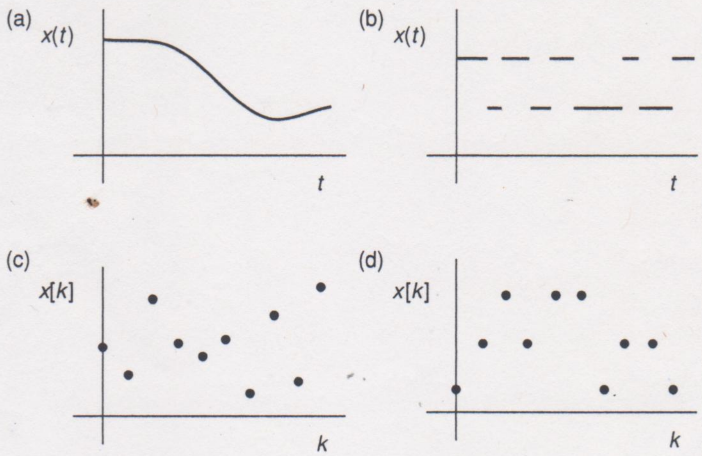

FIGURE 6.6 Example realizations of random processes. (a) Continuous time, continuous amplitude. (b) Continuous time, discrete amplitude (two values). (c) Discrete time, continuous amplitude. (d) Discrete time, discrete amplitude (three values).

In addition to the four cases in the figure, it is possible that the underlying probability space is *mixed* when the random variables have continuous and discrete components. A simple mixed process is given in Example 6.[illegible] where $Y(t)$ has nonzero probability mass at $y = 0$. Other issues and properties of a random process are evident from the previous descriptions. For example, the random variables may change with time (e.g., from Gaussian to exponential), or the same random variable may exist for all $\mathcal{T}$, but one or more of its parameters could change (e.g., Gaussian with time-varying mean). In this chapter, we cover important examples of random processes, discuss their properties and some applications for which they are relevant *models*, and examine the correlation of random variables at different time instants.

We begin with some general properties or *characterizations* of a random process that arise because the process is a collection of random variables indexed by time.

6.5 STATIONARITY

There are several different types of stationarity for a random process. Generally, stationarity refers to the degree to which the probabilistic model of the time-indexed random variables is "constant" with time.

Definition: First-Order Stationary Random process $X(t)$ is *first-order stationary* if the pdf does not depend on time:

$$f_{X(t)}(x) = f_X(x), \quad \text{for all } t \in \mathcal{T}. \tag{6.19}$$

As a result, all moments (raw and central), cumulants, and so on, are constant. For this type of stationarity, we cannot say anything about moments across two or more time instants, such as correlation and covariance.

Definition: *N*th-Order Stationary Random process $X(t)$ is *Nth-order stationary* if the joint probability distribution of the process at N time instants does not change by any time shift τ of all $\{t_1, \ldots, t_N\}$:

$$f_{X(t_1),\ldots,X(t_N)}(x_1, \ldots, x_N) = f_{X(t_1-\tau),\ldots,X(t_N-\tau)}(x_1, \ldots, x_N). \tag{6.20}$$

This type of stationarity is obviously stronger than first-order stationarity; it implies that moments across time instants (up to $N-1$ time differences) are unchanged due to a time shift. We can view Nth-order stationarity as though there is a "sliding window" (of width N) within which the joint pdf does not change.

Definition: Strictly Stationary Random process $X(t)$ is *strictly stationary* if it is Nth-order stationary for all N as $N \to \infty$.

This is the strongest form of stationarity: all statistics of the random process are constant for all time differences and all time shifts. A strictly stationary process implies (6.20) which in turn implies (6.19). For all three cases, the marginal distributions are identical: the type of distribution (Gaussian, Poisson, and so on) is the same and the parameters of the pdf do not change. In general, (6.19) does not imply (6.20). An important exception occurs when the random process is independent such that all joint distributions can be expressed as products of the marginal distributions. This corresponds to an independent and identically distributed (iid) random process, which is discussed in Section 6.6.

In many applications, all three of these definitions of stationarity are often stronger than is needed to characterize a random process. A strictly stationary random process is probably too strong a requirement and is unnecessary for most practical scenarios. A random process that is Nth-order stationary is also quite strong because all moments across N time instants are assumed to be constant, which may be unrealistic in practice. For many problems, we are usually interested only in the first- and second-order moments (especially correlation) and whether or not those are time invariant. This will become evident later when we discuss correlation in detail and its frequency-domain counterpart (via the Fourier transform) known as the power spectral density (PSD).

We repeat the definition of correlation given in Chapter 5 for random variables, applied here to a random process.

Definition: Autocorrelation Function The *autocorrelation function* of random process $X(t)$ is

$$R_{XX}(t_1, t_2) \triangleq \mathcal{E}[X(t_1)X(t_2)], \tag{6.21}$$

which, in general, is a function of the specific values $\{t_1, t_2\}$. The definition for random sequence $X[k]$ is

$$R_{XX}[k_1, k_2] \triangleq \mathcal{E}[X[k_1]X[k_2]]. \tag{6.22}$$

Unlike correlation for random variables, $R_{XX}(t_1, t_2)$ is a *function*: it depends on the particular time instants.

Definition: Uncorrelated Process Random process $X(t)$ is *uncorrelated* if

$$R_{XX}(t_1, t_2) = \begin{cases} \mathcal{E}[X^2(t_1)], & t_1 = t_2 \\ \mathcal{E}[X(t_1)]\mathcal{E}[X(t_2)], & t_1 \neq t_2. \end{cases} \tag{6.23}$$

Likewise for random sequence $X[k]$:

$$R_{XX}[k_1, k_2] = \begin{cases} \mathcal{E}[X^2[k_1]], & k_1 = k_2 \\ \mathcal{E}[X[k_1]]\mathcal{E}[X[k_2]], & k_1 \neq k_2. \end{cases} \tag{6.24}$$

If the process has zero mean, the autocorrelation function is zero, except when its arguments are equal, in which case it is the variance at that particular time.

Definition: Autocovariance Function The *autocovariance function* of random process $X(t)$ is

$$C_{XX}(t_1, t_2) \triangleq \mathcal{E}[(X(t_1) - \mathcal{E}[X(t_1)])(X(t_2) - \mathcal{E}[X(t_2)])]. \tag{6.25}$$

The definition for random sequence $X[k]$ is

$$C_{XX}[k_1, k_2] \triangleq \mathcal{E}[(X[k_1] - \mathcal{E}[X[k_1]])(X[k_2] - \mathcal{E}[X[k_2]])]. \tag{6.26}$$

If the process is uncorrelated, the autocovariance function is zero except when its arguments are equal, which is the variance (even for nonzero mean). The following type of stationarity is often assumed in many engineering applications where systems operate on random signals.

Definition: Wide-Sense Stationary Random process $X(t)$ is *wide-sense stationary* if the mean is constant and the autocovariance function depends only on the time difference and not on specific time instants:

$$\mathcal{E}[X(t)] = \mu_X, \quad \text{for all } t \in \mathcal{T} \tag{6.27}$$

and

$$C_{XX}(t_1, t_2) = C_{XX}(t_2 - t_1), \quad \text{for all } t_1, t_2 \in \mathcal{T}. \tag{6.28}$$

The time difference $t_2 - t_1$ is also called the time *lag*, and when a process is wide-sense stationary, $\tau \triangleq t_2 - t_1$ is usually substituted such that

$$C_{XX}(t_2 - t_1) = C_{XX}(\tau). \tag{6.29}$$

For a random sequence,

$$C_{XX}[k_2 - k_1] = C_{XX}[m], \tag{6.30}$$

where $m \triangleq k_2 - k_1$. The corresponding autocorrelation functions are $R_{XX}(\tau)$ and $R_{XX}[m]$.

For uncorrelated processes, $C_{XX}(\tau) = \sigma_X^2\delta(\tau)$ and $C_{XX}[m] = \sigma_X^2\delta[m]$. These are discussed further in Chapter 8 where *white noise* is defined.

Example 6.5. The autocorrelation function for the random process in Example 6.3 is

$$R_{XX}(t, t+\tau) = \mathcal{E}[A^2]\exp(-\lambda(2t+\tau)) = (1/3)\exp(-\lambda(2t+\tau)), \tag{6.31}$$

which demonstrates that $X(t)$ is *not* wide-sense stationary. The mean also varies with time: $\mu_X(t) = (1/2)\exp(-\lambda t)$.

Wide-sense stationary processes are examined extensively in Chapter 8, and they are generally assumed for the various signals in the application chapters.

From symmetry in the definition, the autocorrelation function is even: $R_{XX}(\tau) = R_{XX}(-\tau)$, and clearly it is nonnegative for lag zero: $R_{XX}(0) = \mathcal{E}[X^2(t)] \geq 0$. The following theorem describes an additional property of the autocorrelation function.

Theorem 6.2. The autocorrelation function is maximum for zero lag:

$$|R_{XX}(\tau)| \leq R_{XX}(0). \tag{6.32}$$

Proof. From the Cauchy–Schwarz inequality in Appendix F,

$$|R_{XX}(\tau)| = |\mathcal{E}[X(t)X(t+\tau)]| \leq \sqrt{\mathcal{E}[X^2(t)]\mathcal{E}[X^2(t+\tau)]}. \tag{6.33}$$

Since the process is wide-sense stationary, $\mathcal{E}[X^2(t)] = \mathcal{E}[X^2(t+\tau)] = R_{XX}(0)$, which completes the proof. □

The previous definitions of correlation and covariance can be extended to two random processes.

