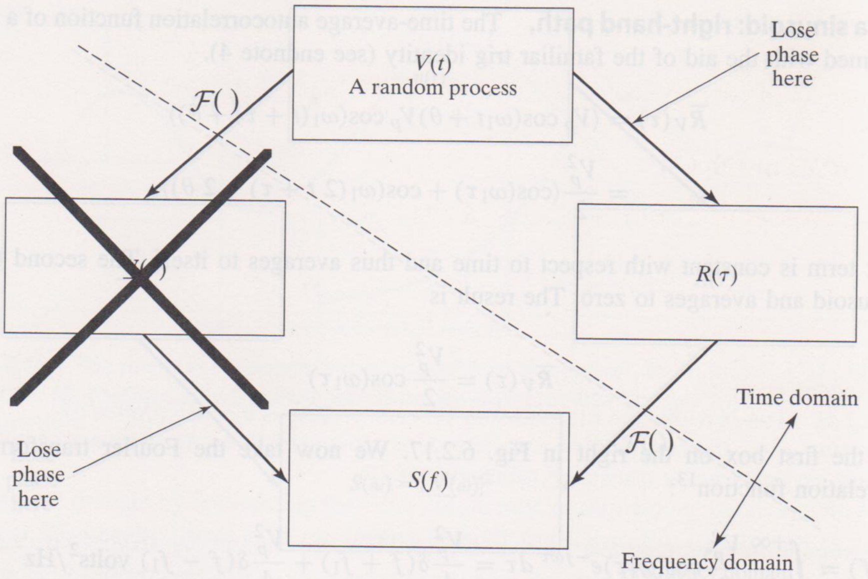

Figure 6.2.18 A random process has no voltage spectrum but does have a power spectrum. The only way to get the power spectrum is through the autocorrelation function.

The definition of power spectral density, PSD. Following the right-hand path, we define the PSD of a WSS random process as

$$S_V(\omega) = \int_{-\infty}^{+\infty} R_V(\tau)e^{-j\omega\tau}\,d\tau \text{ volts}^2/\text{Hz} \tag{6.2.33}$$

where $R_V(\tau) = E[V(t)V(t+\tau)]$, $V(t)$ a WSS random process. The remainder of this section addresses the question, Does the definition of PSD in Eq. (6.2.33) make sense? We have led you to this definition by showing it to be plausible. We will continue to investigate its plausibility through a number of means, to increase your understanding of and intuition about the spectral analysis of random signals.

The inverse transform. Because Fourier transforms come in pairs, we get the inverse transform of Eq. (6.2.33) for free:

$$R_V(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_V(\omega)e^{+j\omega\tau}\,d\omega \text{ volts}^2 \tag{6.2.34}$$

Thus if we know the PSD of a random process, we can derive the autocorrelation function by Eq. (6.2.34).

Ergodic random processes. Recall that an ergodic random process is one in which statistical averages are equal to time averages. Because the right-hand path in Fig. 6.2.18 works for all time functions, it must therefore work for every member of an ergodic random process.

In Sec. 6.1 we presented and investigated four WSS random processes that model four important random signals. We now extend that presentation to include the frequency domain. Our purpose is to demonstrate the plausibility of the PSD concept and to give insight into the properties of these models.

Model of a random asynchronous digital signal

The model. We modeled a random asynchronous digital signal (Sec. 6.1.1) by triggering a toggle flip-flop with a Poisson process of average rate λ. The results were that the model was WSS and that the autocorrelation function was of the form Eq. (6.1.17),

$$R_Q(\tau) = \tfrac{1}{4}e^{-2\lambda|\tau|} + \tfrac{1}{4} \qquad (6.2.39)$$

Although we did not show that this random process is ergodic, we pointed out that the total power, AC power, and DC power were reasonable for a digital signal that spends half its time in the 1 state and half in the 0 state.

The PSD. Using a common Fourier transform pair,[13] we find the PSD of the model to be

$$S(f) = \frac{1}{4}\delta(f) + \frac{\lambda}{(2\lambda)^2 + (2\pi f)^2} \text{ volts}^2/\text{Hz} \qquad (6.2.40)$$

This spectrum is plotted in Fig. 6.2.19.

