/4! = 32e^{-4}/3 \approx 0.1954$.

(c) $P_{N(2)}(2) = 8^2 e^{-8}/2! = 32e^{-8} \approx 0.0107$.

Problem 10.5.2 Solution

Following the instructions given, we express each answer in terms of $N(m)$ which has PMF

$$P_{N(m)}(n) = \begin{cases} (6m)^n e^{-6m}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability of no queries in a one minute interval is $P_{N(1)}(0) = 6^0 e^{-6}/0! = 0.00248$.

(b) The probability of exactly 6 queries arriving in a one minute interval is $P_{N(1)}(6) = 6^6 e^{-6}/6! = 0.161$.

(c) The probability of exactly three queries arriving in a one-half minute interval is $P_{N(0.5)}(3) = 3^3 e^{-3}/3! = 0.224$.

Problem 10.5.3 Solution

Since there is always a backlog an the service times are iid exponential random variables, The time between service completions are a sequence of iid exponential random variables. that is, the service completions are a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is $\lambda = 1/30$ per minute. Since t hours equals $60t$ minutes, the expected number serviced is $\lambda(60t)$ or $2t$. Moreover, the number serviced in the first t hours has the Poisson PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(2t)^n e^{-2t}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 10.5.4 Solution

Since $D(t)$ is a Poisson process with rate 0.1 drops/day, the random variable $D(t)$ is a Poisson random variable with parameter $\alpha = 0.1t$. The PMF of $D(t)$. the number of drops after t days, is

$$P_{D(t)}(d) = \begin{cases} (0.1t)^d e^{-0.1t}/d! & d = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 10.5.5 Solution

Note that it matters whether $t \geq 2$ minutes. If $t \leq 2$, then any customers that have arrived must still be in service. Since a Poisson number of arrivals occur during $(0, t]$,

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 2) \tag{1}$$

For $t \geq 2$, the customers in service are precisely those customers that arrived in the interval $(t-2, t]$. The number of such customers has a Poisson PMF with mean $\lambda[t - (t-2)] = 2\lambda$. The resulting PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (2\lambda)^n e^{-2\lambda}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2) \tag{2}$$

Problem 10.5.6 Solution

The time T between queries are independent exponential random variables with PDF

$$f_T(t) = \begin{cases} (1/8)e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the PDF, we can calculate for $t > 0$,

$$P[T \geq t] = \int_0^t f_T(t')\, dt' = e^{-t/8} \tag{2}$$

Using this formula, each question can be easily answered.

(a) $P[T \geq 4] = e^{-4/8} \approx 0.951$.

(b)

$$P[T \geq 13 | T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} \tag{3}$$

$$= \frac{P[T \geq 13]}{P[T \geq 5]} = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \approx 0.368 \tag{4}$$

(c) Although the time betwen queries are independent exponential random variables, $N(t)$ is not exactly a Poisson random process because the first query occurs at time $t = 0$. Recall that in a Poisson process, the first arrival occurs some time after $t = 0$. However $N(t) - 1$ is a Poisson process of rate 8. Hence, for $n = 0, 1, 2, \ldots$,

$$P[N(t) - 1 = n] = (t/8)^n e^{-t/8}/n! \tag{5}$$

Thus, for $n = 1, 2, \ldots$, the PMF of $N(t)$ is

$$P_{N(t)}(n) = P[N(t) - 1 = n - 1] = (t/8)^{n-1} e^{-t/8}/(n-1)! \tag{6}$$

The complete expression of the PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (t/8)^{n-1} e^{-t/8}/(n-1)! & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 10.5.7 Solution

This proof is just a simplified version of the proof given for Theorem 10.3. The first arrival occurs at time $X_1 > x \geq 0$ iff there are no arrivals in the interval $(0, x]$. Hence, for $x \geq 0$,

$$P[X_1 > x] = P[N(x) = 0] = (\lambda x)^0 e^{-\lambda x}/0! = e^{-\lambda x} \tag{1}$$

Since $P[X_1 \leq x] = 0$ for $x < 0$, the CDF of X_1 is the exponential CDF

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases} \tag{2}$$

Problem 10.5.8 Solution

(a) For $X_i = -\ln U_i$, we can write

$$P[X_i > x] = P[-\ln U_i > x] = P[\ln U_i \leq -x] = P\left[U_i \leq e^{-x}\right] \tag{1}$$

When $x < 0$, $e^{-x} > 1$ so that $P[U_i \leq e^{-x}] = 1$. When $x \geq 0$, we have $0 < e^{-x} \leq 1$, implying $P[U_i \leq e^{-x}] = e^{-x}$. Combining these facts, we have

$$P[X_i > x] = \begin{cases} 1 & x < 0 \\ e^{-x} & x \geq 0 \end{cases} \tag{2}$$

This permits us to show that the CDF of X_i is

$$F_{X_i}(x) = 1 - P[X_i > x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x > 0 \end{cases} \tag{3}$$

We see that X_i has an exponential CDF with mean 1.

(b) Note that $N = n$ iff

$$\prod_{i=1}^{n} U_i \geq e^{-t} > \prod_{i=1}^{n+1} U_i \tag{4}$$

By taking the logarithm of both inequalities, we see that $N = n$ iff

$$\sum_{i=1}^{n} \ln U_i \geq -t > \sum_{i=1}^{n+1} \ln U_i \tag{5}$$

Next, we multiply through by -1 and recall that $X_i = -\ln U_i$ is an exponential random variable. This yields $N = n$ iff

$$\sum_{i=1}^{n} X_i \leq t < \sum_{i=1}^{n+1} X_i \tag{6}$$

Now we recall that a Poisson process $N(t)$ of rate 1 has independent exponential interarrival times $X_1, X_2, \ldots$. That is, the ith arrival occurs at time $\sum_{j=1}^{i} X_j$. Moreover, $N(t) = n$ iff the first n arrivals occur by time t but arrival $n + 1$ occurs after time t. Since the random variable $N(t)$ has a Poisson distribution with mean t, we can write

$$P\left[\sum_{i=1}^{n} X_i \leq t < \sum_{i=1}^{n+1} X_i\right] = P[N(t) = n] = \frac{t^n e^{-t}}{n!}. \tag{7}$$

Problem 10.6.1 Solution

Customers entering (or not entering) the casino is a Bernoulli decomposition of the Poisson process of arrivals at the casino doors. By Theorem 10.6, customers entering the casino are a Poisson process of rate $100/2 = 50$ customers/hour. Thus in the two hours from 5 to 7 PM, the number, N, of customers entering the casino is a Poisson random variable with expected value $\alpha = 2 \cdot 50 = 100$. The PMF of N is

$$P_N(n) = \begin{cases} 100^n e^{-100}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 10.6.2 Solution

In an interval $(t, t+\Delta]$ with an infinitesimal Δ, let A_i denote the event of an arrival of the process $N_i(t)$. Also, let $A = A_1 \cup A_2$ denote the event of an arrival of either process. Since $N_i(t)$ is a Poisson process, the alternative model says that $P[A_i] = \lambda_i \Delta$. Also, since $N_1(t)+N_2(t)$ is a Poisson process, the proposed Poisson process model says

$$P[A] = (\lambda_1 + \lambda_2)\Delta \tag{1}$$

Lastly, the conditional probability of a type 1 arrival given an arrival of either type is

$$P[A_1|A] = \frac{P[A_1 A]}{P[A]} = \frac{P[A_1]}{P[A]} = \frac{\lambda_1 \Delta}{(\lambda_1 + \lambda_2)\Delta} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \tag{2}$$

This solution is something of a cheat in that we have used the fact that the sum of Poisson processes is a Poisson process without using the proposed model to derive this fact.

Problem 10.6.3 Solution

We start with the case when $t \geq 2$. When each service time is equally likely to be either 1 minute or 2 minutes, we have the following situation. Let M_1 denote those customers that arrived in the interval $(t-1, 1]$. All M_1 of these customers will be in the bank at time t and M_1 is a Poisson random variable with mean λ.

Let M_2 denote the number of customers that arrived during $(t-2, t-1]$. Of course, M_2 is Poisson with expected value λ. We can view each of the M_2 customers as flipping a coin to determine whether to choose a 1 minute or a 2 minute service time. Only those customers that chooses a 2 minute service time will be in service at time t. Let M_2' denote those customers choosing a 2 minute service time. It should be clear that M_2' is a Poisson number of Bernoulli random variables. Theorem 10.6 verifies that using Bernoulli trials to decide whether the arrivals of a rate λ Poisson process should be counted yields a Poisson process of rate $p\lambda$. A consequence of this result is that a Poisson number of Bernoulli (success probability p) random variables has Poisson PMF with mean $p\lambda$. In this case, M_2' is Poisson with mean $\lambda/2$. Moreover, the number of customers in service at time t is $N(t) = M_1 + M_2'$. Since M_1 and M_2' are independent Poisson random variables, their sum $N(t)$ also has a Poisson PMF. This was verified in Theorem 6.9. Hence $N(t)$ is Poisson with mean $E[N(t)] = E[M_1] + E[M_2'] = 3\lambda/2$. The PMF of $N(t)$ is

$$P_{N(t)}(n) = \begin{cases} (3\lambda/2)^n e^{-3\lambda/2}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (t \geq 2) \tag{1}$$

Now we can consider the special cases arising when $t < 2$. When $0 \leq t < 1$, every arrival is still in service. Thus the number in service $N(t)$ equals the number of arrivals and has the PMF

