6.3 WSS RANDOM PROCESSES IN LINEAR SYSTEMS

6.3.1 Time- and Frequency-Domain Analysis

We began Sec. 6.2 with a review of spectral analysis. We dealt there with the analysis of signals in linear systems. Generally, we introduced the voltage spectrum, $\underline{\mathbf{V}}(\omega)$, and the system function, $\underline{\mathbf{H}}(j\omega)$, and from these we derived the power spectrum, $S(\omega)$, and the system function for power, $|\underline{\mathbf{H}}(j\omega)|^2$. We also introduced the convolution relationship between the time-domain input signal and the impulse response of the system. We showed that convolution depends on the concept of the impulse response and the linear time-invariant nature of the system. Fourier transform relationships need not be involved.

When we discussed the spectra of random signals, as modeled by WSS random processes, we stressed that no voltage spectrum exists but that a power spectrum does exist. This means that we cannot use the relationship between input and output power spectra based on the reasoning that led to Eq. (6.2.18), at least not until we prove its validity by other means. Such proof we now give based on time-domain analysis. This proof completes the explication of the definition of the PSD as the Fourier transform of the autocorrelation function.

A double convolution. Because the frequency-domain path is not available to us, we take the time-domain approach. We consider a linear system with a random process input, as shown in Fig. 6.3.1.

Because each member function of the input random process is related to a corresponding member function of the output random process by convolution, we attribute the convolution relationship to the entire random process:

$$V_{\text{out}}(t) = \int_{-\infty}^{+\infty} V_{\text{in}}(t-\alpha)h(\alpha)\,d\alpha \tag{6.3.1}$$

We calculate the output autocorrelation function by taking $E[V_{\text{out}}(t)V_{\text{out}}(t+\tau)]$, which involves two convolution integrals and an expectation on the right-hand side. Because integration and expectation are linear operations we take the expectation inside the convolution integrals. We also let the variable of integration in the second convolution integral be $\alpha \to \beta$. The result is

$$E[V_{\text{out}}(t)V_{\text{out}}(t+\tau)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[V_{\text{in}}(t-\alpha)V_{\text{in}}(t-\beta+\tau)]h(\alpha)h(\beta)\,d\alpha\,d\beta \tag{6.3.2}$$

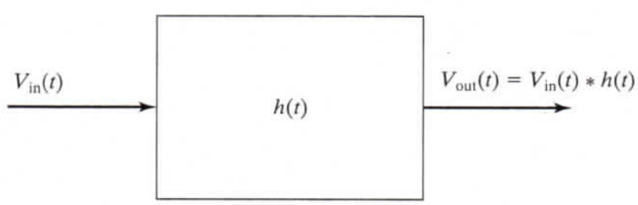

Figure 6.3.1 The input random process and the output random process are related by a convolution relationship.

The two expectations give the input and output autocorrelation functions, except the argument for the input autocorrelation function is the difference between the two arguments, $t-\beta+\tau-(t-\alpha) = \tau+\alpha-\beta$. Thus Eq. (6.3.2) becomes

$$R_{\text{out}}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{\text{in}}(\tau+\alpha-\beta)h(\alpha)h(\beta)\,d\alpha\,d\beta \tag{6.3.3}$$

Equation (6.3.3) is a double convolution. It gives us a means for calculating the output autocorrelation function, given the input autocorrelation function and the impulse response of the system. In certain simple cases this would be a reasonable way to solve a problem; however, our goal is to take the analysis into the frequency domain, where matters are usually simpler. We therefore perform the Fourier transform of Eq. (6.3.3) to obtain the output power spectrum:

$$S_{\text{out}}(\omega) = \int_{-\infty}^{+\infty} R_{\text{out}}(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{\text{in}}(\tau+\alpha-\beta)e^{-j\omega\tau}h(\alpha)h(\beta)\,d\alpha\,d\beta\,d\tau \tag{6.3.4}$$