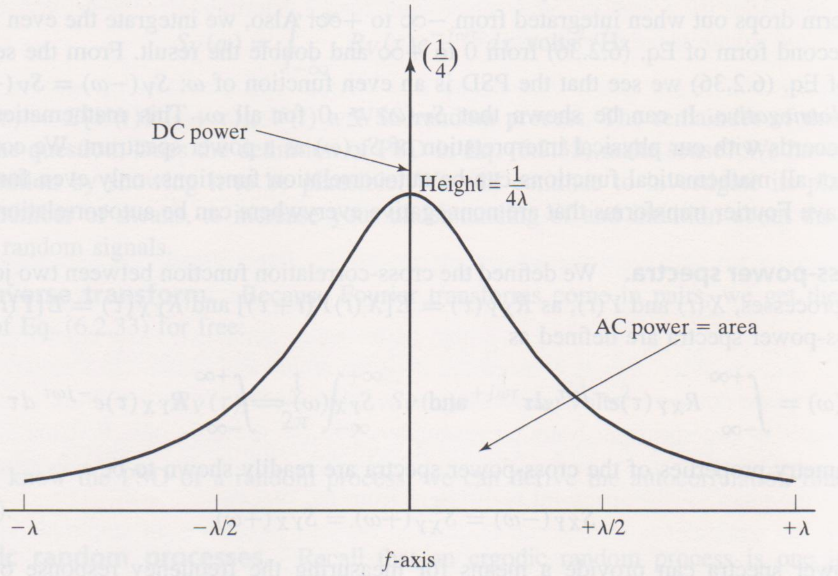

Figure 6.2.19 The spectrum of the asynchronous digital signal. The DC power is represented by an impulse at $f = 0$. The AC power is spread over a continuous spectrum.

```
myanswer =      ?   ;
```

```
Evaluate
```

For the answer, see endnote 16.

Summary. The random process model for an asynchronous digital signal leads to a spectrum with the correct AC and DC power and a reasonable bandwidth. The bandwidth we derived is based on 90% of the AC power.

Model of synchronous digital signal

The model. In Sec. 6.1 we investigated a random process model for a synchronous digital signal. We assumed a clock period of T, equally likely 1s and 0s during each clock cycle, and an arbitrary time origin. The random process proved to be WSS and the autocorrelation function was Eq. (6.1.28)

$$\begin{aligned} R_X(\tau) &= \frac{1}{2} - \frac{1}{4}\frac{|\tau|}{T}, 0 < |\tau| < T \\ &= \frac{1}{4}, \ |\tau| > T \end{aligned} \tag{6.2.44}$$

Using the table of Fourier transform pairs (see endnote 13), we find the PSD to be

$$S_X(f) = \frac{1}{4}\delta(f) + \frac{T}{4}\left(\frac{\sin \pi T f}{\pi T f}\right)^2 \ \text{volts}^2/\text{Hz} \tag{6.2.45}$$

The spectrum in Fig. 6.2.20 shows the DC power as an impulse at $f = 0$, and shows the AC power spread into a continuum of frequencies.

The bandwidth. The bandwidth in hertz can be set equal to the frequency of the first null.

$$B = \frac{1}{T} \tag{6.2.46}$$

We consider that this is a reasonable bandwidth. The digital signal alternates between 1 and 0 with each clock cycle. The fastest-moving signal it can create is when it alternates between 1 and 0 regularly, making a square wave. To get 90% of the power the bandwidth of a square wave goes to the third harmonic, which would be a bandwidth of $\frac{3}{2T}$ in the notation of this problem. This is the worst case and it requires more bandwidth than is given by Eq. (6.2.46). But the alternating structure of the square wave is highly improbable, for at each clock transition the state changes with probability 0.5. We conclude that Eq. (6.2.46) gives a reasonable bandwidth for the random synchronous digital signal. We show in endnote 17 that the power to the first null in the spectrum is 90.4% of the total AC power.