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (0 \leq t \leq 1) \tag{2}$$

When $1 \leq t < 2$, let M_1 denote the number of customers in the interval $(t-1, t]$. All M_1 customers arriving in that interval will be in service at time t. The M_2 customers arriving in the interval $(0, t-1]$ must each flip a coin to decide one a 1 minute or two minute service time. Only those customers choosing the two minute service time will be in service at time t. Since M_2 has a Poisson PMF with mean $\lambda(t-1)$, the number M_2' of those customers in the system at time t has a Poisson PMF with mean $\lambda(t-1)/2$. Finally, the number of customers in service at time t has a Poisson

PMF with expected value $E[N(t)] = E[M_1] + E[M_2'] = \lambda + \lambda(t-1)/2$. Hence, the PMF of $N(t)$ becomes

$$P_{N(t)}(n) = \begin{cases} (\lambda(t+1)/2)^n e^{-\lambda(t+1)/2}/n! & n = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad (1 \le t \le 2) \tag{3}$$

Problem 10.6.4 Solution

Since the arrival times $S_1, \ldots, S_n$ are ordered in time and since a Poisson process cannot have two simultaneous arrivals, the conditional PDF $f_{S_1,\ldots,S_n|N}(S_1,\ldots,S_n|n)$ is nonzero only if $s_1 < s_2 < \cdots < s_n < T$. In this case, consider an arbitrarily small Δ; in particular, $\Delta < \min_i(s_{i+1} - s_i)/2$ implies that the intervals $(s_i, s_i + \Delta]$ are non-overlapping. We now find the joint probability

$$P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n]$$

that each S_i is in the interval $(s_i, s_i + \Delta]$ and that $N = n$. This joint event implies that there were zero arrivals in each interval $(s_i + \Delta, s_{i+1}]$. That is, over the interval $[0, T]$, the Poisson process has exactly one arrival in each interval $(s_i, s_i + \Delta]$ and zero arrivals in the time period $T - \bigcup_{i=1}^n (s_i, s_i + \Delta]$. The collection of intervals in which there was no arrival had a total duration of $T - n\Delta$. Note that the probability of exactly one arrival in the interval $(s_i, s_i + \Delta]$ is $\lambda\Delta e^{-\lambda\delta}$ and the probability of zero arrivals in a period of duration $T - n\Delta$ is $e^{-\lambda(T_n - \Delta)}$. In addition, the event of one arrival in each interval $(s_i, s_i + \Delta)$ and zero events in the period of length $T - n\Delta$ are independent events because they consider non-overlapping periods of the Poisson process. Thus,

$$\begin{aligned} P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n] &= \left(\lambda\Delta e^{-\lambda\Delta}\right)^n e^{-\lambda(T - n\Delta)} && (1) \\ &= (\lambda\Delta)^n e^{-\lambda T} && (2) \end{aligned}$$

Since $P[N = n] = (\lambda T)^n e^{-\lambda T}/n!$, we see that

$$\begin{aligned} &P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n] \\ &\qquad = \frac{P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta, N = n]}{P[N = n]} && (3) \\ &\qquad = \frac{(\lambda\Delta)^n e^{-\lambda T}}{(\lambda T)^n e^{-\lambda T}/n!} && (4) \\ &\qquad = \frac{n!}{T^n}\Delta^n && (5) \end{aligned}$$

Finally, for infinitesimal Δ, the conditional PDF of $S_1, \ldots, S_n$ given $N = n$ satisfies

$$\begin{aligned} f_{S_1,\ldots,S_n|N}(s_1,\ldots,s_n|n)\,\Delta^n &= P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n] && (6) \\ &= \frac{n!}{T^n}\Delta^n && (7) \end{aligned}$$

Since the conditional PDF is zero unless $s_1 < s_2 < \cdots < s_n \le T$, it follows that

$$f_{S_1,\ldots,S_n|N}(s_1,\ldots,s_n|n) = \begin{cases} n!/T^n & 0 \le s_1 < \cdots < s_n \le T, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

If it seems that the above argument had some "hand-waving," we now do the derivation of $P[s_1 < S_1 \le s_1 + \Delta, \ldots, s_n < S_n \le s_n + \Delta | N = n]$ in somewhat excruciating detail. (Feel free to skip the following if you were satisfied with the earlier explanation.)

For the interval $(s,t]$, we use the shorthand notation $0_{(s,t)}$ and $1_{(s,t)}$ to denote the events of 0 arrivals and 1 arrival respectively. This notation permits us to write

$$\begin{aligned} &P\left[s_1 < S_1 \leq s_1+\Delta, \ldots, s_n < S_n \leq s_n+\Delta, N=n\right] \\ &\qquad = P\left[0_{(0,s_1)} 1_{(s_1,s_1+\Delta)} 0_{(s_1+\Delta,s_2)} 1_{(s_2,s_2+\Delta)} 0_{(s_2+\Delta,s_3)} \cdots 1_{(s_n,s_n+\Delta)} 0_{(s_n+\Delta,T)}\right] \end{aligned} \tag{9}$$

The set of events $0_{(0,s_1)}$, $0_{(s_n+\Delta,T)}$, and for $i=1,\ldots,n-1$, $0_{(s_i+\Delta,s_{i+1})}$ and $1_{(s_i,s_i+\Delta)}$ are independent because each devent depend on the Poisson process in a time interval that overlaps none of the other time intervals. In addition, since the Poisson process has rate λ, $P[0_{(s,t)}] = e^{-\lambda(t-s)}$ and $P[1_{(s_i,s_i+\Delta)}] = (\lambda\Delta)e^{-\lambda\Delta}$. Thus,

$$\begin{aligned} &P\left[s_1 < S_1 \leq s_1+\Delta, \ldots, s_n < S_n \leq s_n+\Delta, N=n\right] \\ &\qquad = P\left[0_{(0,s_1)}\right] P\left[1_{(s_1,s_1+\Delta)}\right] P\left[0_{(s_1+\Delta,s_2)}\right] \cdots P\left[1_{(s_n,s_n+\Delta)}\right] P\left[0_{(s_n+\Delta,T)}\right] && (10) \\ &\qquad = e^{-\lambda s_1}\left(\lambda\Delta e^{-\lambda\Delta}\right) e^{-\lambda(s_2-s_1-\Delta)} \cdots \left(\lambda\Delta e^{-\lambda\Delta}\right) e^{-\lambda(T-s_n-\Delta)} && (11) \\ &\qquad = (\lambda\Delta)^n e^{-\lambda T} && (12) \end{aligned}$$

Problem 10.7.1 Solution

From the problem statement, the change in the stock price is $X(8)-X(0)$ and the standard deviation of $X(8)-X(0)$ is 1/2 point. In other words, the variance of $X(8)-X(0)$ is $\text{Var}[X(8)-X(0)] = 1/4$. By the definition of Brownian motion. $\text{Var}[X(8)-X(0)] = 8\alpha$. Hence $\alpha = 1/32$.

Problem 10.7.2 Solution

We need to verify that $Y(t) = X(ct)$ satisfies the conditions given in Definition 10.10. First we observe that $Y(0) = X(c \cdot 0) = X(0) = 0$. Second, we note that since $X(t)$ is Brownian motion process implies that $Y(t) - Y(s) = X(ct) - X(cs)$ is a Gaussian random variable. Further, $X(ct) - X(cs)$ is independent of $X(t')$ for all $t' \leq cs$. Equivalently, we can say that $X(ct) - X(cs)$ is independent of $X(c\tau)$ for all $\tau \leq s$. In other words, $Y(t) - Y(s)$ is independent of $Y(\tau)$ for all $\tau \leq s$. Thus $Y(t)$ is a Brownian motion process.

Problem 10.7.3 Solution

First we observe that $Y_n = X_n - X_{n-1} = X(n) - X(n-1)$ is a Gaussian random variable with mean zero and variance α. Since this fact is true for all n, we can conclude that $Y_1, Y_2, \ldots$ are identically distributed. By Definition 10.10 for Brownian motion, $Y_n = X(n) - X(n-1)$ is independent of $X(m)$ for any $m \leq n-1$. Hence Y_n is independent of $Y_m = X(m) - X(m-1)$ for any $m \leq n-1$. Equivalently, $Y_1, Y_2, \ldots$ is a sequence of independent random variables.

Problem 10.7.4 Solution

Recall that the vector $\mathbf{X}$ of increments has independent components $X_n = W_n - W_{n-1}$. Alternatively, each W_n can be written as the sum

$$\begin{aligned} W_1 &= X_1 && (1) \\ W_2 &= X_1 + X_2 && (2) \\ &\vdots \\ W_k &= X_1 + X_2 + \cdots + X_k. && (3) \end{aligned}$$

In terms of matrices, $\mathbf{W} = \mathbf{A}\mathbf{X}$ where $\mathbf{A}$ is the lower triangular matrix

$$\mathbf{A} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}. \tag{4}$$

Since $E[\mathbf{W}] = \mathbf{A}E[\mathbf{X}] = \mathbf{0}$, it folows from Theorem 5.16 that

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{|\det(\mathbf{A})|} f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right). \tag{5}$$

Since $\mathbf{A}$ is a lower triangular matrix, $\det(\mathbf{A}) = 1$, the product of its diagonal entries. In addition, reflecting the fact that each $X_n = W_n - W_{n-1}$,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 0 & -1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1}\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 - W_1 \\ W_3 - W_2 \\ \vdots \\ W_k - W_{k-1} \end{bmatrix}. \tag{6}$$

Combining these facts with the observation that $f_{\mathbf{X}}(\mathbf{x}) = \prod_{n=1}^{k} f_{X_n}(x_n)$, we can write

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\mathbf{X}}\left(\mathbf{A}^{-1}\mathbf{w}\right) = \prod_{n=1}^{k} f_{X_n}(w_n - w_{n-1}), \tag{7}$$

which completes the missing steps in the proof of Theorem 10.8.