The triple integral divides nicely into three integrals when we make the substitution $x = \tau+\alpha-\beta$. The key factor in this separation is what happens to the exponential term:

$$e^{-j\omega\tau} \rightarrow e^{-j\omega(x-\alpha+\beta)} = e^{-j\omega x}\times e^{+j\omega\alpha}\times e^{-j\omega\beta} \tag{6.3.5}$$

Substitution of Eq. (6.3.5) into Eq. (6.3.4) and separation of the result into three single integrals give the following:

$$S_{\text{out}}(\omega) = \underbrace{\int_{-\infty}^{+\infty} R_{\text{in}}(x)e^{-j\omega x}\,dx}_{S_{\text{in}}(\omega)} \times \underbrace{\int_{-\infty}^{+\infty} h(\alpha)e^{+j\omega\alpha}\,d\alpha}_{\underline{\mathbf{H}}(j\omega)^*} \times \underbrace{\int_{-\infty}^{+\infty} h(\beta)e^{-j\omega\beta}\,d\beta}_{\underline{\mathbf{H}}(j\omega)} \tag{6.3.6}$$

In Eq. (6.3.6) we have identified the first term as the input PSD. The last term is the Fourier transform of the impulse response and thus is the system function, and the middle term is the complex conjugate of the last term. Thus Eq. (6.3.6) reduces to

$$S_{\text{out}}(\omega) = S_{\text{in}}(\omega)\times|\underline{\mathbf{H}}(j\omega)|^2 \tag{6.3.7}$$

which is the same relation, Eq. (6.2.18), we derived for deterministic signals by squaring the voltage spectrum relationship between input and output. Equation (6.3.7) allows us to use frequency-domain techniques for PSDs of WSS random processes and hence is the final vindication of the definition of the PSD as the Fourier transform of the autocorrelation function and its interpretation as a power spectrum.

Input and output DC. Other relationships between input and output can be investigated. For notational simplicity, let the input be $V_{\text{in}}(t) \rightarrow X(t)$ and the output be $V_{\text{out}} \rightarrow Y(t)$. The relationship between input and output given in Eq. (6.3.1) in this notation is

$$Y(t) = \int_{-\infty}^{+\infty} X(t-\alpha)h(\alpha)\,d\alpha \tag{6.3.8}$$

If we take the expectation of Eq. (6.3.8) and take the resulting constant outside the integral, we have

$$\mu_Y = \mu_X \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha \tag{6.3.9}$$

The impulse response, $h(\alpha)$, is related to the system function by the inverse Fourier transform in Eq. (6.2.21). The associated forward transform is

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau}\, d\tau \tag{6.3.10}$$

If we set $\omega = 0$ in Eq. (6.3.10), we have

$$H(0) = \int_{-\infty}^{+\infty} h(\tau)\, d\tau \tag{6.3.11}$$

Thus Eq. (6.3.9) merely tells us that the output DC is the input DC times the system response at DC.

Cross-correlation between input and output. We may cross-correlate the input and output as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int_{-\infty}^{+\infty} E[X(t)X(t-\alpha+\tau)]h(\alpha)\, d\alpha \tag{6.3.12}$$

where we used Eq. (6.3.8) and took the expectation inside the convolution integral. The result is

$$R_{XY}(\tau) = \int_{-\infty}^{+\infty} R_X(\tau - \alpha)h(\alpha)\, d\alpha \tag{6.3.13}$$

Thus the cross-correlation between input and output can be expressed as a convolution between the input autocorrelation function and the impulse response of the system.

Cross-power spectra between input and output. If we take the Fourier transform of Eq. (6.3.13), we have

$$S_{XY}(\omega) = S_X(\omega)H(j\omega) \tag{6.3.14}$$

Note in Eq. (6.3.14) that $S_X(\omega)$ is a real function, but the system response, $H(j\omega)$, has both amplitude and phase, so the cross-power spectrum has both amplitude and phase associated with it.

We will now illustrate some of these general relationships by passing our four models through a low-pass filter.