Definition: Cross-Correlation and Covariance The *cross-correlation function* of random processes $X(t)$ and $Y(t)$ is

$$R_{XY}(t_1, t_2) \triangleq \mathcal{E}[X(t_1)Y(t_2)], \tag{6.34}$$

and the *covariance function* is

$$C_{XY}(t_1, t_2) \triangleq \mathcal{E}[(X(t_1) - \mathcal{E}[X(t_1)])(Y(t_2) - \mathcal{E}[Y(t_2)]]. \tag{6.35}$$

For random sequences $X[k]$ and $X[k]$,

$$R_{XY}[k_1, k_2] \triangleq \mathcal{E}[X[k_1]Y[k_2]], \tag{6.36}$$

$$C_{XY}[k_1, k_2] \triangleq \mathcal{E}[(X[k_1] - \mathcal{E}[X[k_1]])(Y[k_2] - \mathcal{E}[Y[k_2]])]. \tag{6.37}$$

When the processes are wide-sense stationary, these expressions become $R_{XY}(\tau)$, $C_{XY}(\tau)$, $R_{XY}[m]$, and $C_{XY}[m]$, where $\tau \triangleq t_2 - t_1$ and $m \triangleq k_2 - k_1$.

Note that the wide-sense stationary results in the previous definition are *not* symmetric: for example, $R_{XY}(\tau) \neq R_{XY}(-\tau)$; instead $R_{XY}(\tau) = R_{YX}(-\tau)$ and likewise for the other quantities.

For completeness, we also include the definition of cyclostationarity; it is used only briefly later in Chapter 7.

Definition: Cyclostationary Random process $X(t)$ is *wide-sense cyclostationary* if the mean is constant and the autocovariance function is periodic with period T_o:

$$C_{XX}(\tau) = C_{XX}(\tau + T_o), \quad \text{for any } \tau. \tag{6.38}$$

6.6 INDEPENDENT AND IDENTICALLY DISTRIBUTED

Since a random process is viewed as a collection of random variables indexed by time, we can extend the idea of independent random variables to a process. Consider the following example of an independent random sequence.

Example 6.6. At each instant of time $k \in \mathcal{T} = \mathcal{Z}^+$, an outcome $x[k]$ of the random sequence $X[k]$ is drawn from a Bernoulli random variable with fixed parameter p. This is equivalent to a series of coin tosses with $P(H) = p$. At each time instant, the coin is tossed, the outcome is observed, and the result is "appended" to prior outcomes to construct a realization (a discrete-time function). The pdf of the random variable remains unchanged over k, which means the random sequence is first-order stationary. Because each toss is independent, the random sequence is also strictly stationary. Assume the coin has been tossed N times for which $X[k]$ has 2^N possible realizations. An example is shown in Figure 6.7, where $H \equiv 1$ and $T \equiv 0$. The set of 2^N realizations is referred to as the *ensemble* of the random sequence: each realization is an outcome of the underlying probability space. For this discrete and finite example, we can easily express the sample space of the random sequence in terms of $\Omega = \{H, T\}$ for the Bernoulli random variable as the following Cartesian product: $\mathcal{S} = \Omega \times \cdots \times \Omega = \Omega^N$. Its elements $\{\zeta\}$ are the 2^N *permutations*: $\{H, \ldots, H, H\}, \{H, \ldots, H, T\}, \ldots, \{T, \ldots, T, T\}$. Since the sequence is indexed by time (a direction is implied), we must consider all permutations, not just combinations; realization $\{H, \ldots, H, H, T\}$ is different from $\{H, \ldots, H, T, H\}$ even though each set has only one T. Since the random variables are independent and stationary (p does not change), each $\zeta \in \mathcal{S}$ with m heads has probability $p^m q^{N-m}$ with $q \triangleq 1 - p$.

6.1.1 Modeling an Asynchronous Digital Signal

Basic model. We will use a Poisson process as the underlying model for an asynchronous digital signal. The random-process model is based on the Poisson process shown in Fig. 6.1.3.

We model an asynchronous digital signal by using a Poisson point process as the clock input to a toggle flip-flop (FF). The output logic levels are assumed to be 0 and 1 V and the output is preset to 1 at $t = 0$. The circuit is shown in Fig. 6.1.4.

We call this a *semirandom flip-flop* because we have preset the output state to 1. Later, we will eliminate the initial condition to have a fully random FF. The output of the FF for the input sequence in Fig. 6.1.3 is shown in Fig. 6.1.5.

Describing the random processes with PMFs or PDFs. We now describe the state of the random process with a probability mass function, PMF. The output is discrete, $Q(t) = 0$ or 1, and hence a PMF is required. If the output were an analog signal, we would use a PDF.

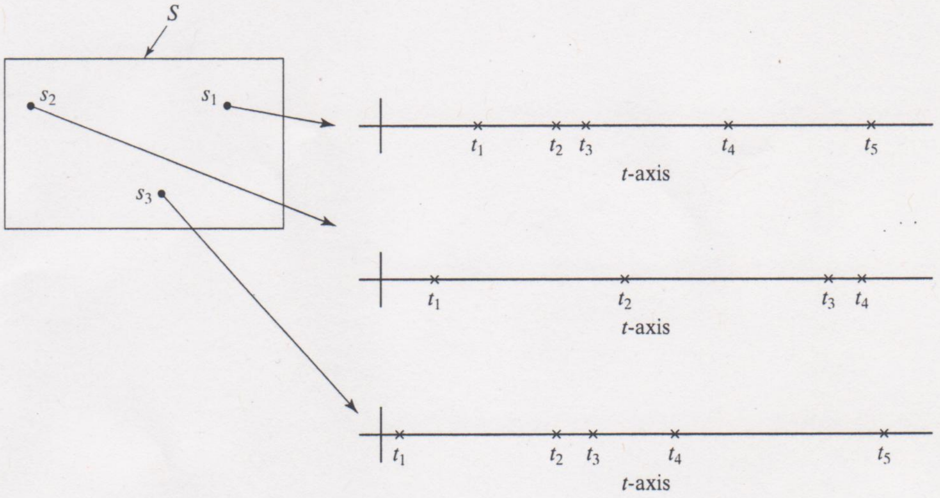

Figure 6.1.3 The Poisson point random process consists of a random sequence of events for each outcome of the chance experiment. We use this process to model an asynchronous digital signal.

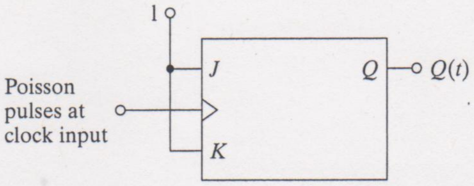

Figure 6.1.4 The output state of the toggle flip-flop will change at every Poisson event at the clock input. Our initial analysis assumes that $Q(0) = 1$ because the output is preset to 1.

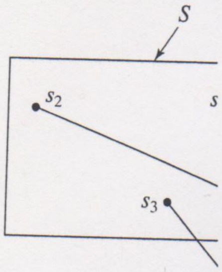

Figure 6.1.5 The outpu
points in Fig. 6.1.3. Note
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First-order PMF.

which takes on values at
[(Fig. 2.2.6)]. The cond
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$$P_{Q(t)}(1) = \ldots$$

We now substitute the Poi

$$P_{Q(t)}(1) = e^{-\lambda t} + \frac{(\lambda t)^{\cdot}}{2!}$$

The series is unfamiliar, bu
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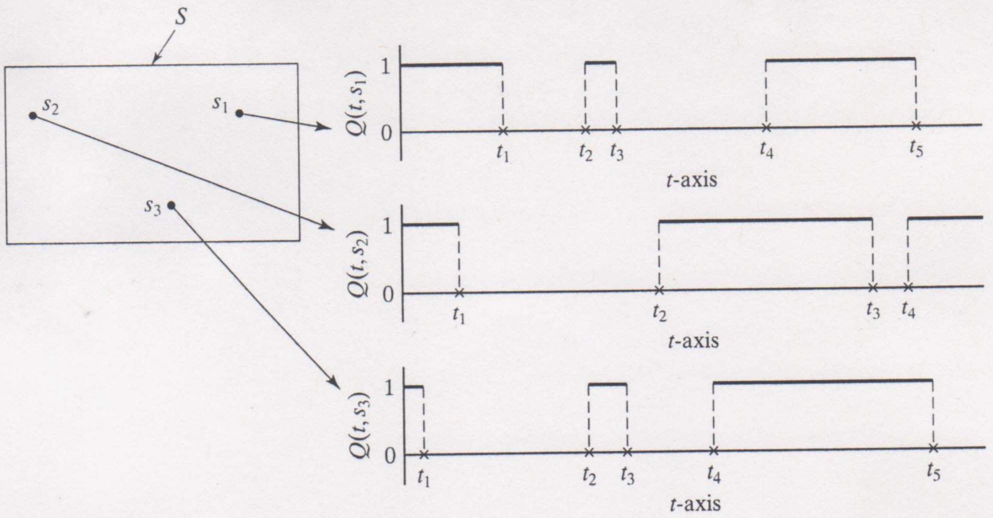

Figure 6.1.5 The output random process is generated by the input random process of Poisson points in Fig. 6.1.3. Note that all output signals begin with $Q(0) = 1$ because we preset the output to 1.

First-order PMF. The first-order PMF gives the output PMF at some time t. By definition,

$$P_{Q(t)}(q) = P[Q(t) = q] \tag{6.1.3}$$

which takes on values at $q = 0$ and $q = 1$ only. Thus $Q(t)$ is a Bernoulli random variable [(Fig. 2.2.6)]. The condition for $Q(t) = 1$ is that there is an even number of clock events between 0 and t. The probability of this event can be calculated from the Poisson process with an average rate of λ as follows:

$$P_{Q(t)}(1) = P[K \text{ even}] = P[K=0] + P[K=2] + P[K=4] + \cdots \tag{6.1.4}$$

We now substitute the Poisson probabilities of Eq. (5.1.12) and obtain the following series:

$$P_{Q(t)}(1) = e^{-\lambda t} + \frac{(\lambda t)^2}{2!}e^{-\lambda t} + \frac{(\lambda t)^4}{4!}e^{-\lambda t} + \cdots = e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] \tag{6.1.5}$$

The series is unfamiliar, but from the standard power-series expansion of the exponential, given in Eq. (5.1.17), we derive the identity

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^{+x} + e^{-x}}{2} \tag{6.1.6}$$

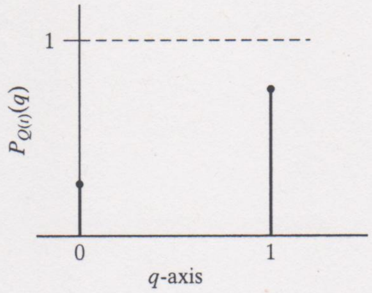

Figure 6.1.6 The PMF of $Q(t)$ at a time when 1 is still favored over 0. As time increases the probabilities become equal.