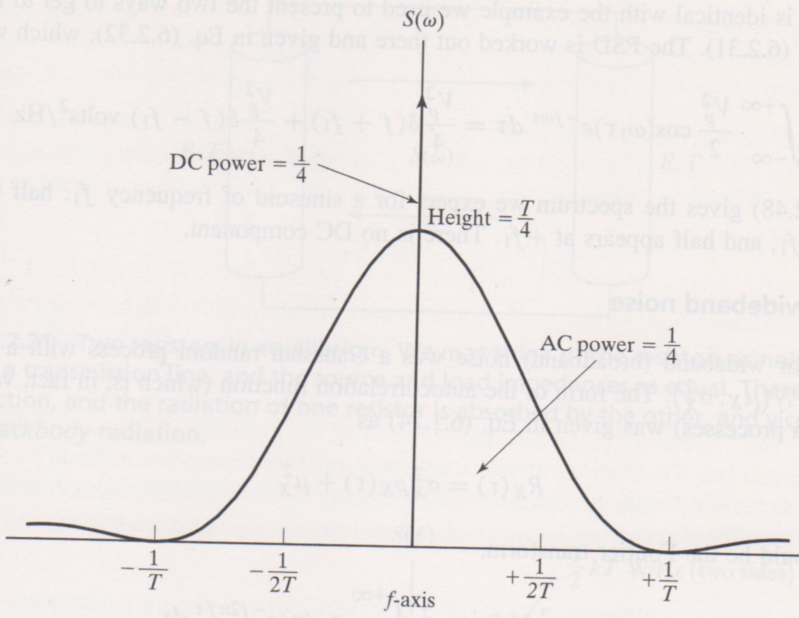

Figure 6.2.20 The spectrum of the model of the synchronous digital signal. As before, the AC and the DC power are the same, as is appropriate for a signal that spends half its time in the 1 state and half in the 0 state. The bandwidth to the first null is $\frac{1}{T}$.

Example 6.2.2: Computer CPU speed
A computer runs with a CPU speed of 560 MHz. What bandwidth is required for the busses bringing the instructions from memory?

Solution The clock period is $T = \frac{1}{560\times10^{+6}}$. The required bandwidth for 90% of the signal power is $B = \frac{1}{T} = 560$ MHz.

Summary. Again, we find that the PSD of the model of the synchronous digital signal gives the right AC and DC powers and a reasonable bandwidth.

Model of a random analog signal

The model. Our model for a random analog signal is a sinusoid of fixed amplitude and frequency but random phase. We showed that this random process is WSS and is also ergodic.

The PSD of the model. The autocorrelation function for the random sinusoid was Eq. (6.1.35):

$$R_V(\tau) = \frac{V_p^2}{2}\cos\omega_1\tau \text{ volts}^2 \tag{6.2.47}$$

This function is identical with the example we used to present the two ways to get to the power spectrum, Eq. (6.2.31). The PSD is worked out there and given in Eq. (6.2.32), which we repeat:

$$S_V(f) = \int_{-\infty}^{+\infty} \frac{V_p^2}{2} \cos(\omega_1 \tau) e^{-j\omega\tau}\, d\tau = \frac{V_p^2}{4}\delta(f + f_1) + \frac{V_p^2}{4}\delta(f - f_1) \text{ volts}^2/\text{Hz} \quad (6.2.48)$$

Equation (6.2.48) gives the spectrum we expect for a sinusoid of frequency f_1: half the power appears at $-f_1$, and half appears at $+f_1$. There is no DC component.

Model for wideband noise

Our model for wideband (broadband) noise was a Gaussian random process with a first-order PDF of $X = N(\mu_X, \sigma_X^2)$. The form of the autocorrelation function (which is, in fact, valid for all WSS random processes) was given in Eq. (6.1.74) as

$$R_X(\tau) = \sigma_X^2 \rho_X(\tau) + \mu_X^2 \quad (6.2.49)$$

The PSD would be the Fourier transform,

$$S(f) = \mu_X^2 \delta(f) + \sigma_X^2 \int_{-\infty}^{+\infty} \rho_X(\tau) e^{-j2\pi f\tau}\, d\tau \quad (6.2.50)$$

