Probability and Stochastic Processes

A Friendly Introduction for Electrical and Computer Engineers

SECOND EDITION

Problem Solutions

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- This solution manual remains under construction. The current count is that 678 (out of 687) problems have solutions. The unsolved problems are

  12.1.7, 12.1.8, 12.5.8, 12.5.9, 12.11.5 – 12.11.9.

  If you volunteer a solution for one of those problems, we'll be happy to include it . . . and, of course, "your wildest dreams will come true."

- Of course, the correctness of every single solution reamins unconfirmed. If you find errors or have suggestions or comments, please send email: ***ryates@winlab.rutgers.edu***.

- If you need to make solution sets for your class, you might like the *Solution Set Constructor* at the instructors site *www.winlab.rutgers.edu/probsolns*. If you need access, send email: ***ryates@winlab.rutgers.edu***.

- MATLAB functions written as solutions to homework problems can be found in the archive `matsoln.zip` (available to instructors) or in the directory `matsoln`. Other MATLAB functions used in the text or in these homework solutions can be found in the archive `matcode.zip` or directory `matcode`. The .m files in `matcode` are available for download from the Wiley website. Two other documents of interest are also available for download:
  - A manual `probmatlab.pdf` describing the `matcode .m` functions is also available.
  - The quiz solutions manual `quizsol.pdf`.

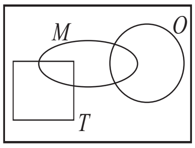
- A web-based solution set constructor for the second edition is available to instructors at ***http://www.winlab.rutgers.edu/probsolns***

- **The next update of this solution manual is likely to occur in January, 2006.**

Problem Solutions – Chapter 1

Problem 1.1.1 Solution

Based on the Venn diagram



the answers are fairly straightforward:

(a) Since $T \cap M \neq \phi$, T and M are not mutually exclusive.

(b) Every pizza is either Regular (R), or Tuscan (T). Hence $R \cup T = S$ so that R and T are collectively exhaustive. Thus its also (trivially) true that $R \cup T \cup M = S$. That is, R, T and M are also collectively exhaustive.

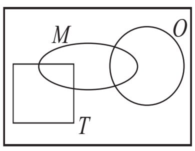
(c) From the Venn diagram, T and O are mutually exclusive. In words, this means that Tuscan pizzas never have onions or pizzas with onions are never Tuscan. As an aside, "Tuscan" is a fake pizza designation; one shouldn't conclude that people from Tuscany actually dislike onions.

(d) From the Venn diagram, $M \cap T$ and O are mutually exclusive. Thus Gerlanda's doesn't make Tuscan pizza with mushrooms and onions.

(e) Yes. In terms of the Venn diagram, these pizzas are in the set $(T \cup M \cup O)^c$.

Problem 1.1.2 Solution

Based on the Venn diagram,



the complete Gerlandas pizza menu is

- Regular without toppings
- Regular with mushrooms
- Regular with onions
- Regular with mushrooms and onions
- Tuscan without toppings
- Tuscan with mushrooms

Problem 1.2.1 Solution

(a) An outcome specifies whether the fax is high (h), medium (m), or low (l) speed, and whether the fax has two (t) pages or four (f) pages. The sample space is

$$S = \{ht, hf, mt, mf, lt, lf\}. \qquad (1)$$

(b) The event that the fax is medium speed is $A_1 = \{mt, mf\}$.

(c) The event that a fax has two pages is $A_2 = \{ht, mt, lt\}$.

(d) The event that a fax is either high speed or low speed is $A_3 = \{ht, hf, lt, lf\}$.

(e) Since $A_1 \cap A_2 = \{mt\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive.

(f) Since
$$A_1 \cup A_2 \cup A_3 = \{ht, hf, mt, mf, lt, lf\} = S, \tag{2}$$
the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.2 Solution

(a) The sample space of the experiment is
$$S = \{aaa, aaf, afa, faa, ffa, faf, aff, fff\}. \tag{1}$$

(b) The event that the circuit from Z fails is
$$Z_F = \{aaf, aff, faf, fff\}. \tag{2}$$
The event that the circuit from X is acceptable is
$$X_A = \{aaa, aaf, afa, aff\}. \tag{3}$$

(c) Since $Z_F \cap X_A = \{aaf, aff\} \neq \phi$, Z_F and X_A are not mutually exclusive.

(d) Since $Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S$, Z_F and X_A are not collectively exhaustive.

(e) The event that more than one circuit is acceptable is
$$C = \{aaa, aaf, afa, faa\}. \tag{4}$$
The event that at least two circuits fail is
$$D = \{ffa, faf, aff, fff\}. \tag{5}$$

(f) Inspection shows that $C \cap D = \phi$ so C and D are mutually exclusive.

(g) Since $C \cup D = S$, C and D are collectively exhaustive.

Problem 1.2.3 Solution

The sample space is
$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}. \tag{1}$$
The event H is the set
$$H = \{A\heartsuit, \ldots, K\heartsuit\}. \tag{2}$$

Problem 1.2.4 Solution

The sample space is

$$S = \left\{ \begin{array}{l} 1/1 \ldots 1/31, 2/1 \ldots 2/29, 3/1 \ldots 3/31, 4/1 \ldots 4/30, \\ 5/1 \ldots 5/31, 6/1 \ldots 6/30, 7/1 \ldots 7/31, 8/1 \ldots 8/31, \\ 9/1 \ldots 9/31, 10/1 \ldots 10/31, 11/1 \ldots 11/30, 12/1 \ldots 12/31 \end{array} \right\}. \tag{1}$$

The event H defined by the event of a July birthday is described by following 31 sample points.

$$H = \{7/1, 7/2, \ldots, 7/31\}. \tag{2}$$

Problem 1.2.5 Solution

Of course, there are many answers to this problem. Here are four event spaces.

1. We can divide students into engineers or non-engineers. Let A_1 equal the set of engineering students and A_2 the non-engineers. The pair $\{A_1, A_2\}$ is an event space.

2. We can also separate students by GPA. Let B_i denote the subset of students with GPAs G satisfying $i - 1 \leq G < i$. At Rutgers, $\{B_1, B_2, \ldots, B_5\}$ is an event space. Note that B_5 is the set of all students with perfect 4.0 GPAs. Of course, other schools use different scales for GPA.

3. We can also divide the students by age. Let C_i denote the subset of students of age i in years. At most universities, $\{C_{10}, C_{11}, \ldots, C_{100}\}$ would be an event space. Since a university may have prodigies either under 10 or over 100, we note that $\{C_0, C_1, \ldots\}$ is always an event space

4. Lastly, we can categorize students by attendance. Let D_0 denote the number of students who have missed zero lectures and let D_1 denote all other students. Although it is likely that D_0 is an empty set, $\{D_0, D_1\}$ is a well defined event space.

Problem 1.2.6 Solution

Let R_1 and R_2 denote the measured resistances. The pair (R_1, R_2) is an outcome of the experiment. Some event spaces include

1. If we need to check that neither resistance is too high, an event space is

$$A_1 = \{R_1 < 100, R_2 < 100\}, \qquad A_2 = \{\text{either } R_1 \geq 100 \text{ or } R_2 \geq 100\}. \tag{1}$$

2. If we need to check whether the first resistance exceeds the second resistance, an event space is

$$B_1 = \{R_1 > R_2\} \qquad B_2 = \{R_1 \leq R_2\}. \tag{2}$$

3. If we need to check whether each resistance doesn't fall below a minimum value (in this case 50 ohms for R_1 and 100 ohms for R_2), an event space is

$$C_1 = \{R_1 < 50, R_2 < 100\}, \qquad C_2 = \{R_1 < 50, R_2 \geq 100\}, \tag{3}$$

$$C_3 = \{R_1 \geq 50, R_2 < 100\}, \qquad C_4 = \{R_1 \geq 50, R_2 \geq 100\}. \tag{4}$$

4. If we want to check whether the resistors in parallel are within an acceptable range of 90 to 110 ohms, an event space is

$$D_1 = \left\{(1/R_1 + 1/R_2)^{-1} < 90\right\}, \tag{5}$$

$$D_2 = \left\{90 \leq (1/R_1 + 1/R_2)^{-1} \leq 110\right\}, \tag{6}$$

$$D_2 = \left\{110 < (1/R_1 + 1/R_2)^{-1}\right\}. \tag{7}$$

Problem 1.3.1 Solution
The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}. \quad (1)$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$. The questions can be answered using Theorem 1.5.

(a) The probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \quad (2)$$

(b) The probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \quad (3)$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \quad (4)$$

Problem 1.3.2 Solution
A sample outcome indicates whether the cell phone is handheld (H) or mobile (M) and whether the speed is fast (F) or slow (W). The sample space is

$$S = \{HF, HW, MF, MW\}. \quad (1)$$

The problem statement tells us that $P[HF] = 0.2$, $P[MW] = 0.1$ and $P[F] = 0.5$. We can use these facts to find the probabilities of the other outcomes. In particular,

$$P[F] = P[HF] + P[MF]. \quad (2)$$

This implies

$$P[MF] = P[F] - P[HF] = 0.5 - 0.2 = 0.3. \quad (3)$$

Also, since the probabilities must sum to 1,

$$P[HW] = 1 - P[HF] - P[MF] - P[MW] = 1 - 0.2 - 0.3 - 0.1 = 0.4. \quad (4)$$

Now that we have found the probabilities of the outcomes, finding any other probability is easy.