Using this identity in Eq. (6.1.5) we obtain

$$P_{Q(t)}(1) = e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] = e^{-\lambda t}\left[\frac{e^{+\lambda t} + e^{-\lambda t}}{2}\right] = \frac{1}{2}[1 + e^{-2\lambda t}] \quad (6.1.7)$$

Equation (6.1.7) gives the result we seek and is easily interpreted in terms of the FF output. Note that for $t = 0$, we have $P_{Q(t)}(1) = 1$, which must be true because we preset the FF to 1. With time increasing, the probability that the output is in the 1 state approaches $P_{Q(t)}(1) \to \frac{1}{2}$, which must be true, since the clock events are random and eventually randomize the output.

The derivation of $P_{Q(t)}(0)$ is similar and leads to a similar result:

$$P_{Q(t)}(0) = \tfrac{1}{2}[1 - e^{-2\lambda t}] \quad (6.1.8)$$

Note that $P_{Q(t)}(1) + P_{Q(t)}(0) = 1$. As stated earlier, $Q(t)$ is a Bernoulli random variable, so the PMF is that shown in Fig. 6.1.6. Figure 6.1.7 shows the PMF of the output state as a function of time.

Figure 6.1.7 confirms what we would expect from the way $Q(t)$ is generated. Initially, the output is preset to 1 on all members of the random process. As the Poisson pulses arrive at the clock input, either earlier or later in the various realizations of the random function, more and more of the outputs change states, and with time the output states become fully randomized.

Second-order PMF. We use the results obtained in Eq. (6.1.7) to derive the second-order PMF of $Q(t)$. By definition, this is

$$P_{Q(t_1)Q(t_2)}(q_1, q_2) = P[(Q(t_1) = q_1) \cap (Q(t_2) = q_2)] \quad (6.1.9)$$

Because $Q(t_1)$ and $Q(t_2)$ are not independent, we consider the four possibilities for $q_1q_2 =$ 00, 01, 10, 11. We will work out the last case because we need it later. The other cases are similar. We pick two times, t_1 and t_2, with $t_2 > t_1$, and determine the probability that the output state is 1 at both times, as shown in Fig. 6.1.8.

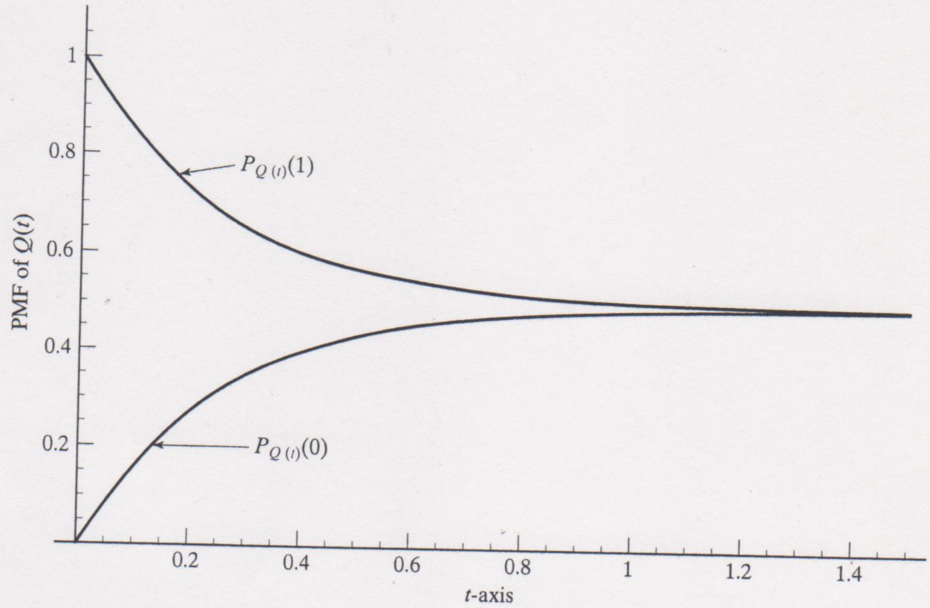

Figure 6.1.7 The PMF of the FF output, $Q(t)$, as a function of time. Initially the output is sure to be 1, but with time the output is randomized by the random input pulses, and eventually 1 and 0 become equally likely. For this part, $\lambda = 2$.

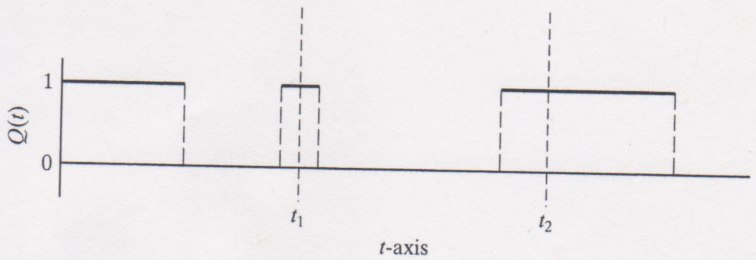

Figure 6.1.8 We pick two times and calculate the probability that $Q(t)$ is 1 at both times, as shown here. The required condition is that there be an even number of transitions between the origin and t_1 and also an even number of transitions between t_1 and t_2.

The probability we will calculate is expressed in Eq. (6.1.10) in terms of a conditional probability:

$$P_{Q(t_1)Q(t_2)}(1, 1) = P[(Q(t_1) = 1) \cap (Q(t_2) = 1)] = P[Q(t_2) = 1 | Q(t_1) = 1] \times P[Q(t_1) = 1] \tag{6.1.10}$$

The conditional probability in Eq. (6.1.10) is the probability of an even number of transitions between t_1 and t_2, which is essentially what we derived in Eqs. (6.1.4)–(6.1.7). Adapting

Eq. (6.1.7), we have

$$P[Q(t_2) = 1|Q(t_1) = 1] = \tfrac{1}{2}[1 + e^{-2\lambda(t_2-t_1)}] \tag{6.1.11}$$

The second term in the second form of Eq. (6.1.10) is exactly what we derived in Eq. (6.1.7), so we obtain

$$P_{Q(t_1)Q(t_2)}(1, 1) = \tfrac{1}{2}[1 + e^{-2\lambda(t_2-t_1)}] \times \tfrac{1}{2}[1 + e^{-2\lambda t_1}] \tag{6.1.12}$$

We have thus determined the second-order PMF of the output of the semirandom flip-flop for one of four possible states. If needed, the others can be derived similarly.

PMFs or expectations? The ultimate and complete description of this random process would be the PMFs to any order desired. Such detail is not required for many basic theoretical and practical applications. Expectations, which represent averages, give much less information about a random process, but the information given is often adequate to design systems for processing a signal. We turn, therefore, to the first- and second-order expectations, which are the mean and the autocorrelation function of $Q(t)$.

The mean of *Q(t)*. By mean, we do not mean the time average but the statistical mean. Look back at Fig. 6.1.5, and consider that you have a vertical line at some time t. The intersection of that line and the random process $Q(t)$ is either 1 or 0 for the individual functions in the random process. The average of those 1s and 0s would be the statistical mean at that time. The mean of a Bernoulli random variable is easily calculated from Eqs. (6.1.7) and (6.1.8) as

$$\begin{aligned}\mu_{Q(t)} &= E[Q(t)] = 0 \times P_{Q(t)}(0) + 1 \times P_{Q(t)}(1) = 0 \times \tfrac{1}{2}[1 - e^{-2\lambda t}] + 1 \times \tfrac{1}{2}[1 + e^{-2\lambda t}] \\ &= \tfrac{1}{2}[1 + e^{-2\lambda t}]\end{aligned} \tag{6.1.13}$$

This result looks like the top curve in Fig. 6.1.7, and this makes sense. Because the FF was preset to 1, the mean should start out at 1, but with time the mean should approach $\frac{1}{2}$ because the output becomes randomized by the Poisson clock pulses.