We cannot do the integral without an explicit form for the coherence function, $\rho_X(\tau)$. Equation (6.2.50) shows the DC component and the AC component. We will work an example in Section 6.3; however, we may make a guess for the bandwidth. We defined a coherence time, τ_c, as the delay beyond which the coherence function remains below 0.1. The coherence time characterizes the time structure of the random process. We would expect the bandwidth to be approximately

$$B \approx \frac{1}{\tau_c} \quad (6.2.51)$$

Next, we discuss the physics, circuit theory, and random-process model for a common form of broadband noise, thermal noise in resistors.

6.2.4 Modeling of Broadband Noise

This is the third time we have taken up the subject of broadband noise. We treated broadband noise in Sec. 6.1 as an important signal to model with a WSS random process. We just discussed the spectrum of broadband noise without giving specifics. We now have the tools for a fuller treatment of this important subject.

Sources of broadband noise. There are a number of physical processes that produce broadband noise in electrical circuits. Thermal motion of carriers in lossy structures such as resistors is a major source. The discrete nature of electrical carriers leads to broadband noise due to partition and recombination processes in semiconductors, and shot noise in vacuum tubes. For several reasons we focus on resistor noise in this section.

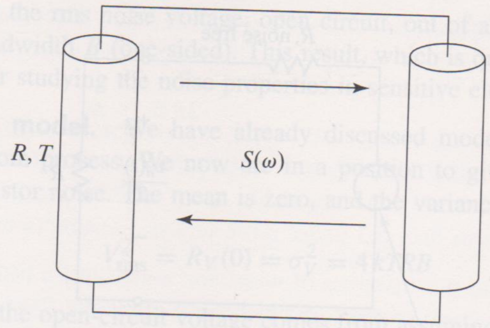

Figure 6.2.21 Two resistors in equilibrium. We may think of the resistors as noise sources, the wires as a transmission line, and the source and load impedances as equal. Therefore there is no reflection, and the radiation of one resistor is absorbed by the other, and vice versa. Thus this is *blackbody* radiation.

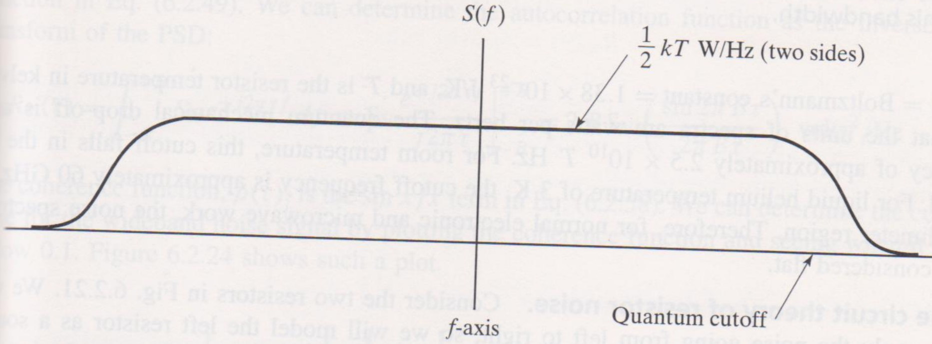

Figure 6.2.22 The emitted spectrum of a resistor is flat throughout the normal electronic frequency range provided the resistor is not cooled to cryogenic temperatures. The magnitude of the emitted spectrum into a matched load is $\frac{1}{2}kT$ W/Hz, where k is Boltzmann's constant, and T is the temperature in kelvin.

Blackbody radiation. Classical physics had some problems understanding the radiation of hot bodies until quantum mechanics yielded adequate models. Resistor noise, also called *Johnson noise*, is blackbody radiation confined to an electrical circuit. Because the noise is produced by the energetic thermal motion of carriers and because collision distances are on the atomic scale, the collision times are extremely short. This leads to a short coherence time and a broad bandwidth. The context and results of the physics of this subject are summarized by the circuit in Fig. 6.2.21. Two resistors of value R and temperature T kelvin (K) are in equilibrium through an equal exchange of energy.