Problem 10.8.1 Solution

The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \tag{1}$$

for $k = 0$, $C_X[m,0] = \text{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]\, E[(X_{m+k} - \mu_X)] = 0 \tag{2}$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{3}$$

Problem 10.8.2 Solution

Recall that $X(t) = t - W$ where $E[W] = 1$ and $E[W^2] = 2$.

(a) The mean is $\mu_X(t) = E[t - W] = t - E[W] = t - 1$.

(b) The autocovariance is

$$\begin{aligned} C_X(t,\tau) &= E[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) && (1)\\ &= E[(t-W)(t+\tau-W)] - (t-1)(t+\tau-1) && (2)\\ &= t(t+\tau) - E[(2t+\tau)W] + E\left[W^2\right] - t(t+\tau) + 2t + \tau - 1 && (3)\\ &= -(2t+\tau)E[W] + 2 + 2t + \tau - 1 && (4)\\ &= 1 && (5) \end{aligned}$$

Problem 10.8.3 Solution

In this problem, the daily temperature process results from

$$C_n = 16\left[1 - \cos\frac{2\pi n}{365}\right] + 4X_n \tag{1}$$

where X_n is an iid random sequence of $N[0,1]$ random variables. The hardest part of this problem is distinguishing between the process C_n and the covariance function $C_C[k]$.

(a) The expected value of the process is

$$E[C_n] = 16E\left[1 - \cos\frac{2\pi n}{365}\right] + 4E[X_n] = 16\left[1 - \cos\frac{2\pi n}{365}\right] \tag{2}$$

(b) The autocovariance of C_n is

$$\begin{aligned} C_C[m,k] &= E\left[\left(C_m - 16\left[1 - \cos\frac{2\pi m}{365}\right]\right)\left(C_{m+k} - 16\left[1 - \cos\frac{2\pi(m+k)}{365}\right]\right)\right] &(3)\\ &= 16E[X_m X_{m+k}] = \begin{cases} 16 & k = 0 \\ 0 & \text{otherwise} \end{cases} &(4) \end{aligned}$$

(c) A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is because day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of "heat waves" or "cold spells" through correlated daily temperatures.

Problem 10.8.4 Solution

By repeated application of the recursion $C_n = C_{n-1}/2 + 4X_n$, we obtain

$$\begin{aligned} C_n &= \frac{C_{n-2}}{4} + 4\left[\frac{X_{n-1}}{2} + X_n\right] &(1)\\ &= \frac{C_{n-3}}{8} + 4\left[\frac{X_{n-2}}{4} + \frac{X_{n-1}}{2} + X_n\right] &(2)\\ &\vdots &(3)\\ &= \frac{C_0}{2^n} + 4\left[\frac{X_1}{2^{n-1}} + \frac{X_2}{2^{n-2}} + \cdots + X_n\right] = \frac{C_0}{2^n} + 4\sum_{i=1}^{n}\frac{X_i}{2^{n-i}} &(4) \end{aligned}$$

(a) Since $C_0, X_1, X_2, \ldots$ all have zero mean,

$$E[C_n] = \frac{E[C_0]}{2^n} + 4\sum_{i=1}^{n}\frac{E[X_i]}{2^{n-i}} = 0 \tag{5}$$

(b) The autocovariance is

$$C_C[m,k] = E\left[\left(\frac{C_0}{2^n} + 4\sum_{i=1}^{n}\frac{X_i}{2^{n-i}}\right)\left(\frac{C_0}{2^m+k} + 4\sum_{j=1}^{m+k}\frac{X_j}{2^{m+k-j}}\right)\right] \tag{6}$$

Since $C_0, X_1, X_2, \ldots$ are independent (and zero mean), $E[C_0 X_i] = 0$. This implies

$$C_C[m,k] = \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16 \sum_{i=1}^{m} \sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i} 2^{m+k-j}} \tag{7}$$

For $i \neq j$, $E[X_i X_j] = 0$ so that only the $i = j$ terms make any contribution to the double sum. However, at this point, we must consider the cases $k \geq 0$ and $k < 0$ separately. Since each X_i has variance 1, the autocovariance for $k \geq 0$ is

$$C_C[m,k] = \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m} \frac{1}{2^{2m+k-2i}} \tag{8}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \sum_{i=1}^{m} (1/4)^{m-i} \tag{9}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1-(1/4)^m}{3/4} \tag{10}$$

For $k < 0$, we can write

$$C_C[m,k] = \frac{E\left[C_0^2\right]}{2^{2m+k}} + 16 \sum_{i=1}^{m} \sum_{j=1}^{m+k} \frac{E\left[X_i X_j\right]}{2^{m-i} 2^{m+k-j}} \tag{11}$$

$$= \frac{1}{2^{2m+k}} + 16 \sum_{i=1}^{m+k} \frac{1}{2^{2m+k-2i}} \tag{12}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^{-k}} \sum_{i=1}^{m+k} (1/4)^{m+k-i} \tag{13}$$

$$= \frac{1}{2^{2m+k}} + \frac{16}{2^k} \frac{1-(1/4)^{m+k}}{3/4} \tag{14}$$

A general expression that's valid for all m and k is

$$C_C[m,k] = \frac{1}{2^{2m+k}} + \frac{16}{2^{|k|}} \frac{1-(1/4)^{\min(m,m+k)}}{3/4} \tag{15}$$

(c) Since $E[C_i] = 0$ for all i, our model has a mean daily temperature of zero degrees Celsius for the entire year. This is not a reasonable model for a year.

(d) For the month of January, a mean temperature of zero degrees Celsius seems quite reasonable. we can calculate the variance of C_n by evaluating the covariance at $n = m$. This yields

$$\mathrm{Var}[C_n] = \frac{1}{4^n} + \frac{16}{4^n} \frac{4(4^n-1)}{3} \tag{16}$$

Note that the variance is upper bounded by

$$\mathrm{Var}[C_n] \leq 64/3 \tag{17}$$

Hence the daily temperature has a standard deviation of $8/\sqrt{3} \approx 4.6$ degrees. Without actual evidence of daily temperatures in January, this model is more difficult to discredit.

Problem 10.8.5 Solution

This derivation of the Poisson process covariance is almost identical to the derivation of the Brownian motion autocovariance since both rely on the use of independent increments. From the definition of the Poisson process, we know that $\mu_N(t) = \lambda t$. When $\tau \geq 0$, we can write

$$C_N(t,\tau) = E\left[N(t)N(t+\tau)\right] - (\lambda t)[\lambda(t+\tau)] \tag{1}$$
$$= E\left[N(t)[(N(t+\tau) - N(t)) + N(t)]\right] - \lambda^2 t(t+\tau) \tag{2}$$
$$= E\left[N(t)[N(t+\tau) - N(t)]\right] + E\left[N^2(t)\right] - \lambda^2 t(t+\tau) \tag{3}$$

By the definition of the Poisson process, $N(t+\tau) - N(t)$ is the number of arrivals in the interval $[t, t+\tau)$ and is independent of $N(t)$ for $\tau > 0$. This implies

$$E\left[N(t)[N(t+\tau) - N(t)]\right] = E\left[N(t)\right] E\left[N(t+\tau) - N(t)\right] = \lambda t[\lambda(t+\tau) - \lambda t] \tag{4}$$

Note that since $N(t)$ is a Poisson random variable, $\text{Var}[N(t)] = \lambda t$. Hence

$$E\left[N^2(t)\right] = \text{Var}[N(t)] + (E\left[N(t)\right]^2 = \lambda t + (\lambda t)^2 \tag{5}$$

Therefore, for $\tau \geq 0$,

$$C_N(t,\tau) = \lambda t[\lambda(t+\tau) - \lambda t) + \lambda t + (\lambda t)^2 - \lambda^2 t(t+\tau) = \lambda t \tag{6}$$

If $\tau < 0$, then we can interchange the labels t and $t+\tau$ in the above steps to show $C_N(t,\tau) = \lambda(t+\tau)$. For arbitrary t and τ, we can combine these facts to write

$$C_N(t,\tau) = \lambda \min(t, t+\tau) \tag{7}$$

Problem 10.9.1 Solution

For an arbitrary set of samples $Y(t_1), \ldots, Y(t_k)$, we observe that $Y(t_j) = X(t_j + a)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}\left(y_1, \ldots, y_k\right) = f_{X(t_1+a),\ldots,X(t_k+a)}\left(y_1, \ldots, y_k\right) \tag{1}$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}\left(y_1, \ldots, y_k\right) = f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}\left(y_1, \ldots, y_k\right) \tag{2}$$

Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}\left(y_1, \ldots, y_k\right) = f_{X(t_1+a),\ldots,X(t_k+a)}\left(y_1, \ldots, y_k\right) \tag{3}$$

This implies

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}\left(y_1, \ldots, y_k\right) = f_{X(t_1+a),\ldots,X(t_k+a)}\left(y_1, \ldots, y_k\right) \tag{4}$$
$$= f_{Y(t_1),\ldots,Y(t_k)}\left(y_1, \ldots, y_k\right) \tag{5}$$

We can conclude that $Y(t)$ is a stationary process.