The autocorrelation function. The autocorrelation function is defined as

$$R_Q(t_1, t_2) = E[Q(t_1)Q(t_2)] \tag{6.1.14}$$

For discrete bivariate random variables, this function is

$$\begin{aligned}R_Q(t_1, t_2) = E[Q(t_1)Q(t_2)] &= \sum_{\text{all states}}\sum q_1 q_2 P_{Q(t_1)Q(t_2)}(q_1, q_2) \\ &= 0 \times 0 \times P_{Q(t_1)Q(t_2)}(0, 0) \\ &\quad + 0 \times 1 \times P_{Q(t_1)Q(t_2)}(0, 1) \\ &\quad + 1 \times 0 \times P_{Q(t_1)Q(t_2)}(1, 0) \\ &\quad + 1 \times 1 \times P_{Q(t_1)Q(t_2)}(1, 1)\end{aligned} \tag{6.1.15}$$

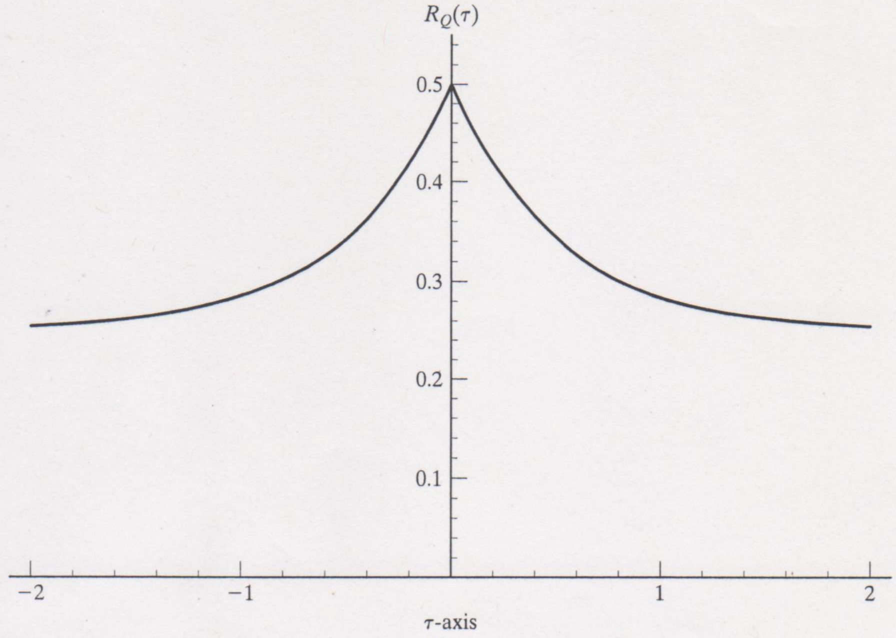

Figure 6.1.9 The autocorrelation function of the output state of the random flip-flop.

Clearly, the only term in the sum that contributes is the last, which requires the probability we calculated in Eq. (6.1.12). The autocorrelation function is

$$R_Q(t_1, t_2) = 1 \times 1 \times \tfrac{1}{2}[1 + e^{-2\lambda(t_2-t_1)}] \times \tfrac{1}{2}[1 + e^{-2\lambda t_1}] \qquad (6.1.16)$$

The result in Eq. (6.1.16) is hard to interpret because it has an early period before the system randomizes. That is why we call this the "semirandom" FF case. We now remove that preset condition to have a fully random FF. We may do this in two ways: we can randomize the initial output state, or we can examine the results far away from the initial time. The latter is easier, so we make the following changes in Eq. (6.1.16)

- Let $t_1 \to$ large.
- Keep $t_2 - t_1 = \text{constant} = \tau$.
- Realize that the expectation is independent of the sign of $t_2 - t_1 = \tau$.

With these changes Eq. (6.1.16) becomes

$$R_Q(\tau) = E[Q(t)Q(t+\tau)] = \tfrac{1}{4}[1 + e^{-2\lambda|\tau|}] \qquad (6.1.17)$$

A plot of Eq. (6.1.17) is shown in Fig. 6.1.9, for $\lambda = 1$.

The autocorrelation is now a function only of time difference, τ. Absolute time does not matter because we have moved far from the time origin, where we preset to 1. Note that under these conditions, the mean is a constant, Eq. (6.1.13).

Example 6.1.2: Micrometeorite counter

A satellite is equipped with a micrometeorite counter. The counter records, on average, 3000 events per day. The input stage of the digital counter changes state for each input. Assuming voltage levels of 0 and 2 voltage, what is the value of the autocorrelation function for that output for a time difference of 15 seconds.

Solution The average rate would be $\lambda = 3000 \times \frac{1}{24} \times \frac{1}{60} = 2.083 \ \frac{\text{events}}{\text{minute}}$. The autocorrelation function for the output is given in Eq. (6.1.17) for logic levels of 0 and 1 V. For logic levels of 0 and 2 V, the autocorrelation function will increase by a factor of 2^2, since $Q(t)$ is multiplied by itself shifted in time. Thus at $\tau = 0.25$ minute, the autocorrelation function has the value $R_Q(0.25) = 2^2 \times \frac{1}{4}(1 + e^{-2\times 2.083\times 0.25}) = 1.353 \ \text{V}^2$.

Definition of wide-sense stationary (WSS) random processes. We now are in a position to define WSS random processes. Let $X(t)$ represent a random process. The definition of WSS is that $X(t)$ satisfies two criteria:

1. The mean is constant, $\mu_X = E[X(t)] = \text{constant}$.
2. The autocorrelation is a function of magnitude of time difference only:[2] $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = f(|t_2 - t_1|)$. Another notation for this expression is that $R_X(\tau) = E[X(t)X(t+\tau)] = f(|\tau|)$, where $f(\cdot)$ is an appropriate function.

Interpretation of WSS random processes. We may interpret WSS random processes as those random processes that look statistically the same at all times, at least as concerns the first- and second-order effects. As we shall see, this means that the total power in the process is constant, and the split between DC and AC power is constant. Thus the DC power and the AC power are constant. This "power" interpretation will be explored later.

The fully random FF random process is WSS. Its mean, Eq. (6.1.13), is constant, $\mu_{Q(t)} \to \frac{1}{2}$ as $\tau \to$ large, and its autocorrelation function is a function of $|\tau|$ only, Eq. (6.1.17). Once we get far away from the time of initialization all that matters is differences in time; absolute time does not matter.

Summary and look ahead. We now have investigated a model for an asynchronous digital signal. Our model is a digital signal that changes states randomly, in accordance with a Poisson process. Some physical realizations of such a signal would be the first stage of a counter monitoring radioactive decay events or monitoring the passage of automobiles on a highway. Our results confirm the WSS nature of the signal. We next develop a model for a synchronous, or clocked, digital signal, then for a random analog signal, and finally for a random noise signal. These remaining sections will be much briefer.

6.1.2 Modeling a Synchronous Digital Signal

The model. Our probability model for a synchronous (clocked) digital signal is shown in Fig. 6.1.10. The probability model shown in Fig. 6.1.10 is random in two ways. The signal is 1 or 0 V in each clock period, with equal probability. The other random aspect is the delay to the beginning of the first full clock period from the arbitrary time origin. This we represent as D in Fig. 6.1.11.

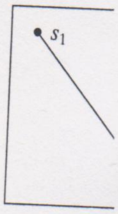

Figure 6.1.10 A probab
each period contains a 1
lead to all such sequence
but have no synchronism

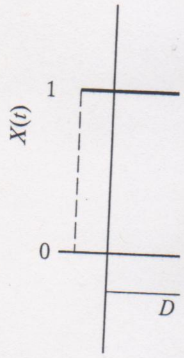

Figure 6.1.11 Definition
clock period, D, from the a
equal probability.

The period, T, in Fig. 6
is uniformly distributed bet

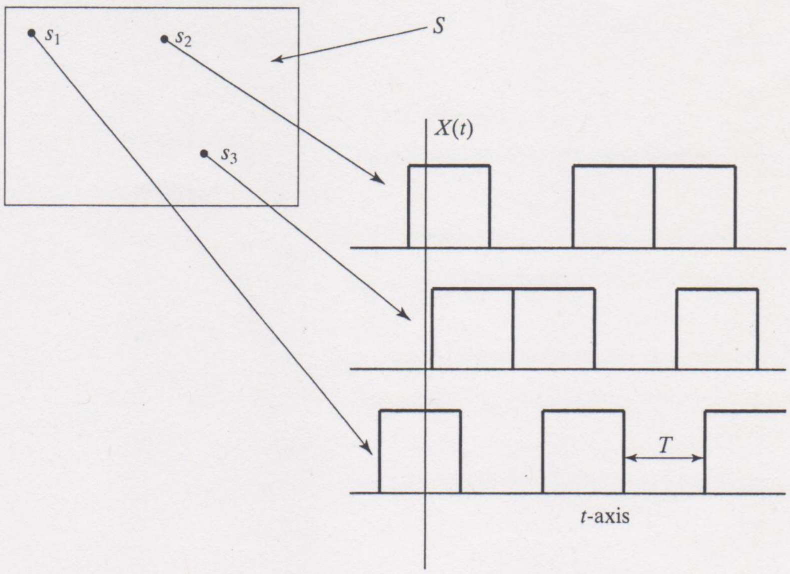

Figure 6.1.10 A probability model for a synchronous digital signal. The clock period is *T*, and each period contains a 1 or 0 with equal probability. The outcomes of the chance experiment lead to all such sequences. Note that the digital functions are synchronous with themselves but have no synchronism with each other. There is therefore no absolute time origin.

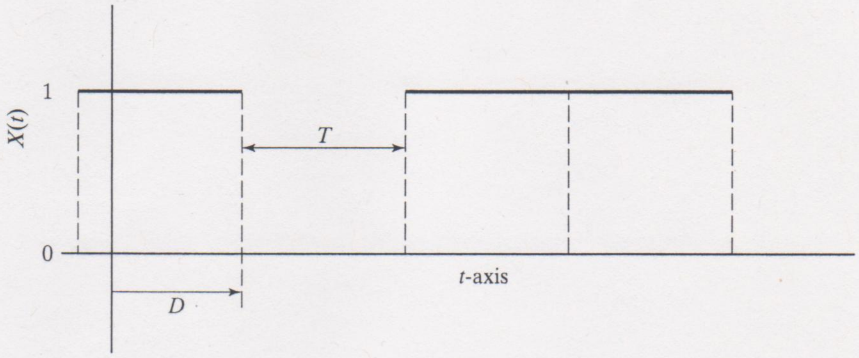

Figure 6.1.11 Definition of the clock period, *T*, and the delay to the beginning of the first full clock period, *D*, from the arbitrary time origin. In each clock period, the signal is 1 or 0 with equal probability.