(a) The probability a cell phone is slow is

$$P[W] = P[HW] + P[MW] = 0.4 + 0.1 = 0.5. \quad (5)$$

(b) The probability that a cell hpone is mobile and fast is $P[MF] = 0.3$.

(c) The probability that a cell phone is handheld is

$$P[H] = P[HF] + P[HW] = 0.2 + 0.4 = 0.6. \quad (6)$$

Problem 1.3.3 Solution

A reasonable probability model that is consistent with the notion of a shuffled deck is that each card in the deck is equally likely to be the first card. Let H_i denote the event that the first card drawn is the ith heart where the first heart is the ace, the second heart is the deuce and so on. In that case, $P[H_i] = 1/52$ for $1 \le i \le 13$. The event H that the first card is a heart can be written as the disjoint union

$$H = H_1 \cup H_2 \cup \cdots \cup H_{13}. \tag{1}$$

Using Theorem 1.1, we have

$$P[H] = \sum_{i=1}^{13} P[H_i] = 13/52. \tag{2}$$

This is the answer you would expect since 13 out of 52 cards are hearts. The point to keep in mind is that this is not just the common sense answer but is the result of a probability model for a shuffled deck and the axioms of probability.

Problem 1.3.4 Solution

Let s_i denote the outcome that the down face has i dots. The sample space is $S = \{s_1, \ldots, s_6\}$. The probability of each sample outcome is $P[s_i] = 1/6$. From Theorem 1.1, the probability of the event E that the roll is even is

$$P[E] = P[s_2] + P[s_4] + P[s_6] = 3/6. \tag{1}$$

Problem 1.3.5 Solution

Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \tag{1}$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \tag{2}$$

The probability of failing requires the student to get a grade less than 4.

$$P[\text{Failing}] = P[3] + P[2] + P[1] + P[0] = 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \tag{3}$$

Problem 1.4.1 Solution

From the table we look to add all the disjoint events that contain H_0 to express the probability that a caller makes no hand-offs as

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5. \tag{1}$$

In a similar fashion we can express the probability that a call is brief by

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}$$

The probability that a call is long or makes at least two hand-offs is

$$\begin{aligned} P[L \cup H_2] &= P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] && (3)\\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. && (4) \end{aligned}$$

Problem 1.4.2 Solution

(a) From the given probability distribution of billed minutes, M, the probability that a call is billed for more than 3 minutes is

$$P[L] = 1 - P[3 \text{ or fewer billed minutes}] \tag{1}$$
$$= 1 - P[B_1] - P[B_2] - P[B_3] \tag{2}$$
$$= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 \tag{3}$$
$$= (1-\alpha)^3 = 0.57. \tag{4}$$

(b) The probability that a call will billed for 9 minutes or less is

$$P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} = 1 - (0.57)^3. \tag{5}$$

Problem 1.4.3 Solution

The first generation consists of two plants each with genotype yg or gy. They are crossed to produce the following second generation genotypes, $S = \{yy, yg, gy, gg\}$. Each genotype is just as likely as any other so the probability of each genotype is consequently 1/4. A pea plant has yellow seeds if it possesses at least one dominant y gene. The set of pea plants with yellow seeds is

$$Y = \{yy, yg, gy\}. \tag{1}$$

So the probability of a pea plant with yellow seeds is

$$P[Y] = P[yy] + P[yg] + P[gy] = 3/4. \tag{2}$$

Problem 1.4.4 Solution

Each statement is a consequence of part 4 of Theorem 1.4.

(a) Since $A \subset A \cup B$, $P[A] \leq P[A \cup B]$.

(b) Since $B \subset A \cup B$, $P[B] \leq P[A \cup B]$.

(c) Since $A \cap B \subset A$, $P[A \cap B] \leq P[A]$.

(d) Since $A \cap B \subset B$, $P[A \cap B] \leq P[B]$.

Problem 1.4.5 Solution

Specifically, we will use Theorem 1.7(c) which states that for any events A and B,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]. \tag{1}$$

To prove the union bound by induction, we first prove the theorem for the case of $n = 2$ events. In this case, by Theorem 1.7(c),

$$P[A_1 \cup A_2] = P[A_1] + P[A_2] - P[A_1 \cap A_2]. \tag{2}$$

By the first axiom of probability, $P[A_1 \cap A_2] \geq 0$. Thus,

$$P[A_1 \cup A_2] \leq P[A_1] + P[A_2]. \tag{3}$$

which proves the union bound for the case $n = 2$. Now we make our induction hypothesis that the union-bound holds for any collection of $n-1$ subsets. In this case, given subsets $A_1, \ldots, A_n$, we define

$$A = A_1 \cup A_2 \cup \cdots \cup A_{n-1}, \qquad B = A_n. \tag{4}$$

By our induction hypothesis,

$$P[A] = P[A_1 \cup A_2 \cup \cdots \cup A_{n-1}] \leq P[A_1] + \cdots + P[A_{n-1}]. \tag{5}$$

This permits us to write

$$\begin{aligned}
P[A_1 \cup \cdots \cup A_n] &= P[A \cup B] && (6)\\
&\leq P[A] + P[B] \qquad \text{(by the union bound for } n = 2\text{)} && (7)\\
&= P[A_1 \cup \cdots \cup A_{n-1}] + P[A_n] && (8)\\
&\leq P[A_1] + \cdots P[A_{n-1}] + P[A_n] && (9)
\end{aligned}$$

which completes the inductive proof.

Problem 1.4.6 Solution

(a) For convenience, let $p_i = P[FH_i]$ and $q_i = P[VH_i]$. Using this shorthand, the six unknowns $p_0, p_1, p_2, q_0, q_1, q_2$ fill the table as

$$\begin{array}{c|ccc} & H_0 & H_1 & H_2 \\ \hline F & p_0 & p_1 & p_2 \\ V & q_0 & q_1 & q_2 \end{array}. \tag{1}$$

However, we are given a number of facts:

$$p_0 + q_0 = 1/3, \qquad p_1 + q_1 = 1/3, \tag{2}$$

$$p_2 + q_2 = 1/3, \qquad p_0 + p_1 + p_2 = 5/12. \tag{3}$$

Other facts, such as $q_0 + q_1 + q_2 = 7/12$, can be derived from these facts. Thus, we have four equations and six unknowns, choosing p_0 and p_1 will specify the other unknowns. Unfortunately, arbitrary choices for either p_0 or p_1 will lead to negative values for the other probabilities. In terms of p_0 and p_1, the other unknowns are

$$q_0 = 1/3 - p_0, \qquad p_2 = 5/12 - (p_0 + p_1), \tag{4}$$

$$q_1 = 1/3 - p_1, \qquad q_2 = p_0 + p_1 - 1/12. \tag{5}$$

Because the probabilities must be nonnegative, we see that

$$0 \leq p_0 \leq 1/3, \tag{6}$$

$$0 \leq p_1 \leq 1/3, \tag{7}$$

$$1/12 \leq p_0 + p_1 \leq 5/12. \tag{8}$$

Although there are an infinite number of solutions, three possible solutions are:

$$p_0 = 1/3, \qquad p_1 = 1/12, \qquad p_2 = 0, \tag{9}$$
$$q_0 = 0, \qquad q_1 = 1/4, \qquad q_2 = 1/3. \tag{10}$$

and

$$p_0 = 1/4, \qquad p_1 = 1/12, \qquad p_2 = 1/12, \tag{11}$$
$$q_0 = 1/12, \qquad q_1 = 3/12, \qquad q_2 = 3/12. \tag{12}$$

and

$$p_0 = 0, \qquad p_1 = 1/12, \qquad p_2 = 1/3, \tag{13}$$
$$q_0 = 1/3, \qquad q_1 = 3/12, \qquad q_2 = 0. \tag{14}$$

(b) In terms of the p_i, q_i notation, the new facts are $p_0 = 1/4$ and $q_1 = 1/6$. These extra facts uniquely specify the probabilities. In this case,

$$p_0 = 1/4, \qquad p_1 = 1/6, \qquad p_2 = 0, \tag{15}$$
$$q_0 = 1/12, \qquad q_1 = 1/6, \qquad q_2 = 1/3. \tag{16}$$

Problem 1.4.7 Solution

It is tempting to use the following proof:

> Since S and ϕ are mutually exclusive, and since $S = S \cup \phi$,
>
> $$1 = P[S \cup \phi] = P[S] + P[\phi]. \tag{1}$$
>
> Since $P[S] = 1$, we must have $P[\phi] = 0$.