Problem 10.9.2 Solution

For an arbitrary set of samples $Y(t_1), \ldots, Y(t_k)$, we observe that $Y(t_j) = X(at_j)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k) = f_{X(at_1),\ldots,X(at_k)}(y_1, \ldots, y_k) \tag{1}$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) = f_{X(at_1+a\tau),\ldots,X(at_k+a\tau)}(y_1, \ldots, y_k) \tag{2}$$

We see that a time offset of τ for the $Y(t)$ process corresponds to an offset of time $\tau' = a\tau$ for the $X(t)$ process. Since $X(t)$ is a stationary process,

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1, \ldots, y_k) &= f_{X(at_1+\tau'),\ldots,X(at_k+\tau')}(y_1, \ldots, y_k) && (3)\\ &= f_{X(at_1),\ldots,X(at_k)}(y_1, \ldots, y_k) && (4)\\ &= f_{Y(t_1),\ldots,Y(t_k)}(y_1, \ldots, y_k) && (5) \end{aligned}$$

We can conclude that $Y(t)$ is a stationary process.

Problem 10.9.3 Solution

For a set of time samples $n_1, \ldots, n_m$ and an offset k, we note that $Y_{n_i+k} = X((n_i + k)\Delta)$. This implies

$$f_{Y_{n_1+k},\ldots,Y_{n_m+k}}(y_1, \ldots, y_m) = f_{X((n_1+k)\Delta),\ldots,X((n_m+k)\Delta)}(y_1, \ldots, y_m) \tag{1}$$

Since $X(t)$ is a stationary process,

$$f_{X((n_1+k)\Delta),\ldots,X((n_m+k)\Delta)}(y_1, \ldots, y_m) = f_{X(n_1\Delta),\ldots,X(n_m\Delta)}(y_1, \ldots, y_m) \tag{2}$$

Since $X(n_i\Delta) = Y_{n_i}$, we see that

$$f_{Y_{n_1+k},\ldots,Y_{n_m+k}}(y_1, \ldots, y_m) = f_{Y_{n_1},\ldots,Y_{n_m}}(y_1, \ldots, y_m) \tag{3}$$

Hence Y_n is a stationary random sequence.

Problem 10.9.4 Solution

Since $Y_n = X_{kn}$,

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1, \ldots, y_m) = f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1, \ldots, y_m) \tag{1}$$

Stationarity of the X_n process implies

$$\begin{aligned} f_{X_{kn_1+kl},\ldots,X_{kn_m+kl}}(y_1, \ldots, y_m) &= f_{X_{kn_1},\ldots,X_{kn_m}}(y_1, \ldots, y_m) && (2)\\ &= f_{Y_{n_1},\ldots,Y_{n_m}}(y_1, \ldots, y_m). && (3) \end{aligned}$$

We combine these steps to write

$$f_{Y_{n_1+l},\ldots,Y_{n_m+l}}(y_1, \ldots, y_m) = f_{Y_{n_1},\ldots,Y_{n_m}}(y_1, \ldots, y_m). \tag{4}$$

Thus Y_n is a stationary process.

Comment: The first printing of the text asks whether Y_n is wide stationary if X_n is wide sense stationary. This fact is also true; however, since wide sense stationarity isn't addressed until the next section, the problem was corrected to ask about stationarity.

Problem 10.9.5 Solution

Given $A = a$, $Y(t) = aX(t)$ which is a special case of $Y(t) = aX(t) + b$ given in Theorem 10.10. Applying the result of Theorem 10.10 with $b = 0$ yields

$$f_{Y(t_1),\ldots,Y(t_n)|A}\left(y_1,\ldots,y_n|a\right) = \frac{1}{a^n} f_{X(t_1),\ldots,X(t_n)}\left(\frac{y_1}{a},\ldots,\frac{y_n}{a}\right) \tag{1}$$

Integrating over the PDF $f_A(a)$ yields

$$\begin{aligned} f_{Y(t_1),\ldots,Y(t_n)}\left(y_1,\ldots,y_n\right) &= \int_0^\infty f_{Y(t_1),\ldots,Y(t_n)|A}\left(y_1,\ldots,y_n|a\right) f_A\left(a\right)\,da && (2)\\ &= \int_0^\infty \frac{1}{a^n} f_{X(t_1),\ldots,X(t_n)}\left(\frac{y_1}{a},\ldots,\frac{y_n}{a}\right) f_A\left(a\right)\,da && (3) \end{aligned}$$

This complicated expression can be used to find the joint PDF of $Y(t_1+\tau),\ldots,Y(t_n+\tau)$:

$$f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}\left(y_1,\ldots,y_n\right) = \int_0^\infty \frac{1}{a^n} f_{X(t_1+\tau),\ldots,X(t_n+\tau)}\left(\frac{y_1}{a},\ldots,\frac{y_n}{a}\right) f_A\left(a\right)\,da \tag{4}$$

Since $X(t)$ is a stationary process, the joint PDF of $X(t_1+\tau),\ldots,X(t_n+\tau)$ is the same as the joint PDf of $X(t_1),\ldots,X(t_n)$. Thus

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}\left(y_1,\ldots,y_n\right) &= \int_0^\infty \frac{1}{a^n} f_{X(t_1+\tau),\ldots,X(t_n+\tau)}\left(\frac{y_1}{a},\ldots,\frac{y_n}{a}\right) f_A\left(a\right)\,da && (5)\\ &= \int_0^\infty \frac{1}{a^n} f_{X(t_1),\ldots,X(t_n)}\left(\frac{y_1}{a},\ldots,\frac{y_n}{a}\right) f_A\left(a\right)\,da && (6)\\ &= f_{Y(t_1),\ldots,Y(t_n)}\left(y_1,\ldots,y_n\right) && (7) \end{aligned}$$

We can conclude that $Y(t)$ is a stationary process.

Problem 10.9.6 Solution

Since $g(\cdot)$ is an unspecified function, we will work with the joint CDF of $Y(t_1+\tau),\ldots,Y(t_n+\tau)$. To show $Y(t)$ is a stationary process, we will show that for all τ,

$$F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}\left(y_1,\ldots,y_n\right) = F_{Y(t_1),\ldots,Y(t_n)}\left(y_1,\ldots,y_n\right) \tag{1}$$

By taking partial derivatives with respect to $y_1,\ldots,y_n$, it should be apparent that this implies that the joint PDF $f_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}(y_1,\ldots,y_n)$ will not depend on τ. To proceed, we write

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}\left(y_1,\ldots,y_n\right) &= P\left[Y(t_1+\tau)\le y_1,\ldots,Y(t_n+\tau)\le y_n\right] && (2)\\ &= P\Bigg[\underbrace{g(X(t_1+\tau))\le y_1,\ldots,g(X(t_n+\tau))\le y_n}_{A_\tau}\Bigg] && (3) \end{aligned}$$

In principle, we can calculate $P[A_\tau]$ by integrating $f_{X(t_1+\tau),\ldots,X(t_n+\tau)}(x_1,\ldots,x_n)$ over the region corresponding to event A_τ. Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau),\ldots,X(t_n+\tau)}\left(x_1,\ldots,x_n\right) = f_{X(t_1),\ldots,X(t_n)}\left(x_1,\ldots,x_n\right) \tag{4}$$

This implies $P[A_\tau]$ does not depend on τ. In particular,

$$\begin{aligned} F_{Y(t_1+\tau),\ldots,Y(t_n+\tau)}\left(y_1,\ldots,y_n\right) &= P\left[A_\tau\right] && (5)\\ &= P\left[g(X(t_1))\le y_1,\ldots,g(X(t_n))\le y_n\right] && (6)\\ &= F_{Y(t_1),\ldots,Y(t_n)}\left(y_1,\ldots,y_n\right) && (7) \end{aligned}$$

Problem 10.10.1 Solution

The autocorrelation function $R_X(\tau) = \delta(\tau)$ is mathematically valid in the sense that it meets the conditions required in Theorem 10.12. That is,

$$R_X(\tau) = \delta(\tau) \geq 0 \tag{1}$$
$$R_X(\tau) = \delta(\tau) = \delta(-\tau) = R_X(-\tau) \tag{2}$$
$$R_X(\tau) \leq R_X(0) = \delta(0) \tag{3}$$

However, for a process $X(t)$ with the autocorrelation $R_X(\tau) = \delta(\tau)$, Definition 10.16 says that the average power of the process is

$$E\left[X^2(t)\right] = R_X(0) = \delta(0) = \infty \tag{4}$$

Processes with infinite average power cannot exist in practice.

Problem 10.10.2 Solution

Since $Y(t) = A + X(t)$, the mean of $Y(t)$ is

$$E\left[Y(t)\right] = E\left[A\right] + E\left[X(t)\right] = E\left[A\right] + \mu_X \tag{1}$$

The autocorrelation of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= E\left[(A + X(t))\,(A + X(t+\tau))\right] && (2)\\ &= E\left[A^2\right] + E\left[A\right]E\left[X(t)\right] + AE\left[X(t+\tau)\right] + E\left[X(t)X(t+\tau)\right] && (3)\\ &= E\left[A^2\right] + 2E\left[A\right]\mu_X + R_X(\tau) && (4) \end{aligned}$$

We see that neither $E[Y(t)]$ nor $R_Y(t,\tau)$ depend on t. Thus $Y(t)$ is a wide sense stationary process.

Problem 10.10.3 Solution

In this problem, we find the autocorrelation $R_W(t,\tau)$ when

$$W(t) = X\cos 2\pi f_0 t + Y \sin 2\pi f_0 t, \tag{1}$$

and X and Y are uncorrelated random variables with $E[X] = E[Y] = 0$.