The period, T, in Fig. 6.1.11 is a known constant, but the delay, D, is a random variable that is uniformly distributed between 0 and T:[3]

$$f_D(d) = \frac{1}{T}, \quad 0 < d \leq T, \text{ zow} \tag{6.1.18}$$

The mean. Again, the random process is 1 or 0 at any time, and we can easily calculate the mean:

$$\mu_{X(t)} = E[X(t)] = 0 \times P[X(t) = 0] + 1 \times \underbrace{P[X(t) = 1]}_{1/2} = \tfrac{1}{2} \text{ volt} \qquad (6.1.19)$$

Thus the mean is $\frac{1}{2}$ V at all times because the signal is equally likely to be 1 or 0.

The autocorrelation function. The autocorrelation function is defined as

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] \qquad (6.1.20)$$

Because $X(t)$ is binary in nature the expectation is similar to Eq. (6.1.15), and the only term that contributes is

$$R_X(t_1, t_2) = 1 \times 1 \times P[(X(t_1) = 1) \cap (X(t_2) = 1)] \qquad (6.1.21)$$

There are two cases to consider.

Case 1: If $t_2 - t_1 > T$, then at least one clock transition between t_1 and t_2 is sure to occur, and the values at $X(t_1)$ and $X(t_2)$ are independent and equally likely to be 1 or 0. In this case Eq. (6.1.21) becomes

$$R_X(t_1, t_2) = 1 \times 1 \times P[(X(t_1) = 1) \cap (X(t_2) = 1)] = 1 \times 1 \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} \qquad (6.1.22)$$

Case 2: If $t_2 - t_1 < T$, then a clock transition may or may not occur between t_1 and t_2. We denote the event of a clock transition occurring in this period as $CT = \{t_1 < \text{clock transition} < t_2\}$. The probability of this event and its complement are

$$P[CT] = \int_{t_1}^{t_2} f_D(d)d(d) = \frac{t_2 - t_1}{T} \quad \text{and} \quad P[\overline{CT}] = 1 - P[CT] = 1 - \frac{t_2 - t_1}{T}, 0 < t_2 - t_1 < T \qquad (6.1.23)$$

In the case where $t_2 - t_1 < T$, we may express the autocorrelation function using the law of total probability [Eq. (1.5.7)] as

$$\begin{aligned} R_X(t_1, t_2) &= 1 \times 1 \times P[(X(t_1) = 1) \cap (X(t_2) = 1)] \\ &= P[(X(t_1) = 1) \cap (X(t_2) = 1) | CT] \times P[CT] \\ &\quad + P[(X(t_1) = 1) \cap (X(t_2) = 1) | \overline{CT}] \times P[\overline{CT}] \end{aligned} \qquad (6.1.24)$$

In the first term, in which a clock transition occurs between t_1 and t_2, $X(t_1)$ and $X(t_2)$ are independent and equally likely to be 1 or 0, and it follows that

$$P[(X(t_1) = 1) \cap (X(t_2) = 1) | CT] \times P[CT] = \frac{1}{2} \times \frac{1}{2} \times \frac{t_2 - t_1}{T} \qquad (6.1.25)$$

where Eq. (6.1.23) was used. In the second term in Eq. (6.1.24), in which a clock transition does not occur between t_1 and t_2, $X(t_1)$ and $X(t_2)$ are the same, with 1 and 0 equally probable, and it follows that

$$P[(X(t_1) = 1) \cap (X(t_2) = 1)|\overline{CT}] \times P[\overline{CT}] = \frac{1}{2} \times \left(1 - \frac{t_2 - t_1}{T}\right) \tag{6.1.26}$$

where again Eq. (6.1.23) was used. Substituting the results of Eqs. (6.1.25) and (6.1.26) into Eq. (6.1.24) and combining terms, we have the result in Eq. (6.1.27) for case 2, $t_2 - t_1 < T$:

$$R_X(t_1, t_2) = \frac{1}{4} \times \frac{t_2 - t_1}{T} + \frac{1}{2} \times \left(1 - \frac{t_2 - t_1}{T}\right) = \frac{1}{2} - \frac{1}{4}\left(\frac{t_2 - t_1}{T}\right) \tag{6.1.27}$$

We now combine Eqs. (6.1.27) and (6.1.22) to get the final result, with the following two additional changes. We note that only the difference between t_2 and t_1 matters; thus we substitute $\tau = t_2 - t_1$. Finally, it does not matter which is greater, t_1 or t_2, because the same probabilities will apply, and thus $|\tau|$ is the true variable of the autocorrelation. The final result is

$$\begin{aligned} R_X(\tau) &= \frac{1}{2} - \frac{1}{4}\frac{|\tau|}{T}, \quad 0 < |\tau| < T \\ &= \frac{1}{4}, \quad |\tau| > T \end{aligned} \tag{6.1.28}$$

This autocorrelation function is plotted in Fig. 6.1.12.

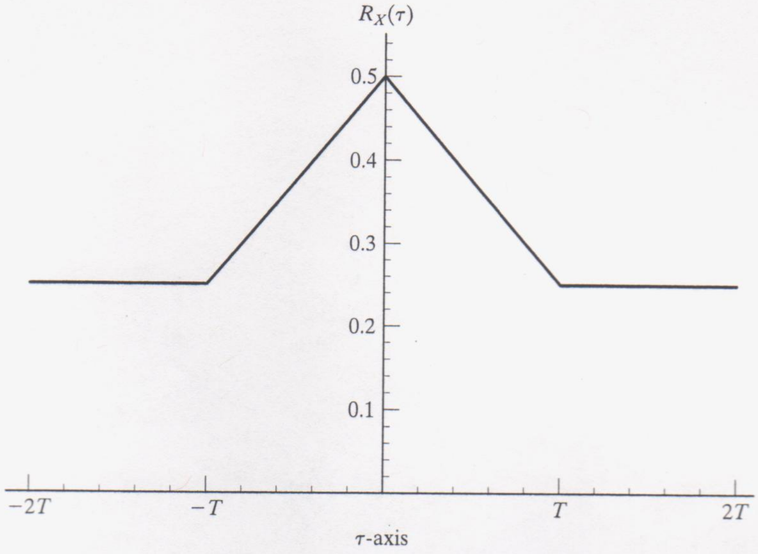

Figure 6.1.12 Autocorrelation function of the model of a synchronous digital signal.

Model is WSS. The model for the synchronous digital signal is WSS, since its mean is constant, Eq. (6.1.19), and its autocorrelation function depends on time differences alone, Eq. (6.1.28). This is a consequence of the stable nature of the statistical properties of the model. The randomization of the synchronous digital signal over the clock period is clearly shown in Fig. 6.1.12.

6.1.3 Modeling a Random Analog Signal

The random sinusoid. In the study of linear systems, we rightly focus much effort in solving problems with sinusoidal waveforms. Sinusoids are used in signal analysis as a basic building block with which more complex waveforms can be analyzed. Sinusoids are also used as carriers for communication signals.

In AC circuit problems, for example, we deal with circuits with known sources, including the amplitude, frequency, and phase of the sinusoidal sources. In contrast, in the real situation that we deal with in a power system we know the frequency quite well, 60 Hz in the United States, and we know the amplitude of the voltage within reasonable bounds, from about $110\sqrt{2}$ to about $125\sqrt{2}$ V, but when we turn on a switch, say to start a dishwasher, we engage the switch at a random time. This is equivalent to turning on the voltage at a random phase. This equivalence is suggested in Fig. 6.1.13.

Semi- and fully random sinusoids. A semirandom sinusoid is a sinusoid turned on at a random time. This model is useful in studying turn-on characteristics of electrical equipment. Here we will study the fully random sinusoid, by which we mean a sinusoid of random phase that exists for all time. Figure 6.1.14 gives the results of a chance experiment that generates six random phases and plots the associated sinusoids.

Properties of the random process. Our main concern is WSS random processes, which require the analysis of the mean and the autocorrelation function. These we can obtain without the PDFs of the random process because we can express this random process as a function of the random variable, Θ, which is the phase of the various sinusoids. Thus we may represent this

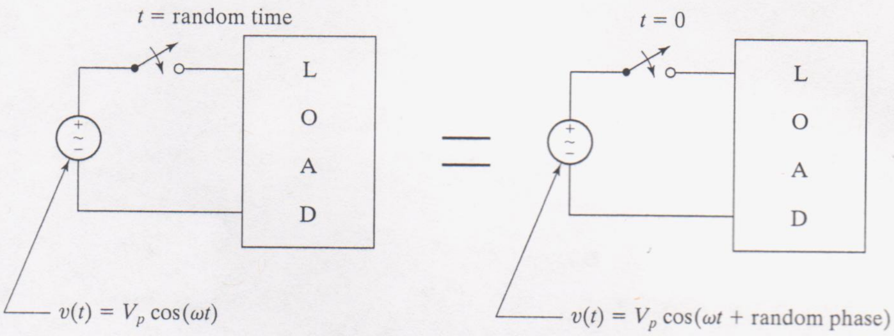

Figure 6.1.13 Turning on a sinusoid at a random time is equivalent to turning on a sinusoid of random phase. The latter would be a semirandom sinusoid. The fully random sinusoid has random phase but no turn-on. This will be our model for a random analog signal.