The above "proof" used the property that for mutually exclusive sets A_1 and A_2,

$$P[A_1 \cup A_2] = P[A_1] + P[A_2]. \tag{2}$$

The problem is that this property is a consequence of the three axioms, and thus must be proven. For a proof that uses just the three axioms, let A_1 be an arbitrary set and for $n = 2, 3, \ldots$, let $A_n = \phi$. Since $A_1 = \cup_{i=1}^{\infty} A_i$, we can use Axiom 3 to write

$$P[A_1] = P[\cup_{i=1}^{\infty} A_i] = P[A_1] + P[A_2] + \sum_{i=3}^{\infty} P[A_i]. \tag{3}$$

By subtracting $P[A_1]$ from both sides, the fact that $A_2 = \phi$ permits us to write

$$P[\phi] + \sum_{n=3}^{\infty} P[A_i] = 0. \tag{4}$$

By Axiom 1, $P[A_i] \geq 0$ for all i. Thus, $\sum_{n=3}^{\infty} P[A_i] \geq 0$. This implies $P[\phi] \leq 0$. Since Axiom 1 requires $P[\phi] \geq 0$, we must have $P[\phi] = 0$.

Problem 1.4.8 Solution

Following the hint, we define the set of events $\{A_i | i = 1, 2, \ldots\}$ such that $i = 1, \ldots, m$, $A_i = B_i$ and for $i > m$, $A_i = \phi$. By construction, $\cup_{i=1}^m B_i = \cup_{i=1}^\infty A_i$. Axiom 3 then implies

$$P[\cup_{i=1}^m B_i] = P[\cup_{i=1}^\infty A_i] = \sum_{i=1}^\infty P[A_i]. \tag{1}$$

For $i > m$, $P[A_i] = P[\phi] = 0$, yielding the claim $P[\cup_{i=1}^m B_i] = \sum_{i=1}^m P[A_i] = \sum_{i=1}^m P[B_i]$.

Note that the fact that $P[\phi] = 0$ follows from Axioms 1 and 2. This problem is more challenging if you just use Axiom 3. We start by observing

$$P[\cup_{i=1}^m B_i] = \sum_{i=1}^{m-1} P[B_i] + \sum_{i=m}^\infty P[A_i]. \tag{2}$$

Now, we use Axiom 3 again on the countably infinite sequence $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^\infty P[A_i] = P[A_m \cup A_{m+1} \cup \cdots] = P[B_m]. \tag{3}$$

Thus, we have used just Axiom 3 to prove Theorem 1.4: $P[\cup_{i=1}^m B_i] = \sum_{i=1}^m P[B_i]$.

Problem 1.4.9 Solution

Each claim in Theorem 1.7 requires a proof from which we can check which axioms are used. However, the problem is somewhat hard because there may still be a simpler proof that uses fewer axioms. Still, the proof of each part will need Theorem 1.4 which we now prove.

For the mutually exclusive events $B_1, \ldots, B_m$, let $A_i = B_i$ for $i = 1, \ldots, m$ and let $A_i = \phi$ for $i > m$. In that case, by Axiom 3,

$$\begin{aligned} P[B_1 \cup B_2 \cup \cdots \cup B_m] &= P[A_1 \cup A_2 \cup \cdots] && (1)\\ &= \sum_{i=1}^{m-1} P[A_i] + \sum_{i=m}^\infty P[A_i] && (2)\\ &= \sum_{i=1}^{m-1} P[B_i] + \sum_{i=m}^\infty P[A_i]. && (3) \end{aligned}$$

Now, we use Axiom 3 again on $A_m, A_{m+1}, \ldots$ to write

$$\sum_{i=m}^\infty P[A_i] = P[A_m \cup A_{m+1} \cup \cdots] = P[B_m]. \tag{4}$$

Thus, we have used just Axiom 3 to prove Theorem 1.4:

$$P[B_1 \cup B_2 \cup \cdots \cup B_m] = \sum_{i=1}^m P[B_i]. \tag{5}$$

(a) To show $P[\phi] = 0$, let $B_1 = S$ and let $B_2 = \phi$. Thus by Theorem 1.4,

$$P[S] = P[B_1 \cup B_2] = P[B_1] + P[B_2] = P[S] + P[\phi]. \tag{6}$$

Thus, $P[\phi] = 0$. Note that this proof uses only Theorem 1.4 which uses only Axiom 3.

(b) Using Theorem 1.4 with $B_1 = A$ and $B_2 = A^c$, we have

$$P[S] = P[A \cup A^c] = P[A] + P[A^c]. \tag{7}$$

Since, Axiom 2 says $P[S] = 1$, $P[A^c] = 1 - P[A]$. This proof uses Axioms 2 and 3.

(c) By Theorem 1.2, we can write both A and B as unions of disjoint events:

$$A = (AB) \cup (AB^c) \qquad B = (AB) \cup (A^cB). \tag{8}$$

Now we apply Theorem 1.4 to write

$$P[A] = P[AB] + P[AB^c], \qquad P[B] = P[AB] + P[A^cB]. \tag{9}$$

We can rewrite these facts as

$$P[AB^c] = P[A] - P[AB], \qquad P[A^cB] = P[B] - P[AB]. \tag{10}$$

Note that so far we have used only Axiom 3. Finally, we observe that $A \cup B$ can be written as the union of mutually exclusive events

$$A \cup B = (AB) \cup (AB^c) \cup (A^cB). \tag{11}$$

Once again, using Theorem 1.4, we have

$$P[A \cup B] = P[AB] + P[AB^c] + P[A^cB] \tag{12}$$

Substituting the results of Equation (10) into Equation (12) yields

$$P[A \cup B] = P[AB] + P[A] - P[AB] + P[B] - P[AB], \tag{13}$$

which completes the proof. Note that this claim required only Axiom 3.

(d) Observe that since $A \subset B$, we can write B as the disjoint union $B = A \cup (A^cB)$. By Theorem 1.4 (which uses Axiom 3),

$$P[B] = P[A] + P[A^cB]. \tag{14}$$

By Axiom 1, $P[A^cB] \geq 0$, hich implies $P[A] \leq P[B]$. This proof uses Axioms 1 and 3.

Problem 1.5.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6. \tag{1}$$

The probability a brief call will have no handoffs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}. \tag{2}$$

(b) The probability of one handoff is $P[H_1] = P[H_1B] + P[H_1L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \tag{3}$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \tag{4}$$

Problem 1.5.2 Solution

Let s_i denote the outcome that the roll is i. So, for $1 \leq i \leq 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability $1/6$, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \tag{1}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \tag{2}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $3/6$. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3. \tag{3}$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{4}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{5}$$

Problem 1.5.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \tag{1}$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \tag{2}$$

Problem 1.5.4 Solution

Define D as the event that a pea plant has two dominant y genes. To find the conditional probability of D given the event Y, corresponding to a plant having yellow seeds, we look to evaluate

$$P[D|Y] = \frac{P[DY]}{P[Y]}. \tag{1}$$

Note that $P[DY]$ is just the probability of the genotype yy. From Problem 1.4.3, we found that with respect to the color of the peas, the genotypes yy, yg, gy, and gg were all equally likely. This implies

$$P[DY] = P[yy] = 1/4 \qquad P[Y] = P[yy, gy, yg] = 3/4. \tag{2}$$

Thus, the conditional probability can be expressed as

$$P[D|Y] = \frac{P[DY]}{P[Y]} = \frac{1/4}{3/4} = 1/3. \tag{3}$$

Problem 1.5.5 Solution

The sample outcomes can be written ijk where the first card drawn is i, the second is j and the third is k. The sample space is

$$S = \{234, 243, 324, 342, 423, 432\}. \tag{1}$$

and each of the six outcomes has probability 1/6. The events $E_1, E_2, E_3, O_1, O_2, O_3$ are

$$E_1 = \{234, 243, 423, 432\}, \qquad O_1 = \{324, 342\}, \tag{2}$$
$$E_2 = \{243, 324, 342, 423\}, \qquad O_2 = \{234, 432\}, \tag{3}$$
$$E_3 = \{234, 324, 342, 432\}, \qquad O_3 = \{243, 423\}. \tag{4}$$

(a) The conditional probability the second card is even given that the first card is even is

$$P[E_2|E_1] = \frac{P[E_2E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2. \tag{5}$$

(b) The conditional probability the first card is even given that the second card is even is

$$P[E_1|E_2] = \frac{P[E_1E_2]}{P[E_2]} = \frac{P[243, 423]}{P[243, 324, 342, 423]} = \frac{2/6}{4/6} = 1/2. \tag{6}$$

(c) The probability the first two cards are even given the third card is even is

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = 0. \tag{7}$$

(d) The conditional probabilities the second card is even given that the first card is odd is

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1. \tag{8}$$

(e) The conditional probability the second card is odd given that the first card is odd is