6.3.2 Modeling a Random Analog Signal Through a Low-Pass Filter

To keep the mathematics manageable, we will consider a low-pass filter for our linear system and treat in turn the four models for random signals that we have been exploring in this chapter.

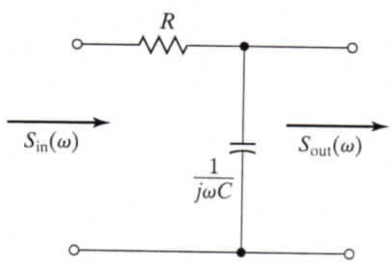

Figure 6.3.2 The *RC* low-pass filter in the frequency domain. The input is a WSS random process, described in the frequency domain by its PSD function.

System properties. The low-pass filter in the frequency domain is shown in Fig. 6.3.2. The power system function for the *RC* low-pass filter is given in Eq. (6.2.19) as

$$|\underline{\mathbf{H}}(j\omega)|^2 = \left|\frac{1}{1+j\omega RC}\right|^2 = \frac{1}{1+(\omega RC)^2} = \frac{1}{1+(f/f_c)^2} \tag{6.3.15}$$

where $f_c = \frac{1}{2\pi RC}$ is the half-power frequency of the filter, also called the *cutoff frequency*. The relationship we will use to determine the output spectrum is given in Eq. (6.3.7) as

$$S_{\text{out}}(\omega) = S_{\text{in}}(\omega) \times |\underline{\mathbf{H}}(j\omega)|^2 \tag{6.3.16}$$

Generally, our method of analysis follows the pattern $R_{\text{in}}(\tau) \hookrightarrow S_{\text{in}}(\omega) \rightarrow S_{\text{out}}(\omega) \rightarrow R_{\text{out}}(t)$. Matters of concern are the DC and AC power in the output, the coherence function, and the bandwidth and coherence time. In Sec. 6.1 we determined the input autocorrelation function for our four models. The $R_{\text{in}}(\tau) \rightarrow S_{\text{in}}(\omega)$ part of the analysis was accomplished in Sec. 6.2 for the four models. We here complete the remainder of the analysis.

Model for a random analog signal

Input characteristics. We begin with the model for a random analog signal because the math is simplest. Our model is a sinusoid with fixed amplitude and frequency but random phase. We showed that this model is WSS and ergodic. The autocorrelation function was given in Eq. (6.1.35) as

$$R_{\text{in}}(\tau) = \frac{V_p^2}{2}\cos\omega_1\tau \text{ volts}^2 \tag{6.3.17}$$

The input PSD function is given in Eq. (6.2.48) as

$$S_{\text{in}}(f) = \frac{V_p^2}{4}\delta(f+f_1) + \frac{V_p^2}{4}\delta(f-f_1) \text{ volts}^2/\text{Hz} \tag{6.3.18}$$

The output PSD function is given by Eq. (6.3.18) multiplied by the power system response, Eq. (6.3.15), with the result

$$S_{\text{out}}(f) = \left[\frac{V_p^2}{4}\delta(f+f_1) + \frac{V_p^2}{4}\delta(f-f_1)\right] \times \frac{1}{1+(f/f_c)^2} \text{ volts}^2/\text{Hz} \tag{6.3.19}$$

The filter function gives its response at $\pm f_1$ and diminishes the amplitude by the factor $\frac{1}{1+(\pm f_1/f_c)^2}$, with the final result

$$S_{\text{out}}(f) = \frac{V_p^2}{4(1+(f_1/f_c)^2)}\delta(f+f_1) + \frac{V_p^2}{4(1+(f_1/f_c)^2)}\delta(f-f_1) \text{ volts}^2/\text{Hz} \tag{6.3.20}$$

Finally, the inverse transform of Eq. (6.3.20) is identical with Eq. (6.3.17), except the amplitude is diminished by the power system response.

$$R_{\text{out}}(\tau) = \frac{V_p^2}{2(1+(f_1/f_c)^2)}\cos 2\pi f_1\tau \text{ volts}^2 \tag{6.3.21}$$