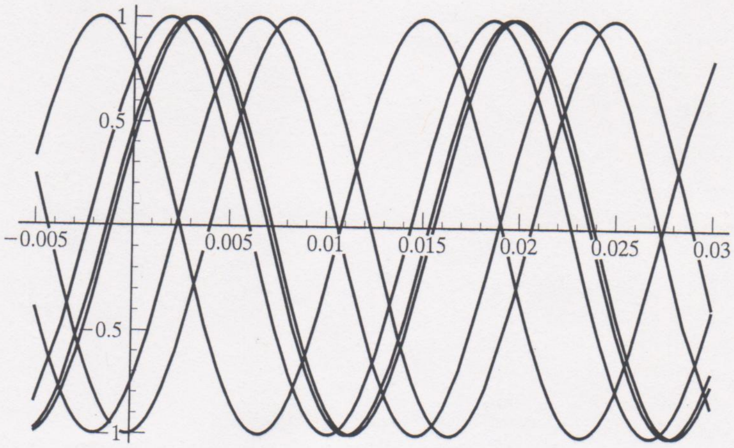

Figure 6.1.14 Six members of the random process modeling a random analog signal. Here we show six sinusoids of random phase. The fully random process treats the phase as a uniformly distributed random variable.

random process simply:

$$V(t, \Theta) = V_p \cos(\omega_1 t + \Theta) \text{ volts} \tag{6.1.29}$$

where V_p is the peak value, ω_1 is the frequency in radians per second (rad/s) and Θ is a uniformly distributed random variable with the PDF

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta \leq 2\pi, \text{ zow} \tag{6.1.30}$$

The mean of *V(t)*. We now can calculate the mean and autocorrelation function of the random process by averaging with respect to Θ, as presented in Eq. (3.1.10). The mean is

$$\mu_{V(t)} = E[V(t, \Theta)] = \int_{-\infty}^{+\infty} V(t, \theta) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} V_p \cos(\omega t + \theta) \frac{1}{2\pi}\, d\theta = 0 \tag{6.1.31}$$

Note that the integration in Eq. (6.1.31) is with respect to θ, not time. The results are zero because we are averaging a sinusoid over a full cycle in θ with ωt fixed.

The autocorrelation function. The autocorrelation function can be determined by the same process:

$$R_V(t_1, t_2) = E[V(t_1, \Theta)V(t_2, \Theta)] = \int_0^{2\pi} V_p \cos(\omega_1 t_1 + \theta) V_p \cos(\omega_1 t_2 + \theta) \frac{1}{2\pi}\, d\theta \tag{6.1.32}$$

The integration in Eq. (6.1.32) can be performed with the aid of a trig identity,[4] which gives the form

$$R_V(t_1, t_2) = \frac{V_p^2}{2} \int_0^{2\pi} [\cos(\omega_1(t_2 - t_1)) + \cos(\omega_1(t_2 + t_1) + 2\theta)] \frac{1}{2\pi}\, d\theta \tag{6.1.33}$$

The second term in Eq. (6.1.33) integrates to zero because the average is performed over two cycles of the sinusoid. The first term in Eq. (6.1.33) has no θ dependence; hence the autocorrelation function is

$$R_V(t_1, t_2) = \frac{V_p^2}{2}\cos(\omega_1(t_2 - t_1)) \int_0^{2\pi} \frac{1}{2\pi}\, d\theta = \frac{V_p^2}{2}\cos(\omega_1(t_2 - t_1)) \tag{6.1.34}$$

If again we let $t_2 - t_1 = \tau$, we have

$$R_V(\tau) = \frac{V_p^2}{2}\cos(\omega_1\tau) \quad \text{volts}^2 \tag{6.1.35}$$

The unit of volts squared relates to power and will be discussed presently.

The fully random sinusoid is WSS. Note that the mean of the fully random sinusoid is zero, which is a constant, and the autocorrelation function is an even function of $t_2 - t_1 = \tau$. Thus the fully random sinusoid is WSS.

Example 6.1.3: The power line

The power input for domestic appliances is 120 V, rms, and a frequency of 60 hertz (Hz), but phase is arbitrary relative to the clocks in your house. What would be the autocorrelation function of the voltage of an appliance output in your house?

Solution The peak voltage would be $120\sqrt{2} = 169.7$ V. The frequency would be $\omega_1 = 2\pi \times 60 = 377.0$ rad/s. Using Eq. (6.1.35) we find the autocorrelation function to be $R_V(\tau) = \frac{(120\sqrt{2})^2}{2}\cos(377\tau)$ volts^2.

You do it. Assume you have an analog clock in your house that has a minute hand 4 in. long. Let $X(t)$ = the horizontal projection of the tip of the minute hand relative to the axis of rotation. Let t be time in minutes from the instant of your birth. Find the autocorrelation of X in feet squared and evaluate at $\tau = 10$ minutes.

```
myanswer = ? ;
```

```
Evaluate
```

For the answer, see endnote 5.

6.1.4 Ergodic Random Processes

Time averages. The following material relates to WSS random processes and to deterministic as well as to random signals. Figure 6.1.15 shows the definitions of the total signal, $v(t)$, the DC component of the signal, V_{DC}, and the AC component of the signal, $v_{AC}(t)$.

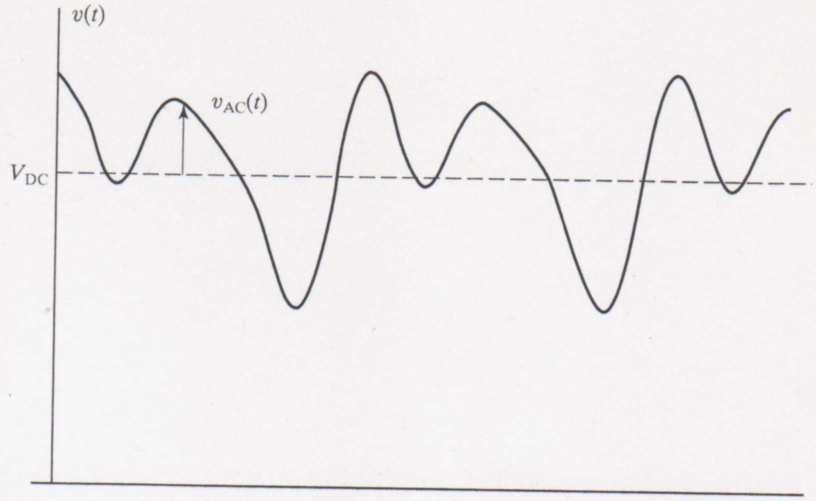

Figure 6.1.15 The definitions of the total signal, $v(t)$, the DC component of the signal, V_{DC}, and the AC component of the signal, $v_{AC}(t)$. Note that in this context "DC" does not mean "direct current" but rather "time-average value," and "AC" does not mean "alternating current" but rather "fluctuating."

For a periodic signal, such as in Fig. 6.1.15, the time average can be determined from one period,

$$\langle v(t)\rangle = \frac{1}{T}\int_0^T v(t)\,dt \tag{6.1.36}$$

but in general one has to average over all time:

$$\langle v(t)\rangle = \lim_{W\to\infty}\frac{1}{W}\int_{-\frac{W}{2}}^{+\frac{W}{2}} v(t)\,dt \tag{6.1.37}$$

where W is the width of a "window" centered on the origin, as shown in Fig. 6.1.16.

"Power" in volts squared. In signal analysis *power* means a measure of the square of the variable. In circuits, the true power would be the voltage squared divided by the impedance level of the circuit in ohms, all multiplied by the power factor if necessary. Here we will deal with power in voltage squared, with the understanding that for real power in watts the impedance level of the circuit must be considered.

The DC value and DC power. The DC value is the time average:

$$V_{DC} = \langle v(t)\rangle \tag{6.1.38}$$

and the DC power is the square of the DC value:

$$P_{DC} = V_{DC}^2 \tag{6.1.39}$$

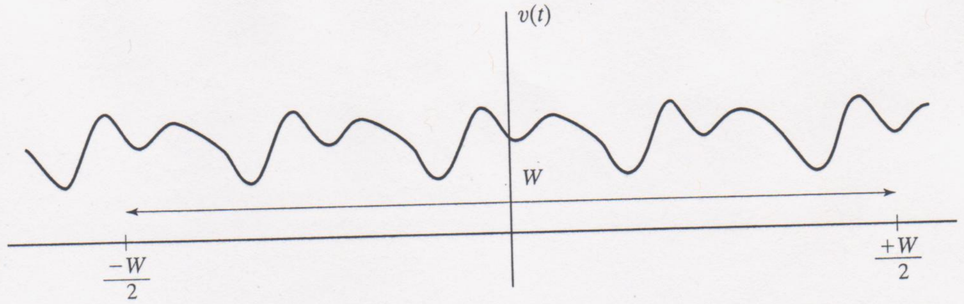

Figure 6.1.16 A "window" over which the function is averaged. The width of the window is allowed to go to infinity, $W \to \infty$, for the time average of the function. This definition of time average works for periodic and random functions.

The AC value and the AC power. The AC value is the total signal minus the DC value:

$$v_{AC}(t) = v(t) - V_{DC} \tag{6.1.40}$$

Note that the time average of the AC value is zero:

$$\langle v_{AC}(t)\rangle = \langle v(t) - V_{DC}\rangle = \langle v(t)\rangle - \langle V_{DC}\rangle = V_{DC} - V_{DC} = 0 \tag{6.1.41}$$

The time average distributes because averaging is a linear operation. The AC power is the time average of the square of the AC component of the signal:

$$P_{AC} = \langle v_{AC}^2(t)\rangle \tag{6.1.42}$$

The total power. The total power in volts squared is the time average of the square of the total voltage, which is also the sum of the DC and AC components, Eq. (6.1.40):

$$P_T = \langle v^2(t)\rangle = \langle (V_{DC} + v_{AC}(t))^2\rangle = \langle V_{DC}^2\rangle + \langle 2V_{DC}v_{AC}(t)\rangle + \langle v_{AC}^2(t)\rangle \tag{6.1.43}$$

The middle term in the expansion vanishes:

$$\langle 2V_{DC}v_{AC}(t)\rangle = 2V_{DC}\langle v_{AC}(t)\rangle = 2V_{DC} \times 0 = 0 \tag{6.1.44}$$

Thus Eq. (6.1.43) reduces to

$$P_T = P_{DC} + P_{AC} \tag{6.1.45}$$

Example 6.1.4: A square wave

Consider a periodic square wave having a period T and an amplitude V, as shown in endnote 6. Find the DC and AC power in this signal.