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0. \tag{9}$$

Problem 1.5.6 Solution

The problem statement yields the obvious facts that $P[L] = 0.16$ and $P[H] = 0.10$. The words "10% of the ticks that had either Lyme disease or HGE carried both diseases" can be written as

$$P[LH|L \cup H] = 0.10. \tag{1}$$

(a) Since $LH \subset L \cup H$,

$$P[LH|L \cup H] = \frac{P[LH \cap (L \cup H)]}{P[L \cup H]} = \frac{P[LH]}{P[L \cup H]} = 0.10. \tag{2}$$

Thus,

$$P[LH] = 0.10P[L \cup H] = 0.10 (P[L] + P[H] - P[LH]). \tag{3}$$

Since $P[L] = 0.16$ and $P[H] = 0.10$,

$$P[LH] = \frac{0.10\,(0.16 + 0.10)}{1.1} = 0.0236. \tag{4}$$

(b) The conditional probability that a tick has HGE given that it has Lyme disease is

$$P[H|L] = \frac{P[LH]}{P[L]} = \frac{0.0236}{0.16} = 0.1475. \quad (5)$$

Problem 1.6.1 Solution

This problems asks whether A and B can be independent events yet satisfy $A = B$? By definition, events A and B are independent if and only if $P[AB] = P[A]P[B]$. We can see that if $A = B$, that is they are the same set, then

$$P[AB] = P[AA] = P[A] = P[B]. \quad (1)$$

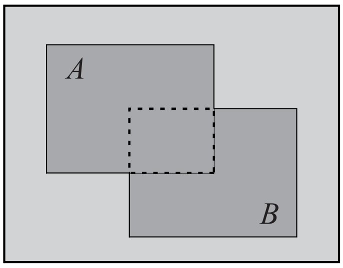
Thus, for A and B to be the same set and also independent,

$$P[A] = P[AB] = P[A]P[B] = (P[A])^2. \quad (2)$$

There are two ways that this requirement can be satisfied:

- $P[A] = 1$ implying $A = B = S$.
- $P[A] = 0$ implying $A = B = \phi$.

Problem 1.6.2 Solution



In the Venn diagram, assume the sample space has area 1 corresponding to probability 1. As drawn, both A and B have area 1/4 so that $P[A] = P[B] = 1/4$. Moreover, the intersection AB has area 1/16 and covers 1/4 of A and 1/4 of B. That is, A and B are independent since

$$P[AB] = P[A]P[B]. \quad (1)$$

Problem 1.6.3 Solution

(a) Since A and B are disjoint, $P[A \cap B] = 0$. Since $P[A \cap B] = 0$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8. \quad (1)$$

A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$P[A \cap B^c] = P[A] = 1/4. \quad (2)$$

It also follows that $P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8$.

(b) Events A and B are dependent since $P[AB] \neq P[A]P[B]$.

(c) Since C and D are independent,

$$P[C \cap D] = P[C]\,P[D] = 15/64. \tag{3}$$

The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{4}$$

It follows that

$$P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] \tag{5}$$
$$= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{6}$$

Using DeMorgan's law, we have

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64. \tag{7}$$

(d) Since $P[C^c D^c] = P[C^c]P[D^c]$, C^c and D^c are independent.

Problem 1.6.4 Solution

(a) Since $A \cap B = \emptyset$, $P[A \cap B] = 0$. To find $P[B]$, we can write

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \tag{1}$$
$$5/8 = 3/8 + P[B] - 0. \tag{2}$$

Thus, $P[B] = 1/4$. Since A is a subset of B^c, $P[A \cap B^c] = P[A] = 3/8$. Furthermore, since A is a subset of B^c, $P[A \cup B^c] = P[B^c] = 3/4$.

(b) The events A and B are dependent because

$$P[AB] = 0 \neq 3/32 = P[A]\,P[B]. \tag{3}$$

(c) Since C and D are independent $P[CD] = P[C]P[D]$. So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \tag{4}$$

In addition, $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$. To find $P[C^c \cap D^c]$, we first observe that

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \tag{5}$$

By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \tag{6}$$

Note that a second way to find $P[C^c \cap D^c]$ is to use the fact that if C and D are independent, then C^c and D^c are independent. Thus

$$P[C^c \cap D^c] = P[C^c]\,P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \tag{7}$$

Finally, since C and D are independent events, $P[C|D] = P[C] = 1/2$.

(d) Note that we found $P[C \cup D] = 5/6$. We can also use the earlier results to show

$$P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c] = 1/2 + (1 - 2/3) - 1/6 = 2/3. \tag{8}$$

(e) By Definition 1.7, events C and D^c are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]\,P[D^c]. \tag{9}$$

Problem 1.6.5 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \tag{1}$$

Each event A_i has probability 1/2. Moreover, each pair of events is independent since

$$P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4. \tag{2}$$

However, the three events A_1, A_2, A_3 are not independent since

$$P[A_1A_2A_3] = 0 \neq P[A_1] P[A_2] P[A_3]. \tag{3}$$

Problem 1.6.6 Solution

There are 16 distinct equally likely outcomes for the second generation of pea plants based on a first generation of $\{rwyg, rwgy, wryg, wrgy\}$. They are listed below

$$\begin{array}{cccc} rryy & rryg & rrgy & rrgg \\ rwyy & rwyg & rwgy & rwgg \\ wryy & wryg & wrgy & wrgg \\ wwyy & wwyg & wwgy & wwgg \end{array} \tag{1}$$

A plant has yellow seeds, that is event Y occurs, if a plant has at least one dominant y gene. Except for the four outcomes with a pair of recessive g genes, the remaining 12 outcomes have yellow seeds. From the above, we see that

$$P[Y] = 12/16 = 3/4 \tag{2}$$

and

$$P[R] = 12/16 = 3/4. \tag{3}$$

To find the conditional probabilities $P[R|Y]$ and $P[Y|R]$, we first must find $P[RY]$. Note that RY, the event that a plant has rounded yellow seeds, is the set of outcomes

$$RY = \{rryy, rryg, rrgy, rwyy, rwyg, rwgy, wryy, wryg, wrgy\}. \tag{4}$$

Since $P[RY] = 9/16$,

$$P[Y|R] = \frac{P[RY]}{P[R]} = \frac{9/16}{3/4} = 3/4 \tag{5}$$

and

$$P[R|Y] = \frac{P[RY]}{P[Y]} = \frac{9/16}{3/4} = 3/4. \tag{6}$$

Thus $P[R|Y] = P[R]$ and $P[Y|R] = P[Y]$ and R and Y are independent events. There are four visibly different pea plants, corresponding to whether the peas are round (R) or not (R^c), or yellow (Y) or not (Y^c). These four visible events have probabilities

$$P[RY] = 9/16 \qquad P[RY^c] = 3/16, \tag{7}$$

$$P[R^cY] = 3/16 \qquad P[R^cY^c] = 1/16. \tag{8}$$

Problem 1.6.7 Solution

(a) For any events A and B, we can write the law of total probability in the form of

$$P[A] = P[AB] + P[AB^c]. \quad (1)$$

Since A and B are independent, $P[AB] = P[A]P[B]$. This implies

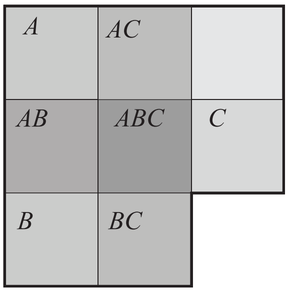
$$P[AB^c] = P[A] - P[A]P[B] = P[A](1 - P[B]) = P[A]P[B^c]. \quad (2)$$

Thus A and B^c are independent.

(b) Proving that A^c and B are independent is not really necessary. Since A and B are arbitrary labels, it is really the same claim as in part (a). That is, simply reversing the labels of A and B proves the claim. Alternatively, one can construct exactly the same proof as in part (a) with the labels A and B reversed.

(c) To prove that A^c and B^c are independent, we apply the result of part (a) to the sets A and B^c. Since we know from part (a) that A and B^c are independent, part (b) says that A^c and B^c are independent.