The spectrum of the exchanged radiation is shown in Fig. 6.2.22. The emitted power spectrum into a matched load is

$$S(f) = \tfrac{1}{2}kT \text{ W/Hz (two sides)} \tag{6.2.52}$$

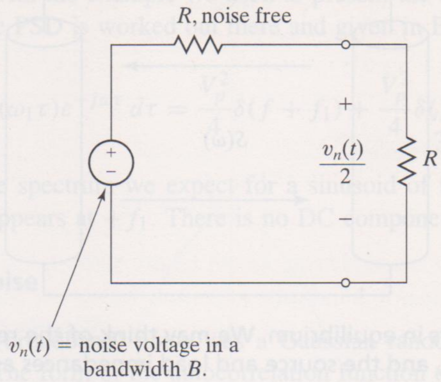

Figure 6.2.23 The Thevenin equivalent circuit for a hot resistor coupled to a matched load. Because any circuit has a limited bandwidth, we have indicated the noise voltage in the bandwidth appropriate to the circuit. Later, we will extend the model to include more details about this bandwidth.

where k = Boltzmann's constant $= 1.38 \times 10^{-23}$ J/K, and T is the resistor temperature in kelvin. Note that the units of spectra are watts per hertz. The quantum mechanical drop-off is at a frequency of approximately $2.5 \times 10^{10}\ T$ Hz. For room temperature, this cutoff falls in the far infrared. For liquid helium temperature of 3 K, the cutoff frequency is approximately 60 GHz, in the millimeter region. Therefore, for normal electronic and microwave work, the noise spectrum can be considered flat.

The circuit theory of resistor noise. Consider the two resistors in Fig. 6.2.21. We will consider only the noise going from left to right, so we will model the left resistor as a source and the right resistor as a load. The Thevenin equivalent circuit allows us to separate the internal noise source from the output impedance, with the model shown in Fig. 6.2.23.

By combining the results of thermal physics with circuit theory we can determine the rms value of the noise voltage. Because the frequency cutoff of thermal noise is far beyond the bandwidth of the circuit, the power in the load from physics is

$$P = \frac{1}{2}kT \times 2B = kTB \quad \text{W} \tag{6.2.53}$$

where B is the bandwidth (one side) of the coupling. From circuit theory we calculate the power in the load as

$$P = \frac{\langle (v_n(t)/2)^2 \rangle}{R} = \frac{V_{\text{rms}}^2}{4R} \tag{6.2.54}$$

Equating the two powers, we have

$$V_{\text{rms}} = \sqrt{4kTRB} \text{ volts} \tag{6.2.55}$$

The output PSD function is given by Eq. (6.3.18) multiplied by the power system response, Eq. (6.3.15), with the result

$$S_{\text{out}}(f) = \left[\frac{V_p^2}{4}\delta(f+f_1) + \frac{V_p^2}{4}\delta(f-f_1)\right] \times \frac{1}{1+(f/f_c)^2} \text{ volts}^2/\text{Hz} \tag{6.3.19}$$

The filter function gives its response at $\pm f_1$ and diminishes the amplitude by the factor $\frac{1}{1+(\pm f_1/f_c)^2}$, with the final result

$$S_{\text{out}}(f) = \frac{V_p^2}{4(1+(f_1/f_c)^2)}\delta(f+f_1) + \frac{V_p^2}{4(1+(f_1/f_c)^2)}\delta(f-f_1) \text{ volts}^2/\text{Hz} \tag{6.3.20}$$

Finally, the inverse transform of Eq. (6.3.20) is identical with Eq. (6.3.17), except the amplitude is diminished by the power system response.

$$R_{\text{out}}(\tau) = \frac{V_p^2}{2(1+(f_1/f_c)^2)}\cos 2\pi f_1 \tau \text{ volts}^2 \tag{6.3.21}$$

Summary. The low-pass filter has the expected effect of reducing the amplitude of the random sinusoid. The magnitude of the effect depends on the relationship between the frequency of the random sinusoid and the cutoff frequency of the filter. For $f_1 \ll f_c$, the filter has little effect; for $f_1 \gg f_c$, the effect will be strong; and for $f_1 = f_c$, the signal power will be reduced by a factor of 2.