We start by writing

$$\begin{aligned} R_W(t,\tau) &= E\left[W(t)W(t+\tau)\right] && (2)\\ &= E\left[(X\cos 2\pi f_0 t + Y\sin 2\pi f_0 t)\,(X\cos 2\pi f_0(t+\tau) + Y\sin 2\pi f_0(t+\tau))\right]. && (3) \end{aligned}$$

Since X and Y are uncorrelated, $E[XY] = E[X]E[Y] = 0$. Thus, when we expand $E[W(t)W(t+\tau)]$ and take the expectation, all of the XY cross terms will be zero. This implies

$$R_W(t,\tau) = E\left[X^2\right]\cos 2\pi f_0 t \cos 2\pi f_0(t+\tau) + E\left[Y^2\right]\sin 2\pi f_0 t \sin 2\pi f_0(t+\tau) \tag{4}$$

Since $E[X] = E[Y] = 0$,

$$E\left[X^2\right] = \text{Var}[X] - (E\left[X\right])^2 = \sigma^2, \qquad E\left[Y^2\right] = \text{Var}[Y] - (E\left[Y\right])^2 = \sigma^2. \tag{5}$$

In addition, from Math Fact B.2, we use the formulas

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{6}$$
$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{7}$$

to write

$$R_W(t,\tau) = \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau + \cos 2\pi f_0(2t+\tau)\right) + \frac{\sigma^2}{2}\left(\cos 2\pi f_0\tau - \cos 2\pi f_0(2t+\tau)\right) \tag{8}$$

$$= \sigma^2 \cos 2\pi f_0\tau \tag{9}$$

Thus $R_W(t,\tau) = R_W(\tau)$. Since

$$E[W(t)] = E[X]\cos 2\pi f_0 t + E[Y]\sin 2\pi f_0 t = 0, \tag{10}$$

we can conclude that $W(t)$ is a wide sense stationary process. However, we note that if $E[X^2] \neq E[Y^2]$, then the $\cos 2\pi f_0(2t+\tau)$ terms in $R_W(t,\tau)$ would not cancel and $W(t)$ would not be wide sense stationary.

Problem 10.10.4 Solution

(a) In the problem statement, we are told that $X(t)$ has average power equal to 1. By Definition 10.16, the average power of $X(t)$ is $E[X^2(t)] = 1$.

(b) Since Θ has a uniform PDF over $[0, 2\pi]$,

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected value of the random phase cosine is

$$E[\cos(2\pi f_c t + \Theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_c t + \theta) f_\Theta(\theta)\, d\theta \tag{2}$$

$$= \int_0^{2\pi} \cos(2\pi f_c t + \theta)\frac{1}{2\pi} d\theta \tag{3}$$

$$= \frac{1}{2\pi}\sin(2\pi f_c t + \theta)\Big|_0^{2\pi} \tag{4}$$

$$= \frac{1}{2\pi}\left(\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)\right) = 0 \tag{5}$$

(c) Since $X(t)$ and Θ are independent,

$$E[Y(t)] = E[X(t)\cos(2\pi f_c t + \Theta)] = E[X(t)]\, E[\cos(2\pi f_c t + \Theta)] = 0 \tag{6}$$

Note that the mean of $Y(t)$ is zero no matter what the mean of $X(t)$ since the random phase cosine has zero mean.

(d) Independence of $X(t)$ and Θ results in the average power of $Y(t)$ being

$$E\left[Y^2(t)\right] = E\left[X^2(t)\cos^2(2\pi f_c t + \Theta)\right] \tag{7}$$

$$= E\left[X^2(t)\right] E\left[\cos^2(2\pi f_c t + \Theta)\right] \tag{8}$$

$$= E\left[\cos^2(2\pi f_c t + \Theta)\right] \tag{9}$$

Note that we have used the fact from part (a) that $X(t)$ has unity average power. To finish the problem, we use the trigonometric identity $\cos^2\phi = (1 + \cos 2\phi)/2$. This yields

$$E\left[Y^2(t)\right] = E\left[\frac{1}{2}\left(1 + \cos(2\pi(2f_c)t + \Theta)\right)\right] = 1/2 \tag{10}$$

Note that $E[\cos(2\pi(2f_c)t + \Theta)] = 0$ by the argument given in part (b) with $2f_c$ replacing f_c.

Problem 10.10.5 Solution

This proof simply parallels the proof of Theorem 10.12. For the first item, $R_X[0] = R_X[m, 0] = E[X_m^2]$. Since $X_m^2 \geq 0$, we must have $E[X_m^2] \geq 0$. For the second item, Definition 10.13 implies that

$$R_X[k] = R_X[m, k] = E[X_m X_{m+k}] = E[X_{m+k} X_m] = R_X[m + k, -k] \tag{1}$$

Since X_m is wide sense stationary, $R_X[m + k, -k] = R_X[-k]$. The final item requires more effort. First, we note that when X_m is wide sense stationary, $\text{Var}[X_m] = C_X[0]$, a constant for all t. Second, Theorem 4.17 says that

$$|C_X[m, k]| \leq \sigma_{X_m} \sigma_{X_{m+k}} = C_X[0]. \tag{2}$$

Note that $C_X[m, k] \leq |C_X[m, k]|$, and thus it follows that

$$C_X[m, k] \leq \sigma_{X_m} \sigma_{X_{m+k}} = C_X[0], \tag{3}$$

(This little step was unfortunately omitted from the proof of Theorem 10.12.) Now for any numbers a, b, and c, if $a \leq b$ and $c \geq 0$, then $(a + c)^2 \leq (b + c)^2$. Choosing $a = C_X[m, k]$, $b = C_X[0]$, and $c = \mu_X^2$ yields

$$\left(C_X[m, m + k] + \mu_X^2\right)^2 \leq \left(C_X[0] + \mu_X^2\right)^2 \tag{4}$$

In the above expression, the left side equals $(R_X[k])^2$ while the right side is $(R_X[0])^2$, which proves the third part of the theorem.

Problem 10.10.6 Solution

The solution to this problem is essentially the same as the proof of Theorem 10.13 except integrals are replaced by sums. First we verify that $\overline{X}_m$ is unbiased:

$$E\left[\overline{X}_m\right] = \frac{1}{2m + 1} E\left[\sum_{n=-m}^{m} X_n\right] \tag{1}$$

$$= \frac{1}{2m + 1} \sum_{n=-m}^{m} E[X_n] = \frac{1}{2m + 1} \sum_{n=-m}^{m} \mu_X = \mu_X \tag{2}$$

To show consistency, it is sufficient to show that $\lim_{m\to\infty} \text{Var}[\overline{X}_m] = 0$. First, we observe that $\overline{X}_m - \mu_X = \frac{1}{2m+1} \sum_{n=-m}^{m} (X_n - \mu_X)$. This implies

$$\text{Var}[\overline{X}(T)] = E\left[\left(\frac{1}{2m + 1} \sum_{n=-m}^{m} (X_n - \mu_X)\right)^2\right] \tag{3}$$

$$= E\left[\frac{1}{(2m + 1)^2} \left(\sum_{n=-m}^{m} (X_n - \mu_X)\right) \left(\sum_{n'=-m}^{m} (X_{n'} - \mu_X)\right)\right] \tag{4}$$

$$= \frac{1}{(2m + 1)^2} \sum_{n=-m}^{m} \sum_{n'=-m}^{m} E[(X_n - \mu_X)(X_{n'} - \mu_X)] \tag{5}$$

$$= \frac{1}{(2m + 1)^2} \sum_{n=-m}^{m} \sum_{n'=-m}^{m} C_X\left[n' - n\right]. \tag{6}$$

We note that

$$\sum_{n'=-m}^{m} C_X\left[n'-n\right] \leq \sum_{n'=-m}^{m} \left|C_X\left[n'-n\right]\right| \tag{7}$$

$$\leq \sum_{n'=-\infty}^{\infty} \left|C_X\left[n'-n\right]\right| = \sum_{k=-\infty}^{\infty} |C_X(k)| < \infty. \tag{8}$$

Hence there exists a constant K such that

$$\mathrm{Var}[\overline{X}_m] \leq \frac{1}{(2m+1)^2} \sum_{n=-m}^{m} K = \frac{K}{2m+1}. \tag{9}$$

Thus $\lim_{m\to\infty} \mathrm{Var}[\overline{X}_m] \leq \lim_{m\to\infty} \frac{K}{2m+1} = 0$.

Problem 10.11.1 Solution

(a) Since $X(t)$ and $Y(t)$ are independent processes,

$$E\left[W(t)\right] = E\left[X(t)Y(t)\right] = E\left[X(t)\right] E\left[Y(t)\right] = \mu_X \mu_Y. \tag{1}$$

In addition,

$$\begin{aligned} R_W(t,\tau) &= E\left[W(t)W(t+\tau)\right] && (2)\\ &= E\left[X(t)Y(t)X(t+\tau)Y(t+\tau)\right] && (3)\\ &= E\left[X(t)X(t+\tau)\right] E\left[Y(t)Y(t+\tau)\right] && (4)\\ &= R_X(\tau)R_Y(\tau) && (5) \end{aligned}$$

We can conclude that $W(t)$ is wide sense stationary.