Solution The DC value is $\frac{V}{2}$, so the DC power is $P_{DC} = \left(\frac{V}{2}\right)^2 = \frac{V^2}{4}$. The AC signal, $v_{AC}(t) = v(t) - \frac{V}{2}$, is another square wave going from $+\frac{V}{2}$ to $-\frac{V}{2}$. Since $P_{AC} = \langle v_{AC}^2(t)\rangle = \left(\pm\frac{V}{2}\right)^2 = \frac{V^2}{4}$ also. Note that the total power, $P_T = \frac{V^2}{2}$, is the time average of the original square wave squared.

You do it. If the square wave has the value V twice as long as it has the value 0, what now is the AC power? Let $V = 1$ V.

```
myanswer = ? ;
```

```
Evaluate
```

For the answer, see endnote 7.

The time-average autocorrelation function. Hitherto we have defined the autocorrelation function as a statistical average, an expectation, as follows:

$$R_V(\tau) = E[V(t)V(t+\tau)] \quad (6.1.46)$$

Thus the (statistical) autocorrelation function involves every member of the family of functions making up the random process. But a time-average autocorrelation function may be defined for a single function. This is true for all periodic deterministic functions, and it is true for each of the "random" functions making up a random process. We replace the expectation operator in Eq. (6.1.46) with the time-average operator for the time-average autocorrelation function:

$$\overline{R}_V(\tau) = \langle v(t)v(t+\tau)\rangle \quad (6.1.47)$$

In words, we average the function times itself shifted τ in the negative t direction. We used the overscore estimator symbol because for a WSS random process the time-average autocorrelation function serves as an estimator for the statistical autocorrelation function.

The time-average autocorrelation function is meaningful for both deterministic and random time functions. In endnote 6 we analyze the time-average autocorrelation function for the square wave used in Example 6.1.4.

Summary. In this section we gave definitions for the DC and AC components of a signal and of the total, DC, and AC power. These definitions apply to both deterministic and random signals. Power is expressed in volts squared.

The ergodic concept. We now have in mind two types of averaging: statistical averaging for the mean and autocorrelation function, and time averaging as described in the previous paragraphs and suggested in Fig. 6.1.17. The ergodic concept addresses the relationship between these two types of averages. In general, a WSS ergodic random process, $X(t)$, has the property that statistical averages are equal to time averages. For the mean

$$\mu_{X(t)} = E[X(t)] \quad \text{and} \quad X_{DC} = \langle X(t)\rangle, \text{ but } E[X(t)] = \langle X(t)\rangle \quad (6.1.48)$$

For the autocorrelation functions, we have the corresponding relation:

$$R_X(\tau) = E[X(t)X(t+\tau)] \quad \text{and} \quad \overline{R}_X(\tau) = \langle X(t)X(t+\tau)\rangle, \text{ but } R_X(\tau) = \overline{R}_X(\tau) \quad (6.1.49)$$

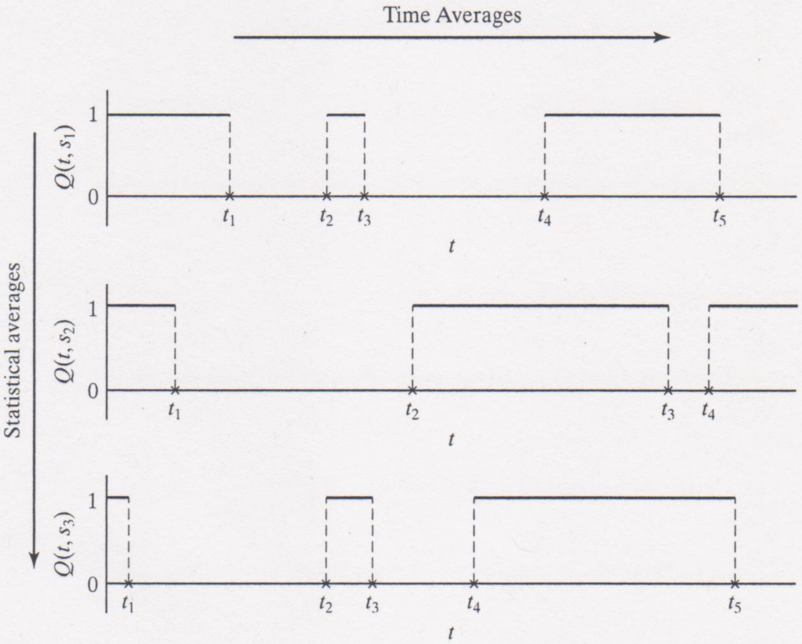

Figure 6.1.17 Time averages must be performed on a single member function of the random process. Statistical averages, the expectation, are performed on the entire random process. The ergodic property requires that the two averages be the same.

Equations (6.1.48) and (6.1.49) are a bit odd in that the statistical averages, $E[X(t)]$ and $E[X(t)X(t+\tau)]$, involve every member function in the random process, whereas the time average, $\langle X(t)\rangle$ and $\langle X(t)X(t+\tau)\rangle$, must of necessity be performed on only one of the member functions of the random process. For Eqs. (6.1.48) and (6.1.49) to be true, the randomness inherent in the entire random process must be present in, and fully represented by, every member function of the random process. The implications of this will be explored after we examine the ergodic property of the fully random sinusoid.

The time-average value and the time-average autocorrelation function of the fully random sinusoid. Because the random sinusoid is a periodic function, we may compute time averages by averaging over one period. Thus the DC value is

$$V_{DC} = \langle V(t,\theta_i)\rangle = \frac{1}{T}\int_0^T V_p \cos(\omega t + \theta_i)\, dt = 0 \tag{6.1.50}$$

Note in Eq. (6.1.50) that we have explicitly stated that the random variable, Θ, takes on a specific value, here called θ_i, for the time average. The time average in Eq. (6.1.50) is similar to the statistical average in Eq. (6.1.31), except that we are now averaging in time. Similarly, the result is zero because we are averaging a sinusoid over one full period:

$$V_{DC} = 0 \tag{6.1.51}$$

The time-average autocorrelation function is

$$\overline{R}_V(\tau) = \langle V(t, \theta_i)V(t + \tau, \theta_i)\rangle = \frac{1}{T}\int_0^T V_p \cos(\omega t + \theta_i)V_p \cos(\omega(t + \tau) + \theta_i)\, dt \tag{6.1.52}$$

Again we use the trig identity for $\cos A \cos B$ (see endnote 4) with the resulting form

$$\overline{R}_V(\tau) = \frac{V_p^2}{2T}\int_0^T [\cos \omega\tau + \cos(\omega(2t + \tau) + 2\theta_i)]\, dt \tag{6.1.53}$$

As before, the second term integrates to zero, and the first is constant with respect to the variable of integration. The result is

$$\overline{R}_V(\tau) = \frac{V_p^2}{2T}\cos \omega\tau \int_0^T dt = \frac{V_p^2}{2}\cos \omega\tau \tag{6.1.54}$$

Comparing time and statistical averages. The time-average value in Eq. (6.1.51) is the same as the statistical expectation in Eq. (6.1.31), and the time-average autocorrelation function in Eq. (6.1.54) is the same as the statistical expectation in Eq. (6.1.35). We therefore conclude that the fully random sinusoid, which is our model for a random analog signal, is not only WSS but is also ergodic.

Why the ergodic property is important. The ergodic property is important to the engineering applications of random processes. Consider the following scenario. You are required to design a system to process a certain type of signal. The signal is an ongoing random signal that looks well behaved, so you take a lot of data and study their properties such as the mean and the autocorrelation function. To design your system, you need to generalize that model into a random process. In that generalization you are assuming the ergodic property. Once your system is designed, it needs to work on specific time signals. In effect, you again needed the ergodic property.

Proving that a random process has the ergodic property. The fully random sinusoid is a random process that is easily shown to be ergodic in mean and autocorrelation function. There are but a few random processes like that. For the rest, the ergodic property is assumed as a matter of convenience, or perhaps necessity. Ultimately, the justification for using the ergodic property is a working system, that is, a good piece of engineering.

Why the autocorrelation function is so important. When we spoke in Sec. 3.5 about correlation between random variables, we criticized the correlation and the covariance as difficult to interpret, and we focused on the correlation coefficient as the major way that the linear relationship between random variables is described. Yet, in dealing with random processes, we focus on the autocorrelation function, which is the correlation of the random process at one time with itself at another time. In this section we show why the autocorrelation function of a WSS random process contains important information about the random process in a concise form. Our purpose here is to show how to read this information from the autocorrelation function.

The definition of the autocorrelation function of a WSS random process, $X(t)$, is

$$R_X(\tau) = E[X(t)X(t+\tau)] \tag{6.1.55}$$

When the WSS random process is also ergodic, then the time-average autocorrelation function is the same:

$$R_X(\tau) = E[X(t)X(t+\tau)] = \langle x(t)x(t+\tau)\rangle \tag{6.1.56}$$

where $x(t)$ is a member function of the random process.

When $\tau = 0$. Setting $\tau = 0$ gives information about the total power in the random process. From Eq. (6.1.56) at $\tau = 0$, we have

$$R_X(0) = E[X^2(t)] = \langle x^2(t)\rangle = P_T \tag{6.1.57}$$

so the autocorrelation at $\tau = 0$ is the total power in the signal. Because $R_X(0)$ also is the mean-square value of the random process, and since the total power is the sum of the AC and DC powers, it follows that

$$R_X(0) = \sigma_X^2 + \mu_X^2 = P_{\text{AC}} + P_{\text{DC}} \tag{6.1.58}$$

But for the ergodic random process, the mean is the same as the time average, or DC value, of the member functions, so we assume for all WSS random process that the variance is the AC power and the square of the mean is the DC power:

$$\sigma_X^2 = P_{\text{AC}} \text{ and } \mu_X^2 = P_{\text{DC}} \tag{6.1.59}$$

When τ gets large. We consider now random processes that lose coherence as τ gets large. This would be true of any random process modeling a physical process without long-term memory. It is true, for example, in our models for synchronous and asynchronous digital signals; however, it is not true for the random sinusoid because this random process never becomes randomized in time. When we let τ get large in Eq. (6.1.55), $X(t)$ and $X(t+\tau)$ become uncorrelated, and the autocorrelation function must approach the square of the mean, as shown by Eq. (3.5.15), when the covariance is zero:[8]

$$\underset{\tau\to\infty}{R_X(\tau)\to} E[X(t)] \times E[X(t+\tau)] = \mu_X^2 = P_{\text{DC}} \tag{6.1.60}$$

From Eq. (6.1.60) it follows that the difference between the autocorrelation at $\tau = 0$ and at $\tau \to \infty$ is the variance, or the AC power:

$$R_X(0) - R_X(\infty) = \sigma_X^2 = P_{\text{AC}} \tag{6.1.61}$$

Thus we can readily derive the total power, the AC power, and the DC power from the autocorrelation function of a WSS ergodic random process.