Summary. The low-pass filter has the expected effect of reducing the amplitude of the random sinusoid. The magnitude of the effect depends on the relationship between the frequency of the random sinusoid and the cutoff frequency of the filter. For $f_1 \ll f_c$, the filter has little effect; for $f_1 \gg f_c$, the effect will be strong; and for $f_1 = f_c$, the signal power will be reduced by a factor of 2. •

6.3.3 Model for Broadband Noise in a Low-Pass Filter

Resistor noise output. The effect of a low-pass filter on broadband noise is best considered as a problem in circuit theory. The question we will address is, How much noise voltage comes out of a real resistor? The problem is stated, and translated into circuit theory, in Fig. 6.3.3.

The source resistance and stray capacitance amount to a low-pass filter. We model the input PSD as white noise of magnitude $S_N = 2kTR$ (two sides) [Eq. (6.2.57)]. The output PSD is therefore

$$S_{\text{out}}(\omega) = S_N \times \frac{1}{1+(\omega RC)^2} \text{ volt}^2/\text{Hz} \tag{6.3.22}$$

and the output autocorrelation function is the inverse Fourier transform. The form of Eq. (6.3.22) requires in the table,[13] $\alpha = \frac{1}{RC}$ and some juggling of the constants, with the result

$$R_{\text{out}}(\tau) = \frac{S_N}{2RC}e^{-|\tau|/RC} \text{ volts}^2 \tag{6.3.23}$$

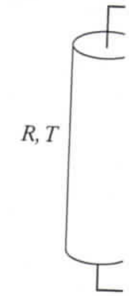

Figure 6.3.3 A resisto equivalent circuit shov resistor source, and a c physical structure.

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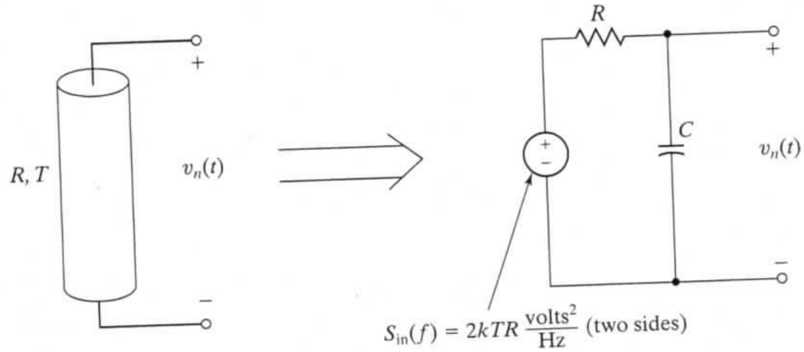

Figure 6.3.3 A resistor has output noise owing to thermal movement of carriers. The equivalent circuit shows the internal noise source, with its PSD, the output impedance of the resistor source, and a capacitance to account for stray, or parasitic, capacitance inherent to the physical structure.

If we insert the value for $S_N = 2kTR$, we find that the output autocorrelation function has the form

$$R_{out}(\tau) = \frac{kT}{C} e^{-|\tau|/RC} \text{ volts}^2 \tag{6.3.24}$$

Character of the output. We consider now the broad features of the output. The output voltage will still be Gaussian and ergodic. There is no mean, but the variance has the value

$$R_{out}(0) = E[V_{out}^2(t)] = \sigma_V^2 = \frac{S_N}{2RC} = \frac{kT}{C} \tag{6.3.25}$$

The rms value of the output noise is therefore

$$V_{out} = \sqrt{\frac{kT}{C}} \text{ rms volts} \tag{6.3.26}$$

For example, at room temperature with 1 pF stray capacitance, we have an rms voltage of 64 μV, independent of R.

A noise bandwidth, B_n, is found by equating the total noise power given in Eq. (6.3.23) to

$$S_N \times 2B_n = \frac{S_N}{2RC} \tag{6.3.27}$$

It follows that

$$B_n = \frac{1}{4RC} \text{ Hz} \tag{6.3.28}$$

The half-power point of the low-pass filter is $f_c = \frac{1}{2\pi RC}$, and therefore the noise bandwidth is about 1.6 times the half-power frequency.