(b) To examine whether $X(t)$ and $W(t)$ are jointly wide sense stationary, we calculate

$$R_{WX}(t,\tau) = E\left[W(t)X(t+\tau)\right] = E\left[X(t)Y(t)X(t+\tau)\right]. \tag{6}$$

By independence of $X(t)$ and $Y(t)$,

$$R_{WX}(t,\tau) = E\left[X(t)X(t+\tau)\right] E\left[Y(t)\right] = \mu_Y R_X(\tau). \tag{7}$$

Since $W(t)$ and $X(t)$ are both wide sense stationary and since $R_{WX}(t,\tau)$ depends only on the time difference τ, we can conclude from Definition 10.18 that $W(t)$ and $X(t)$ are jointly wide sense stationary.

Problem 10.11.2 Solution

To show that $X(t)$ and $X_i(t)$ are jointly wide sense stationary, we must first show that $X_i(t)$ is wide sense stationary and then we must show that the cross correlation $R_{XX_i}(t,\tau)$ is only a function of the time difference τ. For each $X_i(t)$, we have to check whether these facts are implied by the fact that $X(t)$ is wide sense stationary.

(a) Since $E[X_1(t)] = E[X(t+a)] = \mu_X$ and

$$R_{X_1}(t,\tau) = E[X_1(t)X_1(t+\tau)] \tag{1}$$
$$= E[X(t+a)X(t+\tau+a)] \tag{2}$$
$$= R_X(\tau), \tag{3}$$

we have verified that $X_1(t)$ is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_1}(t,\tau) = E[X(t)X_1(t+\tau)] \tag{4}$$
$$= E[X(t)X(t+\tau+a)] \tag{5}$$
$$= R_X(\tau+a). \tag{6}$$

Since $R_{XX_1}(t,\tau)$ depends on the time difference τ but not on the absolute time t, we conclude that $X(t)$ and $X_1(t)$ are jointly wide sense stationary.

(b) Since $E[X_2(t)] = E[X(at)] = \mu_X$ and

$$R_{X_2}(t,\tau) = E[X_2(t)X_2(t+\tau)] \tag{7}$$
$$= E[X(at)X(a(t+\tau))] \tag{8}$$
$$= E[X(at)X(at+a\tau)] = R_X(a\tau), \tag{9}$$

we have verified that $X_2(t)$ is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_2}(t,\tau) = E[X(t)X_2(t+\tau)] \tag{10}$$
$$= E[X(t)X(a(t+\tau))] \tag{11}$$
$$= R_X((a-1)t+\tau). \tag{12}$$

Except for the trivial case when $a = 1$ and $X_2(t) = X(t)$, $R_{XX_2}(t,\tau)$ depends on both the absolute time t and the time difference τ, we conclude that $X(t)$ and $X_2(t)$ are not jointly wide sense stationary.

Problem 10.11.3 Solution

(a) $Y(t)$ has autocorrelation function

$$R_Y(t,\tau) = E[Y(t)Y(t+\tau)] \tag{1}$$
$$= E[X(t-t_0)X(t+\tau-t_0)] \tag{2}$$
$$= R_X(\tau). \tag{3}$$

(b) The cross correlation of $X(t)$ and $Y(t)$ is

$$R_{XY}(t,\tau) = E[X(t)Y(t+\tau)] \tag{4}$$
$$= E[X(t)X(t+\tau-t_0)] \tag{5}$$
$$= R_X(\tau-t_0). \tag{6}$$

(c) We have already verified that $R_Y(t,\tau)$ depends only on the time difference τ. Since $E[Y(t)] = E[X(t-t_0)] = \mu_X$, we have verified that $Y(t)$ is wide sense stationary.

(d) Since $X(t)$ and $Y(t)$ are wide sense stationary and since we have shown that $R_{XY}(t,\tau)$ depends only on τ, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Comment: This problem is badly designed since the conclusions don't depend on the specific $R_X(\tau)$ given in the problem text. (Sorry about that!)

Problem 10.12.1 Solution

Writing $Y(t+\tau) = \int_0^{t+\tau} N(v)\,dv$ permits us to write the autocorrelation of $Y(t)$ as

$$R_Y(t,\tau) = E[Y(t)Y(t+\tau)] = E\left[\int_0^t \int_0^{t+\tau} N(u)N(v)\,dv\,du\right] \quad (1)$$
$$= \int_0^t \int_0^{t+\tau} E[N(u)N(v)]\,dv\,du \quad (2)$$
$$= \int_0^t \int_0^{t+\tau} \alpha\delta(u-v)\,dv\,du. \quad (3)$$

At this point, it matters whether $\tau \geq 0$ or if $\tau < 0$. When $\tau \geq 0$, then v ranges from 0 to $t+\tau$ and at some point in the integral over v we will have $v = u$. That is, when $\tau \geq 0$,

$$R_Y(t,\tau) = \int_0^t \alpha\,du = \alpha t. \quad (4)$$

When $\tau < 0$, then we must reverse the order of integration. In this case, when the inner integral is over u, we will have $u = v$ at some point so that

$$R_Y(t,\tau) = \int_0^{t+\tau} \int_0^t \alpha\delta(u-v)\,du\,dv = \int_0^{t+\tau} \alpha\,dv = \alpha(t+\tau). \quad (5)$$

Thus we see the autocorrelation of the output is

$$R_Y(t,\tau) = \alpha \min\{t, t+\tau\} \quad (6)$$

Perhaps surprisingly, $R_Y(t,\tau)$ is what we found in Example 10.19 to be the autocorrelation of a Brownian motion process. In fact, Brownian motion is the integral of the white noise process.

Problem 10.12.2 Solution

Let $\mu_i = E[X(t_i)]$.

(a) Since $C_X(t_1, t_2 - t_1) = \rho\sigma_1\sigma_2$, the covariance matrix is

$$\mathbf{C} = \begin{bmatrix} C_X(t_1,0) & C_X(t_1,t_2-t_1) \\ C_X(t_2,t_1-t_2) & C_X(t_2,0) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \quad (1)$$

Since $\mathbf{C}$ is a 2×2 matrix, it has determinant $|\mathbf{C}| = \sigma_1^2\sigma_2^2(1-\rho^2)$.

(b) Is is easy to verify that

$$\mathbf{C}^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} \dfrac{1}{\sigma_1^2} & \dfrac{-\rho}{\sigma_1\sigma_2} \\ \dfrac{-\rho}{\sigma_1\sigma_2} & \dfrac{1}{\sigma_1^2} \end{bmatrix} \quad (2)$$

(c) The general form of the multivariate density for $X(t_1), X(t_2)$ is

$$f_{X(t_1),X(t_2)}(x_1,x_2) = \frac{1}{(2\pi)^{k/2}|\mathbf{C}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})'\mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{X})} \quad (3)$$

where $k=2$ and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}'$ and $\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}'$. Hence,

$$\frac{1}{(2\pi)^{k/2}|\mathbf{C}|^{1/2}} = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}}. \quad (4)$$

Furthermore, the exponent is

$$
\begin{aligned}
&-\frac{1}{2}(\bar{x}-\bar{\mu}_X)^\top \mathbf{C}^{-1}(\bar{x}-\bar{\mu}_X) \\
&\qquad = -\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix} \frac{1}{1-\rho^2} \begin{bmatrix} \dfrac{1}{\sigma_1^2} & \dfrac{-\rho}{\sigma_1\sigma_2} \\ \dfrac{-\rho}{\sigma_1\sigma_2} & \dfrac{1}{\sigma_1^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix} && (5) \\
&\qquad = -\frac{\left(\dfrac{x_1-\mu_1}{\sigma_1}\right)^2 - \dfrac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\dfrac{x_2-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)} && (6)
\end{aligned}
$$

Plugging in each piece into the joint PDF $f_{X(t_1),X(t_2)}(x_1,x_2)$ given above, we obtain the bivariate Gaussian PDF.

Problem 10.12.3 Solution

Let $\mathbf{W} = \begin{bmatrix} W(t_1) & W(t_2) & \cdots & W(t_n) \end{bmatrix}'$ denote a vector of samples of a Brownian motion process. To prove that $W(t)$ is a Gaussian random process, we must show that $\mathbf{W}$ is a Gaussian random vector. To do so, let

$$
\begin{aligned}
\mathbf{X} &= \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}' && (1) \\
&= \begin{bmatrix} W(t_1) & W(t_2)-W(t_1) & W(t_3)-W(t_2) & \cdots & W(t_n)-W(t_{n-1}) \end{bmatrix}' && (2)
\end{aligned}
$$

denote the vector of increments. By the definition of Brownian motion, $X_1,\ldots,X_n$ is a sequence of independent Gaussian random variables. Thus $\mathbf{X}$ is a Gaussian random vector. Finally,

$$
\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} = \begin{bmatrix} X_1 \\ X_1+X_2 \\ \vdots \\ X_1+\cdots+X_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}}_{\mathbf{A}} \mathbf{X}. \qquad (3)
$$

Since $\mathbf{X}$ is a Gaussian random vector and $\mathbf{W} = \mathbf{AX}$ with $\mathbf{A}$ a rank n matrix, Theorem 5.16 implies that $\mathbf{W}$ is a Gaussian random vector.