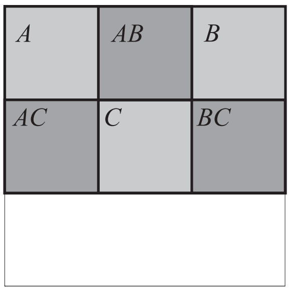
Problem 1.6.8 Solution



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area 1/2 and thus probability 1/2. Moreover, the three way intersection ABC has probability 1/8. Thus A, B, and C are mutually independent since

$$P[ABC] = P[A]P[B]P[C]. \quad (1)$$

Problem 1.6.9 Solution



In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, A, B, and C each have area 1/3 and thus probability 1/3. The three way intersection ABC has zero probability, implying A, B, and C are not mutually independent since

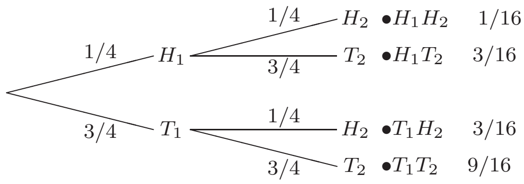
$$P[ABC] = 0 \neq P[A]P[B]P[C]. \quad (1)$$

However, AB, BC, and AC each has area 1/9. As a result, each pair of events is independent since

$$P[AB] = P[A]P[B], \qquad P[BC] = P[B]P[C], \qquad P[AC] = P[A]P[C]. \quad (2)$$

Problem 1.7.1 Solution

A sequential sample space for this experiment is



(a) From the tree, we observe

$$P[H_2] = P[H_1H_2] + P[T_1H_2] = 1/4. \tag{1}$$

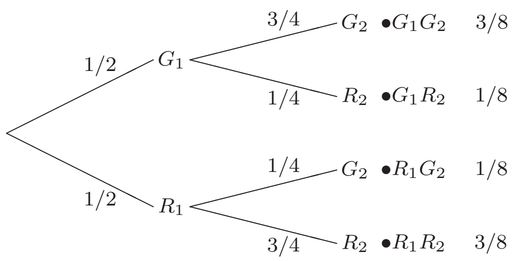
This implies

$$P[H_1|H_2] = \frac{P[H_1H_2]}{P[H_2]} = \frac{1/16}{1/4} = 1/4. \tag{2}$$

(b) The probability that the first flip is heads and the second flip is tails is $P[H_1T_2] = 3/16$.

Problem 1.7.2 Solution

The tree with adjusted probabilities is



From the tree, the probability the second light is green is

$$P[G_2] = P[G_1G_2] + P[R_1G_2] = 3/8 + 1/8 = 1/2. \tag{1}$$

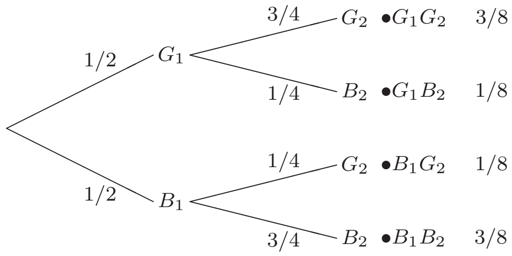
The conditional probability that the first light was green given the second light was green is

$$P[G_1|G_2] = \frac{P[G_1G_2]}{P[G_2]} = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = 3/4. \tag{2}$$

Finally, from the tree diagram, we can directly read that $P[G_2|G_1] = 3/4$.

Problem 1.7.3 Solution

Let G_i and B_i denote events indicating whether free throw i was good (G_i) or bad (B_i). The tree for the free throw experiment is

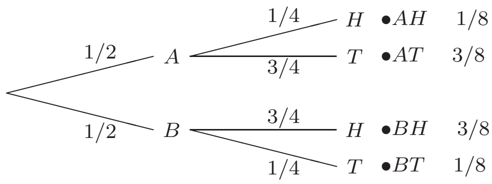


The game goes into overtime if exactly one free throw is made. This event has probability

$$P[O] = P[G_1B_2] + P[B_1G_2] = 1/8 + 1/8 = 1/4. \tag{1}$$

Problem 1.7.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Problem 1.7.5 Solution

The $P[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $P[+|H^c]$. Since the test is correct 99% of the time,

$$P[-|H] = P[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$P[H|+] = \frac{P[+,H]}{P[+]} = \frac{P[+,H]}{P[+,H] + P[+,H^c]}. \tag{2}$$

We can use Bayes' formula to evaluate these joint probabilities.

$$P[H|+] = \frac{P[+|H]\,P[H]}{P[+|H]\,P[H] + P[+|H^c]\,P[H^c]} \tag{3}$$

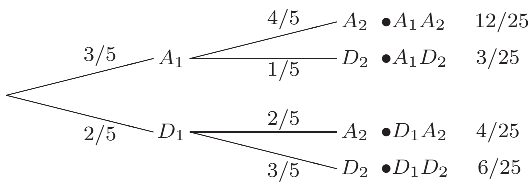
$$= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \tag{4}$$

$$= 0.0194. \tag{5}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 1.7.6 Solution

Let A_i and D_i indicate whether the ith photodetector is acceptable or defective.



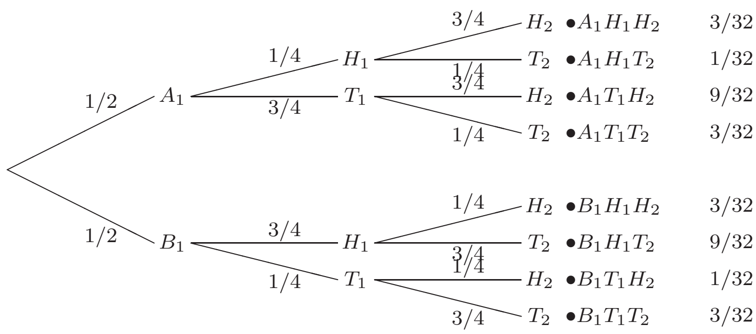
(a) We wish to find the probability $P[E_1]$ that exactly one photodetector is acceptable. From the tree, we have

$$P[E_1] = P[A_1D_2] + P[D_1A_2] = 3/25 + 4/25 = 7/25. \tag{1}$$

(b) The probability that both photodetectors are defective is $P[D_1D_2] = 6/25$.

Problem 1.7.7 Solution

The tree for this experiment is



The event H_1H_2 that heads occurs on both flips has probability

$$P[H_1H_2] = P[A_1H_1H_2] + P[B_1H_1H_2] = 6/32. \tag{1}$$

The probability of H_1 is

$$P[H_1] = P[A_1H_1H_2] + P[A_1H_1T_2] + P[B_1H_1H_2] + P[B_1H_1T_2] = 1/2. \tag{2}$$

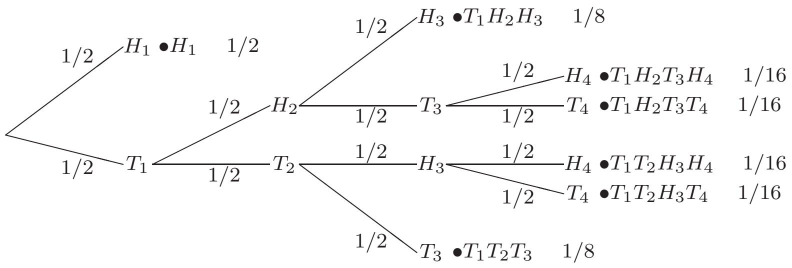
Similarly,

$$P[H_2] = P[A_1H_1H_2] + P[A_1T_1H_2] + P[B_1H_1H_2] + P[B_1T_1H_2] = 1/2. \tag{3}$$

Thus $P[H_1H_2] \neq P[H_1]P[H_2]$, implying H_1 and H_2 are not independent. This result should not be surprising since if the first flip is heads, it is likely that coin B was picked first. In this case, the second flip is less likely to be heads since it becomes more likely that the second coin flipped was coin A.

Problem 1.7.8 Solution

(a) The primary difficulty in this problem is translating the words into the correct tree diagram. The tree for this problem is shown below.



(b) From the tree,

$$P[H_3] = P[T_1H_2H_3] + P[T_1T_2H_3H_4] + P[T_1T_2H_3H_4] \tag{1}$$
$$= 1/8 + 1/16 + 1/16 = 1/4. \tag{2}$$

Similarly,

$$P[T_3] = P[T_1H_2T_3H_4] + P[T_1H_2T_3T_4] + P[T_1T_2T_3] \tag{3}$$
$$= 1/8 + 1/16 + 1/16 = 1/4. \tag{4}$$

(c) The event that Dagwood must diet is

$$D = (T_1H_2T_3T_4) \cup (T_1T_2H_3T_4) \cup (T_1T_2T_3). \tag{5}$$

The probability that Dagwood must diet is

$$P[D] = P[T_1H_2T_3T_4] + P[T_1T_2H_3T_4] + P[T_1T_2T_3] \tag{6}$$
$$= 1/16 + 1/16 + 1/8 = 1/4. \tag{7}$$

The conditional probability of heads on flip 1 given that Dagwood must diet is

$$P[H_1|D] = \frac{P[H_1D]}{P[D]} = 0. \tag{8}$$

Remember, if there was heads on flip 1, then Dagwood always postpones his diet.

(d) From part (b), we found that $P[H_3] = 1/4$. To check independence, we calculate

$$P[H_2] = P[T_1H_2H_3] + P[T_1H_2T_3] + P[T_1H_2T_4T_4] = 1/4 \tag{9}$$
$$P[H_2H_3] = P[T_1H_2H_3] = 1/8. \tag{10}$$

Now we find that

$$P[H_2H_3] = 1/8 \neq P[H_2]P[H_3]. \tag{11}$$

Hence, H_2 and H_3 are dependent events. In fact, $P[H_3|H_2] = 1/2$ while $P[H_3] = 1/4$. The reason for the dependence is that given H_2 occurred, then we know there will be a third flip which may result in H_3. That is, knowledge of H_2 tells us that the experiment didn't end after the first flip.