6.3.3 Model for Broadband Noise in a Low-Pass Filter

Resistor noise output. The effect of a low-pass filter on broadband noise is best considered as a problem in circuit theory. The question we will address is, How much noise voltage comes out of a real resistor? The problem is stated, and translated into circuit theory, in Fig. 6.3.3. The source resistance and stray capacitance amount to a low-pass filter. We model the input PSD as white noise of magnitude $S_N = 2kTR$ (two sides) [Eq. (6.2.57)]. The output PSD is therefore

$$S_{\text{out}}(\omega) = S_N \times \frac{1}{1+(\omega RC)^2} \text{ volt}^2/\text{Hz} \tag{6.3.22}$$

and the output autocorrelation function is the inverse Fourier transform. The form of Eq. (6.3.22) requires in the table,[13] $\alpha = \frac{1}{RC}$ and some juggling of the constants, with the result

$$R_{\text{out}}(\tau) = \frac{S_N}{2RC}e^{-|\tau|/RC} \text{ volts}^2 \tag{6.3.23}$$

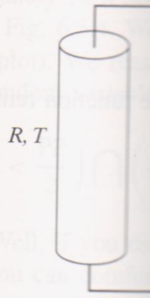

Figure 6.3.3 A resistor h… equivalent circuit shows t… resistor source, and a capa… physical structure.

If we insert the value for S…

Character of the ou…

voltage will still be Gaussi…

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For example, at room temp… independent of R.

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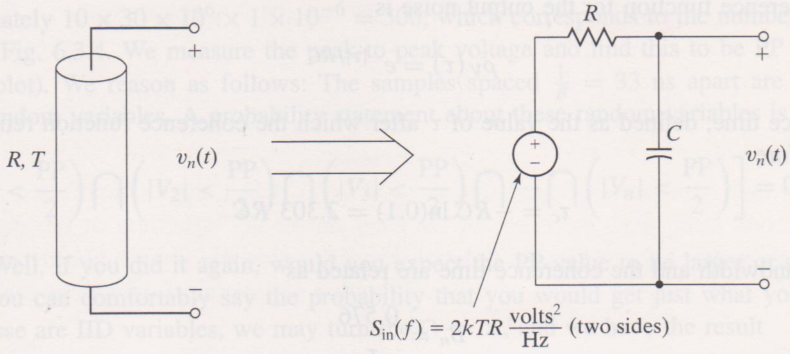

Figure 6.3.3 A resistor has output noise owing to thermal movement of carriers. The equivalent circuit shows the internal noise source, with its PSD, the output impedance of the resistor source, and a capacitance to account for stray, or parasitic, capacitance inherent to the physical structure.

If we insert the value for $S_N = 2kTR$, we find that the output autocorrelation function has the form

$$R_{\text{out}}(\tau) = \frac{kT}{C} e^{-|\tau|/RC} \text{ volts}^2 \tag{6.3.24}$$

Character of the output. We consider now the broad features of the output. The output voltage will still be Gaussian and ergodic. There is no mean, but the variance has the value

$$R_{\text{out}}(0) = E[V_{\text{out}}^2(t)] = \sigma_V^2 = \frac{S_N}{2RC} = \frac{kT}{C} \tag{6.3.25}$$

The rms value of the output noise is therefore

$$V_{\text{out}} = \sqrt{\frac{kT}{C}} \text{ rms volts} \tag{6.3.26}$$

For example, at room temperature with 1 pF stray capacitance, we have an rms voltage of 64 μV, independent of R.

A noise bandwidth, B_n, is found by equating the total noise power given in Eq. (6.3.23) to

$$S_N \times 2B_n = \frac{S_N}{2RC} \tag{6.3.27}$$

It follows that

$$B_n = \frac{1}{4RC} \text{ Hz} \tag{6.3.28}$$

The half-power point of the low-pass filter is $f_c = \frac{1}{2\pi RC}$, and therefore the noise bandwidth is about 1.6 times the half-power frequency.