Problem 10.13.1 Solution

From the instructions given in the problem, the program `noisycosine.m` will generate the four plots.

```
n=1000; t=0.001*(-n:n);
w=gaussrv(0,0.01,(2*n)+1);
%Continuous Time, Continuous Value
xcc=2*cos(2*pi*t) + w';
plot(t,xcc);
xlabel('\it t');ylabel('\it X_{cc}(t)');
axis([-1 1 -3 3]);
figure; %Continuous Time, Discrete Value
xcd=round(xcc); plot(t,xcd);
xlabel('\it t');ylabel('\it X_{cd}(t)');
axis([-1 1 -3 3]);
figure; %Discrete time, Continuous Value
ts=subsample(t,100); xdc=subsample(xcc,100);
plot(ts,xdc,'b.');
xlabel('\it t');ylabel('\it X_{dc}(t)');
axis([-1 1 -3 3]);
figure; %Discrete Time, Discrete Value
xdd=subsample(xcd,100); plot(ts,xdd,'b.');
xlabel('\it t');ylabel('\it X_{dd}(t)');
axis([-1 1 -3 3]);
```

In `noisycosine.m`, we use a function `subsample.m` to obtain the discrete time sample functions. In fact, `subsample` is hardly necessary since it's such a simple one-line MATLAB function:

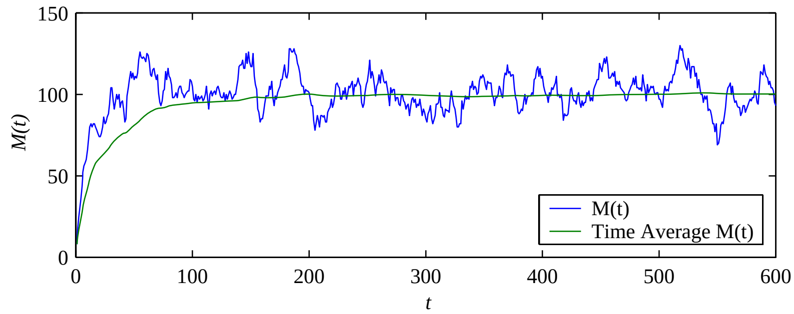
```
function y=subsample(x,n)
%input x(1), x(2) ...
%output y(1)=x(1), y(2)=x(1+n), y(3)=x(2n+1)
y=x(1:n:length(x));
```

However, we use it just to make `noisycosine.m` a little more clear.

Problem 10.13.2 Solution

```
>> t=(1:600)';
>> M=simswitch(10,0.1,t);
>> Mavg=cumsum(M)./t;
>> plot(t,M,t,Mavg);
```

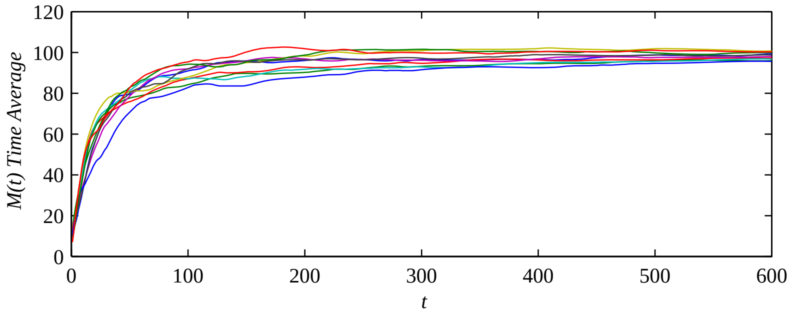
These commands will simulate the switch for 600 minutes, producing the vector `M` of samples of $M(t)$ each minute, the vector `Mavg` which is the sequence of time average estimates, and a plot resembling this one:



From the figure, it appears that the time average is converging to a value in th neighborhood of 100. In particular, because the switch is initially empty with $M(0) = 0$, it takes a few hundred minutes for the time average to climb to something close to 100. Following the problem instructions, we can write the following short program to examine ten simulation runs:

```
function Mavg=simswitchavg(T,k)
%Usage: Mavg=simswitchavg(T,k)
%simulate k runs of duration T of the
%telephone switch in Chapter 10
%and plot the time average of each run
t=(1:k)';
%each column of Mavg is a time average sample run
Mavg=zeros(T,k);
for n=1:k,
    M=simswitch(10,0.1,t);
    Mavg(:,n)=cumsum(M)./t;
end
plot(t,Mavg);
```

The command `simswitchavg(600,10)` produced this graph:



From the graph, one can see that even after $T = 600$ minutes, each sample run produces a time average $\overline{M}_{600}$ around 100. Note that in Chapter 12, we will able Markov chains to prove that the expected number of calls in the switch is in fact 100. However, note that even if T is large, $\overline{M}_T$ is still a random variable. From the above plot, one might guess that $\overline{M}_{600}$ has a standard deviation of perhaps $\sigma = 2$ or $\sigma = 3$. An exact calculation of the variance of $\overline{M}_{600}$ is fairly difficult because it is a sum of dependent random variables, each of which has a PDF that is in itself reasonably difficult to calculate.

Problem 10.13.3 Solution

In this problem, our goal is to find out the average number of ongoing calls in the switch. Before we use the approach of Problem 10.13.2, its worth a moment to consider the physical situation. In particular, calls arrive as a Poisson process of rate $\lambda = 100$ call/minute and each call has duration of *exactly* one minute. As a result, if we inspect the system at an arbitrary time t at least one minute past initialization, the number of calls at the switch will be exactly the number of calls N_1 that arrived in the previous minute. Since calls arrive as a Poisson proces of rate $\lambda = 100$ calls/minute. N_1 is a Poisson random variable with $E[N_1] = 100$.

In fact, this should be true for every inspection time t. Hence it should surprising if we compute the time average and find the time average number in the queue to be something other than 100. To check out this quickie analysis, we use the method of Problem 10.13.2. However, unlike Problem 10.13.2, we cannot directly use the function `simswitch.m` because the call duration are no longer exponential random variables. Instead, we must modify `simswitch.m` for the deterministic one minute call durations, yielding the function `simswitchd.m`:

```
function M=simswitchd(lambda,T,t)
%Poisson arrivals, rate lambda
%Deterministic (T) call duration
%For vector t of times
%M(i) = no. of calls at time t(i)
s=poissonarrivals(lambda,max(t));
y=s+T;
A=countup(s,t);
D=countup(y,t);
M=A-D;
```

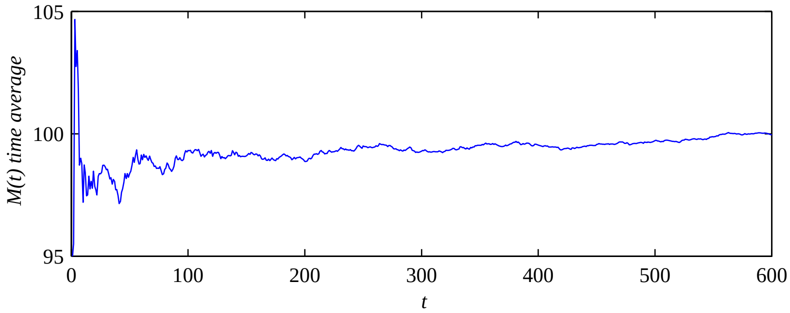
Note that if you compare `simswitch.m` in the text with `simswitchd.m` here, two changes occurred. The first is that the exponential call durations are replaced by the deterministic time T. The other change is that `count(s,t)` is replaced by `countup(s,t)`. In fact, `n=countup(x,y)` does exactly the same thing as `n=count(x,y)`; in both cases, `n(i)` is the number of elements less than or equal to $y(i)$. The difference is that `countup` requires that the vectors `x` and `y` be nondecreasing.

Now we use the same procedure as in Problem 10.13.2 and form the time average

$$\overline{M}(T) = \frac{1}{T}\sum_{t=1}^{T} M(t). \tag{1}$$

```
>> t=(1:600)';
>> M=simswitchd(100,1,t);
>> Mavg=cumsum(M)./t;
>> plot(t,Mavg);
```

We form and plot the time average using these commands will yield a plot vaguely similar to that shown below.



We used the word "vaguely" because at $t = 1$, the time average is simply the number of arrivals in the first minute, which is a Poisson ($\alpha = 100$) random variable which has not been averaged. Thus, the left side of the graph will be random for each run. As expected, the time average appears to be converging to 100.

Problem 10.13.4 Solution

The random variable S_n is the sum of n exponential (λ) random variables. That is, S_n is an Erlang (n, λ) random variable. Since $K = 1$ if and only if $S_n > T$, $P[K = 1] = P[S_n > T]$. Typically, $P[K = 1]$ is fairly high because

$$E[S_n] = \frac{n}{\lambda} = \frac{\lceil 1.1\lambda T \rceil}{\lambda} \approx 1.1T. \tag{1}$$

Increasing n increases $P[K = 1]$; however, `poissonarrivals` then does more work generating exponential random variables. Although we don't want to generate more exponential random variables than necessary, if we need to generate a lot of arrivals (ie a lot of exponential interarrival times), then MATLAB is typically faster generating a vector of them all at once rather than generating them one at a time. Choosing $n = \lceil 1.1\lambda T \rceil$ generates about 10 percent more exponential random variables than we typically need. However, as long as $P[K = 1]$ is high, a ten percent penalty won't be too costly.