The coherence function for the output noise is

$$\rho_V(\tau) = e^{-|\tau|/RC} \tag{6.3.29}$$

The coherence time, defined as the value of τ after which the coherence function remains below 0.1, is therefore

$$\tau_c = -RC\ln(0.1) = 2.303\ RC \tag{6.3.30}$$

The noise bandwidth and the coherence time are related as

$$B_n = \frac{0.576}{\tau_c} \tag{6.3.31}$$

An application of this theory. Let us say we have a sample of wideband noise and we wish to estimate the variance. Of course, with a true rms voltmeter that has adequate bandwidth one can measure the noise directly, but we will use an oscilloscope. Our method will be to display the noise on the scope and estimate its peak-to-peak (PP) value. The scope trace looks like that plotted in Fig. 6.3.4.

Let us say the known bandwidth of the signal, either from the source or from the scope, is B, and that the trace speed is S cm/s. Because the coherence time is roughly $\frac{1}{B}$, the 10-cm trace contains approximately $n = 10B/S$ independent samples of the voltage. For example, if the scope bandwidth is 30 MHz, and the trace speed is 1 cm/μs, the number of independent samples

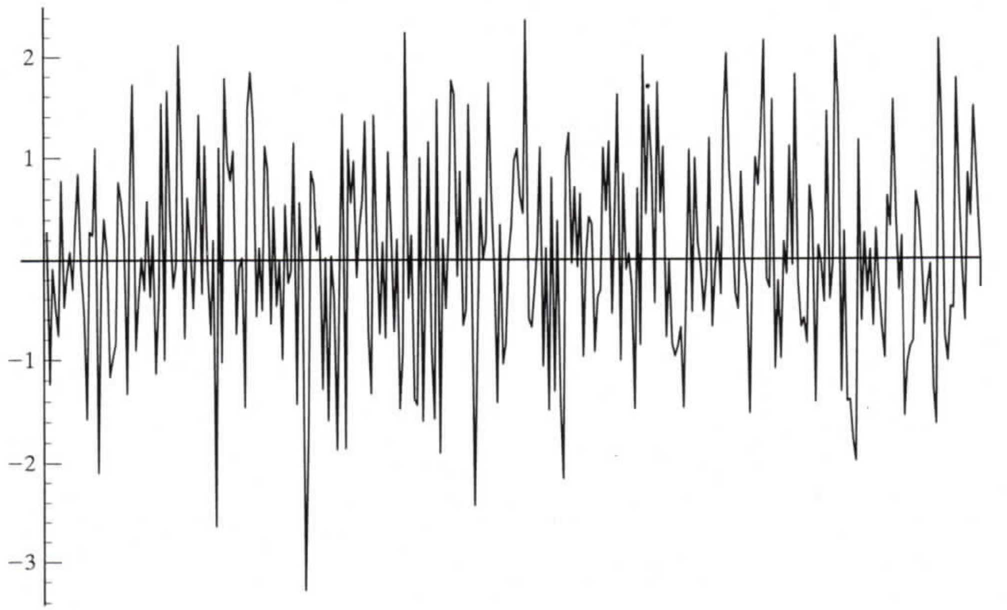

Figure 6.3.4 Simulated noise as it would appear on the screen of an oscilloscope. There are $n = 300$ independent samples in this scan.

is approximately $10 \times 30 \times 10^6 \times 1 \times 10^{-6} = 300$, which corresponds to the number of samples pictured in Fig. 6.3.4. We measure the peak-to-peak voltage and find this to be PP volts (about 5.5 in the plot). We reason as follows: The samples spaced $\frac{1}{B} = 33$ ns apart are IID random Gaussian random variables. A probability statement about these random variables is

$$P\left[\left(|V_1| < \frac{\text{PP}}{2}\right) \cap \left(|V_2| < \frac{\text{PP}}{2}\right) \cap \left(|V_3| < \frac{\text{PP}}{2}\right) \cap \cdots \cap \left(|V_n| < \frac{\text{PP}}{2}\right)\right] = 0.5 \quad (6.3.32)$$