The coherence function. The correlation coefficient between two random variables, X and Y, is defined in Eq. (3.5.17) as

$$\rho_{XY} = \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} \tag{6.1.62}$$

Consider now the random process $X(t)$. If we let $X \to X(t)$ and $Y \to X(t+\tau)$, the result is

$$\rho_X(\tau) = \frac{E[X(t)X(t+\tau)] - \mu_X \mu_X}{\sigma_X \sigma_X} = \frac{R_X(\tau) - R_X(\infty)}{R_X(0) - R_X(\infty)} \tag{6.1.63}$$

The function $\rho_X(\tau)$ we will call the *coherence function* because it gives the correlation coefficient between the WSS random process at some time and itself at some increment of time τ later or earlier. Note that $\rho_X(0) = 1$.

The coherence time. A WSS random process that decorrelates as τ increases has a characteristic time during which it maintains a degree of coherence; or one could also say that after a period of time it decorrelates. We define a coherence time as that time in τ beyond which the correlation coefficient remains below 0.1. Thus the coherence time, τ_c, is defined as

$$|\rho_X(\tau)| \leq 0.1 \text{ for all } \tau > \tau_c \tag{6.1.64}$$

We could also call the coherence time the *decorrelation time,* since this is the time when the random process loses most of its self-correlation, or even *correlation time,* since this is the time period over which the random process retains a measure of correlation. We give examples later.

A standard form for the autocorrelation function. If we solve Eq. (6.1.63) for $R_X(\tau)$ and use Eqs. (6.1.61) and (6.1.60) we obtain the form

$$R_X(\tau) = \sigma_X^2 \rho_X(\tau) + \mu_X^2 = P_{AC}\rho_X(\tau) + P_{DC} \tag{6.1.65}$$

Thus the autocorrelation function contains information about the total power, the AC power, the DC power, and the coherence properties of the random process. We now apply Eq. (6.1.65) to the three random process models that we have analyzed in this section.

The asynchronous digital signal model. The model for the asynchronous random process, the fully random flip-flop, has an autocorrelation function of Eq. (6.1.66), repeated from Eq. (6.1.17),

$$R_Q(\tau) = \tfrac{1}{4}[1 + e^{-2\lambda|\tau|}] = \underbrace{\tfrac{1}{4}}_{P_{AC}}\underbrace{e^{-2\lambda|\tau|}}_{\rho_Q(\tau)} + \underbrace{\tfrac{1}{4}}_{P_{DC}} \tag{6.1.66}$$

Here the autocorrelation function is readily placed into the form of Eq. (6.1.65), and the various components are evident. Because $Q(t)$ varies randomly between 1 and 0, with equal probability, the DC value is $\frac{1}{2}$, and hence the DC power is $\frac{1}{4}$. Because 1^2 is also 1, the total power also must be $\frac{1}{2}$, and hence the AC power must be $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. These values are identified in Eq. (6.1.66). The coherence function shows a steady loss of coherence due to the random clock pulses. The coherence time is found to be

$$e^{-2\lambda|\tau_c|} = 0.1 \Rightarrow \tau_c = \frac{\ln 10}{2\lambda} = \frac{1.15}{\lambda} \quad (6.1.67)$$

The synchronous model. The model for the synchronous random process yields an autocorrelation function in Eq. (6.1.28) of

$$\begin{aligned} R_X(\tau) &= \frac{1}{2} - \frac{1}{4}\frac{|\tau|}{T}, \quad 0 < |\tau| < T \\ &= \frac{1}{4}, \quad |\tau| > T \end{aligned} \quad (6.1.68)$$

which can be placed in the form

$$\begin{aligned} R_X(\tau) &= \frac{1}{4}\left[1 - \frac{|\tau|}{T}\right] + \frac{1}{4}, \quad 0 < |\tau| < T \\ &= \frac{1}{4}, \quad |\tau| > T \end{aligned}$$

Here again we have a DC power of $\frac{1}{4}$, an AC power of $\frac{1}{4}$, and a coherence function of

$$\rho_X(\tau) = 1 - \frac{|\tau|}{T}, \quad 0 < |\tau| \leq T, \text{ zow} \quad (6.1.69)$$

The coherence time for this model is

$$\rho_X(\tau_c) = 1 - \frac{|\tau_c|}{T} = 0.1 \Rightarrow \tau_c = 0.9T \quad (6.1.70)$$

The signal loses all coherence in one clock period, and our definition gives an answer very close to this value.

The analog model. Although the random sinusoid does not lose coherence, the model in Eq. (6.1.65) still fits very well. The autocorrelation function is given in Eq. (6.1.35) as

$$R_V(\tau) = \frac{V_p^2}{2}\cos\omega\tau \quad \text{V}^2 \quad (6.1.71)$$

which we may place into the form of Eq. (6.1.65) as

$$R_V(\tau) = \underbrace{\frac{V_p^2}{2}}_{P_{AC}} \underbrace{\cos\omega\tau}_{\rho_V(\tau)} + \underbrace{0}_{P_{DC}} \quad (6.1.72)$$

Note the AC and DC power are clearly represented, and the coherence function is

$$\rho_V(\tau) = \cos\omega\tau \quad (6.1.73)$$

This coherence function never goes to zero, which tell us that the random sinusoid never loses coherence. No coherence time exists.

Summary. We
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6.1.5 Model for Wid

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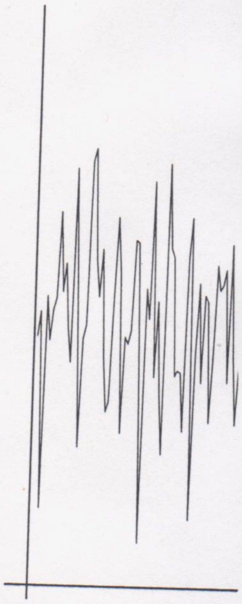

Figure 6.1.18 A noise sig
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Summary. We examined three random processes that model digital and analog signals, defined WSS and ergodic random processes, discussed time averages and statistical averages, and showed how to extract from the autocorrelation function important information about power and coherence properties. In Sec. 6.2, Spectral Analysis of Random Signals, we show that additional important information is contained in the autocorrelation function; but first we must give one further model, that for Gaussian noise.

6.1.5 Model for Wideband Noise

Many physical processes generate wideband noise, also called *broadband* noise. Examples are resistor thermal noise, antenna noise from atmospheric and extraterrestrial sources, and a variety of noise mechanisms in electronic devices due to the discrete nature of electricity. Figure 6.1.18 shows something of what such noise looks like.

The random process model. A random process model for such noise would consist of a large number of outcomes, each with its associated noise signal. We make the following assumptions about this random process.

1. *The noise is Gaussian.* This assumption is appropriate because of the complex mechanisms associated with the generation of such noise. This is logical and is experimentally verifiable in most cases.

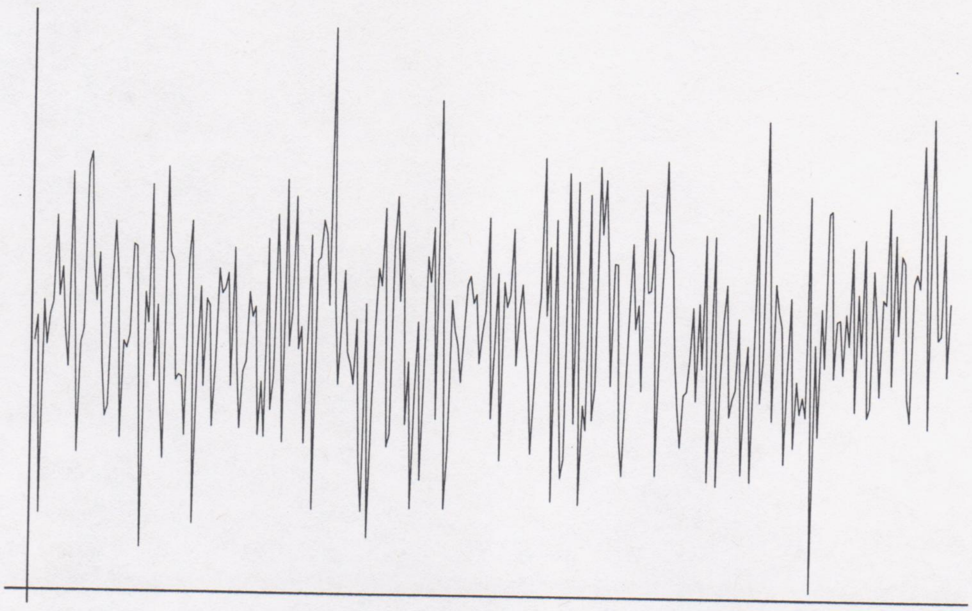

Figure 6.1.18 A noise signal. Such signals are associated with the thermal motion of carriers, discrete carrier phenomena, and other natural sources of noise. Sensitive analog electronic equipment inherently adds such noise to the signal.