Problem 1.7.9 Solution

(a) We wish to know what the probability that we find no good photodiodes in n pairs of diodes. Testing each pair of diodes is an independent trial such that with probability p, both diodes of a pair are bad. From Problem 1.7.6, we can easily calculate p.

$$p = P[\text{both diodes are defective}] = P[D_1D_2] = 6/25. \tag{1}$$

The probability of Z_n, the probability of zero acceptable diodes out of n pairs of diodes is p^n because on each test of a pair of diodes, both must be defective.

$$P[Z_n] = \prod_{i=1}^{n} p = p^n = \left(\frac{6}{25}\right)^n \tag{2}$$

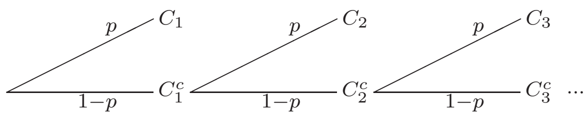
(b) Another way to phrase this question is to ask how many pairs must we test until $P[Z_n] \leq 0.01$. Since $P[Z_n] = (6/25)^n$, we require

$$\left(\frac{6}{25}\right)^n \leq 0.01 \quad \Rightarrow \quad n \geq \frac{\ln 0.01}{\ln 6/25} = 3.23. \tag{3}$$

Since n must be an integer, $n = 4$ pairs must be tested.

Problem 1.7.10 Solution

The experiment ends as soon as a fish is caught. The tree resembles



From the tree, $P[C_1] = p$ and $P[C_2] = (1-p)p$. Finally, a fish is caught on the nth cast if no fish were caught on the previous $n-1$ casts. Thus,

$$P[C_n] = (1-p)^{n-1}p. \tag{1}$$

Problem 1.8.1 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 1.8.2 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.

Problem 1.8.3 Solution

(a) The experiment of picking two cards and recording them in the order in which they were selected can be modeled by two sub-experiments. The first is to pick the first card and record it, the second sub-experiment is to pick the second card without replacing the first and recording it. For the first sub-experiment we can have any one of the possible 52 cards for a total of 52 possibilities. The second experiment consists of all the cards minus the one that was picked first(because we are sampling without replacement) for a total of 51 possible outcomes. So the total number of outcomes is the product of the number of outcomes for each sub-experiment.

$$52 \cdot 51 = 2652 \text{ outcomes.} \tag{1}$$

(b) To have the same card but different suit we can make the following sub-experiments. First we need to pick one of the 52 cards. Then we need to pick one of the 3 remaining cards that are of the same type but different suit out of the remaining 51 cards. So the total number outcomes is

$$52 \cdot 3 = 156 \text{ outcomes.} \tag{2}$$

(c) The probability that the two cards are of the same type but different suit is the number of outcomes that are of the same type but different suit divided by the total number of outcomes involved in picking two cards at random from a deck of 52 cards.

$$P\left[\text{same type, different suit}\right] = \frac{156}{2652} = \frac{1}{17}. \tag{3}$$

(d) Now we are not concerned with the ordering of the cards. So before, the outcomes $(K\heartsuit, 8\diamondsuit)$ and $(8\diamondsuit, K\heartsuit)$ were distinct. Now, those two outcomes are not distinct and are only considered to be the single outcome that a King of hearts and 8 of diamonds were selected. So every pair of outcomes before collapses to a single outcome when we disregard ordering. So we can redo parts (a) and (b) above by halving the corresponding values found in parts (a) and (b). The probability however, does not change because both the numerator and the denominator have been reduced by an equal factor of 2, which does not change their ratio.

Problem 1.8.4 Solution

We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

1. Choose 1 of the 10 pitchers. There are $N_1 = \binom{10}{1} = 10$ ways to do this.
2. Choose 1 of the 15 field players to be the designated hitter (DH). There are $N_2 = \binom{15}{1} = 15$ ways to do this.
3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are $N_3 = \binom{14}{8}$ to do this.
4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are $N_4 = 9!$ ways to do this.

So the total number of different starting lineups when the DH is selected among the field players is

$$N = N_1N_2N_3N_4 = (10)(15)\binom{14}{8}9! = 163{,}459{,}296{,}000. \tag{1}$$

Note that this overestimates the number of combinations the manager must really consider because most field players can play only one or two positions. Although these constraints on the manager reduce the number of possible lineups, it typically makes the manager's job more difficult. As for the counting, we note that our count did not need to specify the positions played by the field players. Although this is an important consideration for the manager, it is not part of our counting of different lineups. In fact, the 8 nonpitching field players are allowed to switch positions at any time in the field. For example, the shortstop and second baseman could trade positions in the middle of an inning. Although the DH can go play the field, there are some coomplicated rules about this. Here is an an excerpt from Major league Baseball Rule 6.10:

> The Designated Hitter may be used defensively, continuing to bat in the same position in the batting order, but the pitcher must then bat in the place of the substituted defensive player, unless more than one substitution is made, and the manager then must designate their spots in the batting order.

If you're curious, you can find the complete rule on the web.

Problem 1.8.5 Solution

When the DH can be chosen among all the players, including the pitchers, there are two cases:

- The DH is a field player. In this case, the number of possible lineups, N_F, is given in Problem 1.8.4. In this case, the designated hitter must be chosen from the 15 field players. We repeat the solution of Problem 1.8.4 here: We can break down the experiment of choosing a starting lineup into a sequence of subexperiments:

  1. Choose 1 of the 10 pitchers. There are $N_1 = \binom{10}{1} = 10$ ways to do this.
  2. Choose 1 of the 15 field players to be the designated hitter (DH). There are $N_2 = \binom{15}{1} = 15$ ways to do this.
  3. Of the remaining 14 field players, choose 8 for the remaining field positions. There are $N_3 = \binom{14}{8}$ to do this.
  4. For the 9 batters (consisting of the 8 field players and the designated hitter), choose a batting lineup. There are $N_4 = 9!$ ways to do this.

  So the total number of different starting lineups when the DH is selected among the field players is

  $$N = N_1N_2N_3N_4 = (10)(15)\binom{14}{8}9! = 163{,}459{,}296{,}000. \tag{1}$$

- The DH is a pitcher. In this case, there are 10 choices for the pitcher, 10 choices for the DH among the pitchers (including the pitcher batting for himself), $\binom{15}{8}$ choices for the field players, and 9! ways of ordering the batters into a lineup. The number of possible lineups is

  $$N' = (10)(10)\binom{15}{8}9! = 233, 513, 280, 000. \tag{2}$$

The total number of ways of choosing a lineup is $N + N' = 396{,}972{,}576{,}000$.

Problem 1.8.6 Solution

(a) We can find the number of valid starting lineups by noticing that the swingman presents three situations: (1) the swingman plays guard, (2) the swingman plays forward, and (3) the swingman doesn't play. The first situation is when the swingman can be chosen to play the guard position, and the second where the swingman can only be chosen to play the forward position. Let N_i denote the number of lineups corresponding to case i. Then we can write the total number of lineups as $N_1 + N_2 + N_3$. In the first situation, we have to choose 1 out of 3 centers, 2 out of 4 forwards, and 1 out of 4 guards so that

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72. \tag{1}$$

In the second case, we need to choose 1 out of 3 centers, 1 out of 4 forwards and 2 out of 4 guards, yielding

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72. \tag{2}$$

Finally, with the swingman on the bench, we choose 1 out of 3 centers, 2 out of 4 forward, and 2 out of four guards. This implies

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108, \tag{3}$$

and the total number of lineups is $N_1 + N_2 + N_3 = 252$.

Problem 1.8.7 Solution

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k > 0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}. \tag{1}$$

By careful choice of n and k, we can choose p close to 0.01. For example,

n	9	11	14	17
k	0	1	2	3
p	0.0079	0.012	0.0105	0.0090

(2)

A gamecard with $N = 14$ boxes and $5 + k = 7$ shaded boxes would be quite reasonable.

Problem 1.9.1 Solution

(a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$P[00111] = (0.8)^2(0.2)^3 = 0.00512. \tag{1}$$

(b) The probability that a code word has exactly three 1's is

$$P[\text{three 1's}] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \tag{2}$$

Problem 1.9.2 Solution

Given that the probability that the Celtics win a single championship in any given year is 0.32, we can find the probability that they win 8 straight NBA championships.

$$P[\text{8 straight championships}] = (0.32)^8 = 0.00011. \tag{1}$$

The probability that they win 10 titles in 11 years is

$$P[\text{10 titles in 11 years}] = \binom{11}{10}(.32)^{10}(.68) = 0.00084. \tag{2}$$

The probability of each of these events is less than 1 in 1000! Given that these events took place in the relatively short fifty year history of the NBA, it should seem that these probabilities should be much higher. What the model overlooks is that the sequence of 10 titles in 11 years started when Bill Russell joined the Celtics. In the years with Russell (and a strong supporting cast) the probability of a championship was much higher.