Why 0.5? Well, if you did it again, would you expect the PP value to be larger or smaller? Not knowing, you can comfortably say the probability that you would get just what you got is 0.5. Because these are IID variables, we may turn the $\cap \rightarrow \times$, and we have the result

$$P\left[|V| < \frac{\text{PP}}{2}\right]^n = 0.5 \text{ or } 2\Phi\left(\frac{\text{PP}}{2\sigma_V}\right) - 1 = \sqrt[n]{0.5} \quad (6.3.33)$$

Here we took the nth root of the equation and used the standard formula [Eq. (3.4.9)] for the central probability of a Gaussian random variable. We may solve Eq. (6.3.33) for σ_V, which is the standard deviation, or the rms value, of the noise. The formula is a bit awkward and is best illustrated by an example. We have 300 samples and a PP value of approximately 5.5. For $n = 300$, we have

$$\frac{\text{PP}}{2\sigma_V} = \Phi^{-1}\left(\frac{1 + \sqrt[300]{0.5}}{2}\right) = \Phi^{-1}(0.9988) \approx 3.09 \quad (6.3.34)$$

where we used the inverse CDF table [see endnote 34 in Chapter 3]. We solve Eq. (6.3.34) for σ_V with the result that $\sigma_V \approx \frac{5.5}{6.18} = 0.89$. (We generated Fig. 6.3.4 with a standard deviation of 1, so this is acceptable agreement.)

Summary. We showed how to estimate the standard deviation, or the rms value, of wide-band noise based on the peak-to-peak value and the number of independent samples. The results amount to dividing the PP noise by a factor in the range of 6 to 7. Use 6 if you have hundreds of samples, 7 if you have thousands.

6.3.4 Model for an Asynchronous Digital Signal in a Low-Pass Filter

Model. Our model for an asynchronous digital signal was a flip-flop triggered by a Poisson process with an average rate of λ. The random process was WSS with an autocorrelation function, Eq. (6.1.17), of

$$R_{\text{in}}(\tau) = \frac{1}{4}[1 + e^{-2\lambda|\tau|}] \text{ volts}^2 \quad (6.3.35)$$

The PSD is given in Eq. (6.2.40):

$$S_{\text{in}}(f) = \frac{1}{4}\delta(f) + \frac{\lambda}{(2\lambda)^2 + (2\pi f)^2} \text{ volts}^2/\text{Hz} \quad (6.3.36)$$

We pass this signal through the low-pass filter to produce the output spectrum

$$S_{\text{out}}(f) = \frac{1}{4}\delta(f) + \frac{\lambda}{(2\lambda)^2 + (2\pi f)^2} \times \frac{1}{1 + (f/f_c)^2} \text{ volts}^2/\text{Hz} \tag{6.3.37}$$

We leave the first term alone because this is a DC term, which passes unaffected through the filter. The second term gives the effect of the filter on the AC spectrum of the digital signal. We now determine the autocorrelation function by performing the inverse Fourier transform on Eq. (6.3.37). For this purpose we make a partial-fraction expansion of the second term, which is of the form

$$\frac{1}{x^2 + a^2} \times \frac{1}{x^2 + b^2} = \frac{1}{b^2 - a^2}\left[\frac{1}{x^2 + a^2} - \frac{1}{x^2 + b^2}\right],$$
$$\text{where } x = 2\pi f,\ a = 2\lambda, \text{ and } b = 2\pi f_c \tag{6.3.38}$$