When n is small, it doesn't much matter if we are efficient because the amount of calculation is small. The question that must be addressed is to estimate $P[K = 1]$ when n is large. In this case, we can use the central limit theorem because S_n is the sum of n exponential random variables. Since $E[S_n] = n/\lambda$ and $\text{Var}[S_n] = n/\lambda^2$,

$$P[S_n > T] = P\left[\frac{S_n - n/\lambda}{\sqrt{n/\lambda^2}} > \frac{T - n/\lambda}{\sqrt{n/\lambda^2}}\right] \approx Q\left(\frac{\lambda T - n}{\sqrt{n}}\right) \tag{2}$$

To simplify our algebra, we assume for large n that $0.1\lambda T$ is an integer. In this case, $n = 1.1\lambda T$ and

$$P[S_n > T] \approx Q\left(-\frac{0.1\lambda T}{\sqrt{1.1\lambda T}}\right) = \Phi\left(\sqrt{\frac{\lambda T}{110}}\right) \tag{3}$$

Thus for large λT, $P[K = 1]$ is very small. For example, if $\lambda T = 1{,}000$, $P[S_n > T] \approx \Phi(3.01) = 0.9987$. If $\lambda T = 10{,}000$, $P[S_n > T] \approx \Phi(9.5)$.

Problem 10.13.5 Solution

Following the problem instructions, we can write the function `newarrivals.m`. For convenience, here are `newarrivals` and `poissonarrivals` side by side.

```
function s=newarrivals(lam,T)
%Usage s=newarrivals(lam,T)
%Returns Poisson arrival times
%s=[s(1) ... s(n)] over [0,T]
n=poissonrv(lam*T,1);
s=sort(T*rand(n,1));
```

```
function s=poissonarrivals(lam,T)
%arrival times s=[s(1) ... s(n)]
%  s(n)<= T < s(n+1)
n=ceil(1.1*lam*T);
s=cumsum(exponentialrv(lam,n));
while (s(length(s))< T),
  s_new=s(length(s))+ ...
    cumsum(exponentialrv(lam,n));
  s=[s; s_new];
end
s=s(s<=T);
```

Clearly the code for `newarrivals` is shorter, more readable, and perhaps, with the help of Problem 10.6.4, more logical than `poissonarrivals`. Unfortunately this doesn't mean the code runs better. Here are some `cputime` comparisons:

```
>> t=cputime;s=poissonarrivals(1,100000);t=cputime-t
t =
    0.1110
>> t=cputime;s=newarrivals(1,100000);t=cputime-t
t =
    0.5310
>> t=cputime;poissonrv(100000,1);t=cputime-t
t =
    0.5200
>>
```

Unfortunately, these results were highly repeatable. The function `poissonarrivals` generated 100,000 arrivals of a rate 1 Poisson process required roughly 0.1 seconds of cpu time. The same task took `newarrivals` about 0.5 seconds, or roughly 5 times as long! In the `newarrivals` code, the culprit is the way `poissonrv` generates a single Poisson random variable with expected value 100,000. In this case, `poissonrv` generates the first 200,000 terms of the Poisson PMF! This required calculation is so large that it dominates the work need to generate 100,000 uniform random numbers. In fact, this suggests that a more efficient way to generate a Poisson (α) random variable N is to generate arrivals of a rate α Poisson process until the Nth arrival is after time 1.

Problem 10.13.6 Solution

We start with `brownian.m` to simulate the Brownian motion process with barriers, Since the goal is to estimate the barrier probability $P[|X(t)| = b]$, we don't keep track of the value of the process over all time. Also, we simply assume that a unit time step $\tau = 1$ for the process. Thus, the process starts at $n = 0$ at position $W_0 = 0$ at each step n, the position, if we haven't reached a barrier, is $W_n = W_{n-1} + X_n$, where $X_1, \ldots, X_T$ are iid Gaussian $(0, \sqrt{\alpha})$ random variables. Accounting for the effect of barriers,

$$W_n = \max(\min(W_{n-1} + X_n, b), -b). \tag{1}$$

To implement the simulation, we can generate the vector `x` of increments all at once. However to check at each time step whether we are crossing a barrier, we need to proceed sequentially. (This is analogous to the problem in Quiz 10.13.)

In brownbarrier shown below, pb(1) tracks how often the process touches the left barrier at $-b$ while pb(2) tracks how often the right side barrier at b is reached. By symmetry, $P[X(t) = b] = P[X(t) = -b]$. Thus if T is chosen very large, we should expect pb(1)=pb(2). The extent to which this is not the case gives an indication of the extent to which we are merely estimating the barrier probability. Here is the code and for each $T \in \{10{,}000, 100{,}000, 1{,}000{,}000\}$, here two sample runs:

```
function pb=brownwall(alpha,b,T)
%pb=brownwall(alpha,b,T)
%Brownian motion, param. alpha
%walls at [-b, b], sampled
%unit of time until time T
%each Returns vector pb:
%pb(1)=fraction of time at -b
%pb(2)=fraction of time at b
T=ceil(T);
x=sqrt(alpha).*gaussrv(0,1,T);
w=0;pb=zeros(1,2);
for k=1:T,
    w=w+x(k);
    if (w <= -b)
        w=-b;
        pb(1)=pb(1)+1;
    elseif (w >= b)
        w=b;
        pb(2)=pb(2)+1;
    end
end
pb=pb/T;
```

```
>> pb=brownwall(0.01,1,1e4)
pb =
    0.0301    0.0353
>> pb=brownwall(0.01,1,1e4)
pb =
    0.0417    0.0299
>> pb=brownwall(0.01,1,1e5)
pb =
    0.0333    0.0360
>> pb=brownwall(0.01,1,1e5)
pb =
    0.0341    0.0305
>> pb=brownwall(0.01,1,1e6)
pb =
    0.0323    0.0342
>> pb=brownwall(0.01,1,1e6)
pb =
    0.0333    0.0324
>>
```

The sample runs show that for $\alpha = 0.1$ and $b = 1$ that the

$$P[X(t) = -b] \approx P[X(t) = b] \approx 0.03. \tag{2}$$

Otherwise, the numerical simulations are not particularly instructive. Perhaps the most important thing to understand is that the Brownian motion process with barriers is *very* different from the ordinary Brownian motion process. Remember that for ordinary Brownian motion, the variance of $X(t)$ always increases linearly with t. For the process with barriers, $X^2(t) \leq b^2$ and thus $\text{Var}[X(t)] \leq b^2$. In fact, for the process with barriers, the PDF of $X(t)$ converges to a limit as t becomes large. If you're curious, you shouldn't have much trouble digging in the library to find out more.

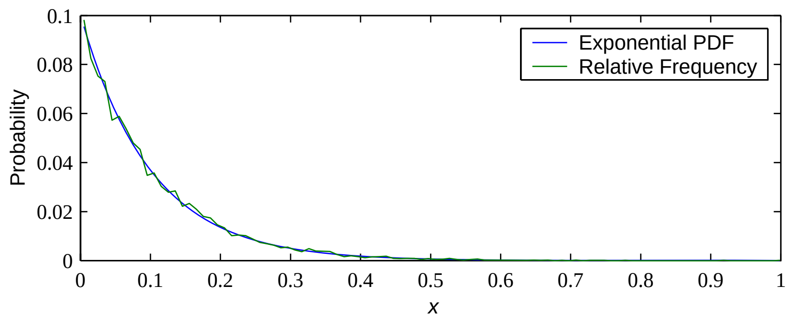
Problem 10.13.7 Solution

In this problem, we start with the simswitch.m code to generate the vector of departure times y. We then construct the vector I of inter-departure times. The command hist,20 will generate a 20 bin histogram of the departure times. The fact that this histogram resembles an exponential PDF suggests that perhaps it is reasonable to try to match the PDF of an exponential (μ) random variable against the histogram.

In most problems in which one wants to fit a PDF to measured data, a key issue is how to choose the parameters of the PDF. In this problem, choosing μ is simple. Recall that the switch has a Poisson arrival process of rate λ so interarrival times are exponential (λ) random variables. If $1/\mu < 1/\lambda$, then the average time between departures from the switch is less than the average time between arrivals to the switch. In this case, calls depart the switch faster than they arrive which is impossible because each departing call was an arriving call at an earlier time. Similarly, if $1/\mu > 1/\lambda$, then calls would be departing from the switch more slowly than they arrived. This can happen to an overloaded switch; however, it's impossible in this system because each arrival departs after an exponential time. Thus the only possibility is that $1/\mu = 1/\lambda$. In the program `simswitchdepart.m`, we plot a histogram of departure times for a switch with arrival rate λ against the scaled exponential (λ) PDF $\lambda e^{-\lambda x} b$ where b is the histogram bin size. Here is the code:

```
function I=simswitchdepart(lambda,mu,T)
%Usage: I=simswitchdepart(lambda,mu,T)
%Poisson arrivals, rate lambda
%Exponential (mu) call duration
%Over time [0,T], returns I,
%the vector of inter-departure times
%M(i) = no. of calls at time t(i)
s=poissonarrivals(lambda,T);
y=s+exponentialrv(mu,length(s));
y=sort(y);
n=length(y);
I=y-[0; y(1:n-1)]; %interdeparture times
imax=max(I);b=ceil(n/100);
id=imax/b; x=id/2:id:imax;
pd=hist(I,x); pd=pd/sum(pd);
px=exponentialpdf(lambda,x)*id;
plot(x,px,x,pd);
xlabel('\it x');ylabel('Probability');
legend('Exponential PDF','Relative Frequency');
```

Here is an example of the output corresponding to `simswitchdepart(10,1,1000)`.



As seen in the figure, the match is quite good. Although this is not a carefully designed statistical test of whether the inter-departure times are exponential random variables, it is enough evidence that one may want to pursue whether such a result can be proven.

In fact, the switch in this problem is an example of an $M/M/\infty$ queuing system for which it has been shown that not only do the inter-departure have an exponential distribution, but the steady-state departure process is a Poisson process. For the curious reader, details can be found, for example, in the text *Discrete Stochastic Processes* by Gallager.