Problem 1.9.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights, G, Y, and R obey the multinomial probability law:

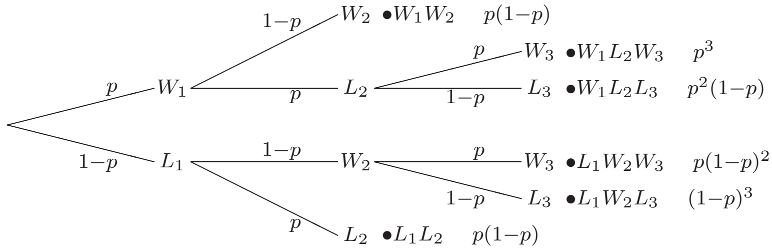
$$P[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)\left(\frac{7}{16}\right)^2. \tag{1}$$

The probability that the number of green lights equals the number of red lights

$$\begin{aligned} P[G = R] &= P[G = 1, R = 1, Y = 3] + P[G = 2, R = 2, Y = 1] + P[G = 0, R = 0, Y = 5] && (2)\\ &= \frac{5!}{1!1!3!}\left(\frac{7}{16}\right)\left(\frac{7}{16}\right)\left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right) + \frac{5!}{0!0!5!}\left(\frac{1}{8}\right)^5 && (3)\\ &\approx 0.1449. && (4) \end{aligned}$$

Problem 1.9.4 Solution

For the team with the homecourt advantage, let W_i and L_i denote whether game i was a win or a loss. Because games 1 and 3 are home games and game 2 is an away game, the tree is



The probability that the team with the home court advantage wins is

$$P[H] = P[W_1W_2] + P[W_1L_2W_3] + P[L_1W_2W_3] \tag{1}$$
$$= p(1-p) + p^3 + p(1-p)^2. \tag{2}$$

Note that $P[H] \le p$ for $1/2 \le p \le 1$. Since the team with the home court advantage would win a 1 game playoff with probability p, the home court team is less likely to win a three game series than a 1 game playoff!

Problem 1.9.5 Solution

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3 \qquad P[G_2] = 2/3. \tag{1}$$

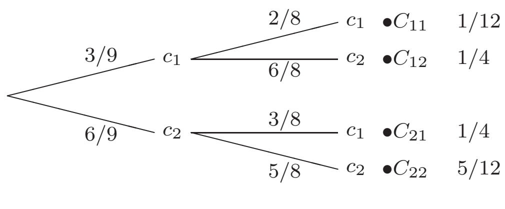
In addition, the problem statement tells us that

$$P[K|G_1] = 1/2 \qquad P[K|G_2] = 1/3. \tag{2}$$

Combining these facts using the Law of Total Probability yields

$$P[K] = P[K|G_1]P[G_1] + P[K|G_2]P[G_2] \tag{3}$$
$$= (1/2)(1/3) + (1/3)(2/3) = 7/18. \tag{4}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$P[K_1K_2|C_{11}] = (1/2)^2 \qquad P[K_1K_2|C_{12}] = (1/2)(1/3) \tag{5}$$
$$P[K_1K_2|C_{21}] = (1/3)(1/2) \qquad P[K_1K_2|C_{22}] = (1/3)^2 \tag{6}$$

By the law of total probability,

$$P[K_1K_2] = P[K_1K_2|C_{11}]\,P[C_{11}] + P[K_1K_2|C_{12}]\,P[C_{12}] \quad (7)$$
$$+ P[K_1K_2|C_{21}]\,P[C_{21}] + P[K_1K_2|C_{22}]\,P[C_{22}] \quad (8)$$
$$= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = 15/96. \quad (9)$$

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P[K_2|C_{ij}]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2, \qquad P[K_2|C_{12}] = 1/3, \quad (10)$$
$$P[K_2|C_{21}] = 1/2, \qquad P[K_2|C_{22}] = 1/3. \quad (11)$$

By the law of total probability,

$$P[K_2] = P[K_2|C_{11}]\,P[C_{11}] + P[K_2|C_{12}]\,P[C_{12}]$$
$$+ P[K_2|C_{21}]\,P[C_{21}] + P[K_2|C_{22}]\,P[C_{22}] \quad (12)$$
$$= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}. \quad (13)$$

We observe that K_1 and K_2 are not independent since

$$P[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = P[K_1]\,P[K_2]. \quad (14)$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss.

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is 1/2. If the kicker is from group 2, the success probability is 1/3. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

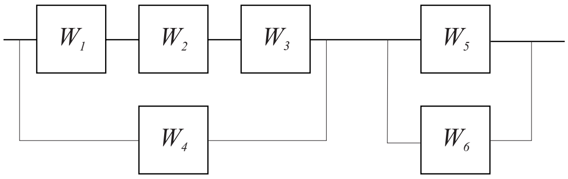
$$P[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5, \qquad P[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5. \quad (15)$$

We use the Law of Total Probability to find

$$P[M] = P[M|G_1]\,P[G_1] + P[M|G_2]\,P[G_2] \quad (16)$$
$$= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right). \quad (17)$$

Problem 1.10.1 Solution

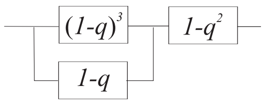
From the problem statement, we can conclude that the device components are configured in the following way.



To find the probability that the device works, we replace series devices 1, 2, and 3, and parallel devices 5 and 6 each with a single device labeled with the probability that it works. In particular,

$$P[W_1W_2W_3] = (1-q)^3, \tag{1}$$
$$P[W_5 \cup W_6] = 1 - P[W_5^cW_6^c] = 1 - q^2. \tag{2}$$

This yields a composite device of the form



The probability $P[W']$ that the two devices in parallel work is 1 minus the probability that neither works:

$$P[W'] = 1 - q(1-(1-q)^3). \tag{3}$$

Finally, for the device to work, both composite device in series must work. Thus, the probability the device works is

$$P[W] = [1 - q(1-(1-q)^3)][1-q^2]. \tag{4}$$

Problem 1.10.2 Solution

Suppose that the transmitted bit was a 1. We can view each repeated transmission as an independent trial. We call each repeated bit the receiver decodes as 1 a success. Using $S_{k,5}$ to denote the event of k successes in the five trials, then the probability k 1's are decoded at the receiver is

$$P[S_{k,5}] = \binom{5}{k} p^k(1-p)^{5-k}, \quad k = 0,1,\ldots,5. \tag{1}$$

The probability a bit is decoded correctly is

$$P[C] = P[S_{5,5}] + P[S_{4,5}] = p^5 + 5p^4(1-p) = 0.91854. \tag{2}$$

The probability a deletion occurs is

$$P[D] = P[S_{3,5}] + P[S_{2,5}] = 10p^3(1-p)^2 + 10p^2(1-p)^3 = 0.081. \tag{3}$$

The probability of an error is

$$P[E] = P[S_{1,5}] + P[S_{0,5}] = 5p(1-p)^4 + (1-p)^5 = 0.00046. \tag{4}$$

Note that if a 0 is transmitted, then 0 is sent five times and we call decoding a 0 a success. You should convince yourself that this a symmetric situation with the same deletion and error probabilities. Introducing deletions reduces the probability of an error by roughly a factor of 20. However, the probability of successfull decoding is also reduced.

Problem 1.10.3 Solution

Note that each digit 0 through 9 is mapped to the 4 bit binary representation of the digit. That is, 0 corresponds to 0000, 1 to 0001, up to 9 which corresponds to 1001. Of course, the 4 bit binary numbers corresponding to numbers 10 through 15 go unused, however this is unimportant to our problem. the 10 digit number results in the transmission of 40 bits. For each bit, an independent trial determines whether the bit was correct, a deletion, or an error. In Problem 1.10.2, we found the probabilities of these events to be

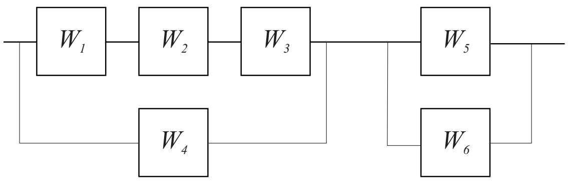
$$P[C] = \gamma = 0.91854, \qquad P[D] = \delta = 0.081, \qquad P[E] = \epsilon = 0.00046. \tag{1}$$

Since each of the 40 bit transmissions is an independent trial, the joint probability of c correct bits, d deletions, and e erasures has the multinomial probability

$$P[C = c, D = d, E = d] = \begin{cases} \frac{40!}{c!d!e!}\gamma^c\delta^d\epsilon^e & c + d + e = 40; c, d, e \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 1.10.4 Solution

From the statement of Problem 1.10.1, the configuration of device components is



By symmetry, note that the reliability of the system is the same whether we replace component 1, component 2, or component 3. Similarly, the reliability is the same whether we replace component 5 or component 6. Thus we consider the following cases:

I **Replace component 1** In this case

$$P[W_1W_2W_3] = (1 - \frac{q}{2})(1 - q)^2, \qquad P[W_4] = 1 - q, \qquad P[W_5 \cup W_6] = 1 - q^2. \tag{1}$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = 1 - \frac{q^2}{2}(5 - 4q + q^2). \tag{2}$$