Using this partial fraction expansion, we obtain the form

$$S_{\text{out}}(f) = \frac{1}{4}\delta(f) + \frac{4\pi^2\lambda f_c^2}{(2\pi f_c)^2 - (2\lambda)^2}\left[\frac{1}{(2\pi f)^2 + (2\lambda)^2} - \frac{1}{(2\pi f)^2 + (2\pi f_c)^2}\right] \text{ volts}^2/\text{Hz} \tag{6.3.39}$$

From the table of Fourier transform pairs, we find the inverse transform to be[13]

$$R_{\text{out}}(\tau) = \frac{1}{4} + \frac{4\pi^2\lambda f_c^2}{(2\pi f_c)^2 - (2\lambda)^2}\left[\frac{1}{4\lambda}e^{-2\lambda|\tau|} - \frac{1}{4\pi f_c}e^{-2\pi f_c|\tau|}\right] \tag{6.3.40}$$

Equation (6.3.40) is not easily interpreted. We consider two extremes.

Signal bandwidth ≪ filter bandwidth. If $2\lambda \ll 2\pi f_c$, then the first term in the bracket dominates the second term in the bracket, and the second term in the denominator out front is negligible. In that case, the output autocorrelation function approaches the input,

$$\underset{2\lambda \ll 2\pi f_c}{R_{\text{out}}(\tau) \rightarrow} \frac{1}{4} + \frac{1}{4}e^{-2\lambda|\tau|} = R_{\text{in}}(\tau) \tag{6.3.41}$$

This is what we would expect to happen. Essentially, the bandwidth of the low-pass filter is much wider than the spectrum in the input and thus the signal goes through the filter with little change. The frequency-domain picture of this situation is shown in Fig. 6.3.5, where $2\lambda = \frac{1}{20}2\pi f_c$.

The time-domain picture shows the input pulses slightly rounded due to the loss of high frequencies. Figure 6.3.6 shows the output when the filter bandwidth is much larger than the signal bandwidth, $\lambda = \frac{\pi}{20} f_c$. The main effect is that the pulse corners are rounded.

Figure 6.3.5 The signa
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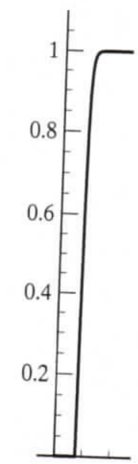

Figure 6.3.6 The outpu
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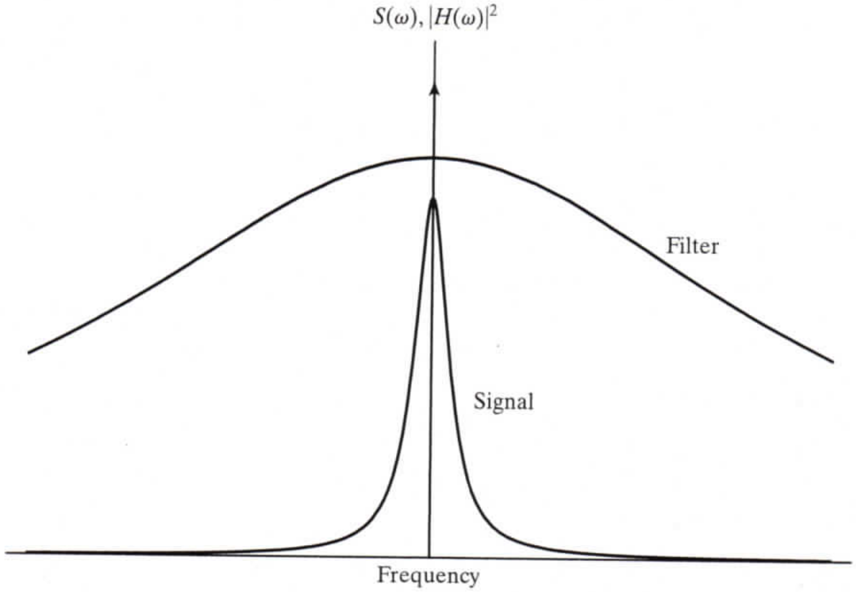

Figure 6.3.5 The signal bandwidth is 5% of the filter half-power bandwidth. The filter has little effect on the signal in this case. The vertical scale is arbitrary.