In this case, the probability the system works is

$$P[W_I] = P[W_1W_2W_3 \cup W_4] P[W_5 \cup W_6] = \left[1 - \frac{q^2}{2}(5 - 4q + q^2)\right](1 - q^2). \tag{3}$$

II **Replace component 4** In this case,

$$P[W_1W_2W_3] = (1 - q)^3, \qquad P[W_4] = 1 - \frac{q}{2}, \qquad P[W_5 \cup W_6] = 1 - q^2. \tag{4}$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = 1 - \frac{q}{2} + \frac{q}{2}(1 - q)^3. \tag{5}$$

In this case, the probability the system works is

$$P[W_{II}] = P[W_1W_2W_3 \cup W_4] P[W_5 \cup W_6] = \left[1 - \frac{q}{2} + \frac{q}{2}(1 - q)^3\right](1 - q^2). \tag{6}$$

III **Replace component 5** In this case,

$$P[W_1W_2W_3] = (1-q)^3, \qquad P[W_4] = 1-q, \qquad P[W_5 \cup W_6] = 1 - \frac{q^2}{2}. \tag{7}$$

This implies

$$P[W_1W_2W_3 \cup W_4] = 1 - (1 - P[W_1W_2W_3])(1 - P[W_4]) = (1-q)\left[1 + q(1-q)^2\right]. \tag{8}$$

In this case, the probability the system works is

$$P[W_{III}] = P[W_1W_2W_3 \cup W_4]\, P[W_5 \cup W_6] \tag{9}$$

$$= (1-q)\left(1 - \frac{q^2}{2}\right)\left[1 + q(1-q)^2\right]. \tag{10}$$

From these expressions, its hard to tell which substitution creates the most reliable circuit. First, we observe that $P[W_{II}] > P[W_I]$ if and only if

$$1 - \frac{q}{2} + \frac{q}{2}(1-q)^3 > 1 - \frac{q^2}{2}(5 - 4q + q^2). \tag{11}$$

Some algebra will show that $P[W_{II}] > P[W_I]$ if and only if $q^2 < 2$, which occurs for all nontrivial (i.e., nonzero) values of q. Similar algebra will show that $P[W_{II}] > P[W_{III}]$ for all values of $0 \leq q \leq 1$. Thus the best policy is to replace component 4.

Problem 1.11.1 Solution

We can generate the 200×1 vector $\mathbf{T}$, denoted `T` in MATLAB, via the command

```
T=50+ceil(50*rand(200,1))
```

Keep in mind that `50*rand(200,1)` produces a 200×1 vector of random numbers, each in the interval $(0, 50)$. Applying the ceiling function converts these random numbers to rndom integers in the set $\{1, 2, \ldots, 50\}$. Finally, we add 50 to produce random numbers between 51 and 100.

Problem 1.11.2 Solution

Rather than just solve the problem for 50 trials, we can write a function that generates vectors $\mathbf{C}$ and $\mathbf{H}$ for an arbitrary number of trials n. The code for this task is

```
function [C,H]=twocoin(n);
C=ceil(2*rand(n,1));
P=1-(C/4);
H=(rand(n,1)< P);
```

The first line produces the $n \times 1$ vector `C` such that `C(i)` indicates whether coin 1 or coin 2 is chosen for trial i. Next, we generate the vector `P` such that `P(i)=0.75` if `C(i)=1`; otherwise, if `C(i)=2`, then `P(i)=0.5`. As a result, `H(i)` is the simulated result of a coin flip with heads, corresponding to `H(i)=1`, occurring with probability `P(i)`.

Problem 1.11.3 Solution

Rather than just solve the problem for 100 trials, we can write a function that generates n packets for an arbitrary number of trials n. The code for this task is

```
function C=bit100(n);
% n is the number of 100 bit packets sent
B=floor(2*rand(n,100));
P=0.03-0.02*B;
E=(rand(n,100)< P);
C=sum((sum(E,2)<=5));
```

First, `B` is an $n \times 100$ matrix such that `B(i,j)` indicates whether bit `i` of packet `j` is zero or one. Next, we generate the $n \times 100$ matrix `P` such that `P(i,j)=0.03` if `B(i,j)=0`; otherwise, if `B(i,j)=1`, then `P(i,j)=0.01`. As a result, `E(i,j)` is the simulated error indicator for bit `i` of packet `j`. That is, `E(i,j)=1` if bit `i` of packet `j` is in error; otherwise `E(i,j)=0`. Next we sum across the rows of `E` to obtain the number of errors in each packet. Finally, we count the number of packets with 5 or more errors.

For $n = 100$ packets, the packet success probability is inconclusive. Experimentation will show that `C=97`, `C=98`, `C=99` and `C=100` correct packets are typica values that might be observed. By increasing n, more consistent results are obtained. For example, repeated trials with $n = 100,000$ packets typically produces around $C = 98,400$ correct packets. Thus 0.984 is a reasonable estimate for the probability of a packet being transmitted correctly.

Problem 1.11.4 Solution

To test n 6-component devices, (such that each component works with probability q) we use the following function:

```
function N=reliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
W=rand(n,6)>q;
D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
D=D&(W(:,5)|W(:,6));
N=sum(D);
```

The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical operators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Lastly, we count the number `N` of working devices. The following code snippet produces ten sample runs, where each sample run tests `n=100` devices for $q = 0.2$.

```
>> for n=1:10, w(n)=reliable6(100,0.2); end
>> w
w =
    82   87   87   92   91   85   85   83   90   89
>>
```

As we see, the number of working devices is typically around 85 out of 100. Solving Problem 1.10.1, will show that the probability the device works is actually 0.8663.

Problem 1.11.5 Solution

The code

```
function n=countequal(x,y)
%Usage:  n=countequal(x,y)
%n(j)= # elements of x = y(j)
[MX,MY]=ndgrid(x,y);
%each column of MX = x
%each row of MY = y
n=(sum((MX==MY),1))';
```

for `countequal` is quite short (just two lines excluding comments) but needs some explanation. The key is in the operation

`[MX,MY]=ndgrid(x,y).`

The MATLAB built-in function `ndgrid` facilitates plotting a function $g(x, y)$ as a surface over the x, y plane. The x, y plane is represented by a grid of all pairs of points $x(i), y(j)$. When `x` has n elements, and `y` has m elements, `ndgrid(x,y)` creates a grid (an $n \times m$ array) of all possible pairs `[x(i) y(j)]`. This grid is represented by two separate $n \times m$ matrices: `MX` and `MY` which indicate the `x` and `y` values at each grid point. Mathematically,

`MX(i,j) = x(i),    MY(i,j)=y(j).`

Next, `C=(MX==MY)` is an $n \times m$ array such that `C(i,j)=1` if `x(i)=y(j)`; otherwise `C(i,j)=0`. That is, the jth column of `C` indicates indicates which elements of `x` equal `y(j)`. Lastly, we sum along each column j to count number of elements of `x` equal to `y(j)`. That is, we sum along column j to count the number of occurrences (in `x`) of `y(j)`.

Problem 1.11.6 Solution

For arbitrary number of trials n and failure probability q, the following functions evaluates replacing each of the six components by an ultrareliable device.

```
function N=ultrareliable6(n,q);
% n is the number of 6 component devices
%N is the number of working devices
for r=1:6,
    W=rand(n,6)>q;
    R=rand(n,1)>(q/2);
    W(:,r)=R;
    D=(W(:,1)&W(:,2)&W(:,3))|W(:,4);
    D=D&(W(:,5)|W(:,6));
    N(r)=sum(D);
end
```

This above code is based on the code for the solution of Problem 1.11.4. The $n \times 6$ matrix `W` is a *logical* matrix such that `W(i,j)=1` if component `j` of device `i` works properly. Because `W` is a logical matrix, we can use the MATLAB logical operators `|` and `&` to implement the logic requirements for a working device. By applying these logical opeators to the $n \times 1$ columns of `W`, we simulate the test of `n` circuits. Note that `D(i)=1` if device `i` works. Otherwise, `D(i)=0`. Note that in the code, we first generate the matrix `W` such that each component has failure probability q. To simulate the replacement of the jth device by the ultrareliable version by replacing the jth column of `W` by the column vector `R` in which a device has failure probability $q/2$. Lastly, for each column replacement, we count the number `N` of working devices. A sample run for $n = 100$ trials and $q = 0.2$ yielded these results:

```
>> ultrareliable6(100,0.2)
ans =
    93    89    91    92    90    93
```

From the above, we see, for example, that replacing the third component with an ultrareliable component resulted in 91 working devices. The results are fairly inconclusive in that replacing devices 1, 2, or 3 should yield the same probability of device failure. If we experiment with $n = 10,000$ runs, the results are more definitive:

```
>> ultrareliable6(10000,0.2)
ans =
        8738        8762        8806        9135        8800        8796
>> >> ultrareliable6(10000,0.2)
ans =
        8771        8795        8806        9178        8886        8875
>>
```

In both cases, it is clear that replacing component 4 maximizes the device reliability. The somewhat complicated solution of Problem 1.10.4 will confirm this observation.