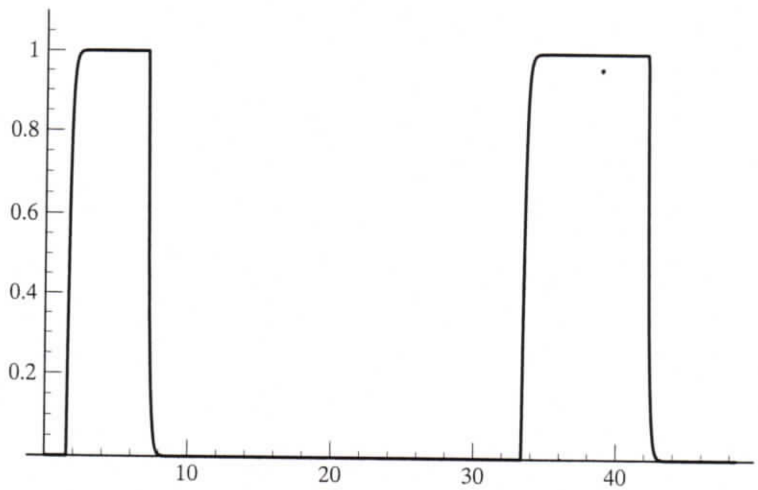

Figure 6.3.6 The output of the low-pass filter with an asynchronous digital signal input. The filter bandwidth is much broader than the signal bandwidth, so the effect of the filter is to round the corners slightly.

Another approach of WSS in Linear Systems

4. for all $N > 0$, all $t_1 < t_2 < \ldots < t_N$ and all complex $a_1, a_2, \ldots, a_N$,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l^* R_{XX}(t_k - t_l) \geq 0.$$

This was shown in Section 7.1 to be a necessary condition for a given function $g(t, s) = g(t-s)$ to be an autocorrelation function. We will show that this property is also a sufficient condition, so that positive semidefiniteness actually *characterizes* autocorrelation functions. In general, however, it is very difficult to check property 4 directly.

To start off, we can specialize the results of Theorems 7.3-1 and 7.3-2, which were derived for the general case, to LSI systems. Rewriting Equation 7.3-2 we have

$$\begin{aligned} E[Y(t)] &= L\{\mu_X(t)\} \\ &= \int_{-\infty}^{\infty} \mu_X(\tau)h(t-\tau)\,d\tau \\ &\triangleq \mu_X(t) * h(t). \end{aligned}$$

Using Theorem 7.3-2 and Equations 7.3-3 and 7.3-4, we get also

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{+\infty} h^*(\tau_2) R_{XX}(t_1, t_2 - \tau_2)d\tau_2,$$

and

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} h(\tau_1) R_{XY}(t_1 - \tau_1, t_2)d\tau_1,$$

which can be written in convolution operator notation as

$$R_{XY}(t_1, t_2) = h^*(t_2) * R_{XX}(t_1, t_2),$$

where the convolution is along the t_2-axis, and

$$R_{YY}(t_1, t_2) = h(t_1) * R_{XY}(t_1, t_2),$$

where the convolution is along the t_1-axis. Combining these two equations, we get $R_{YY}(t_1, t_2) = h(t_1) * R_{XX}(t_1, t_2) * h^*(t_2)$.

Wide-Sense Stationary Case

If we input the stationary random process $X(t)$ to an LSI system with impulse response $h(t)$, then the output random process can be expressed as the convolution integral,

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau, \tag{7.5-1}$$

when this integral exists. Computing the mean of the output process $Y(t)$, we get

$$\begin{aligned} E[Y(t)] &= \int_{-\infty}^{+\infty} h(\tau)E[X(t-\tau)]d\tau \quad \text{by Theorem 7.3-1,} \\ &= \int_{-\infty}^{+\infty} h(\tau)\mu_X d\tau = \mu_X \int_{-\infty}^{+\infty} h(\tau)d\tau, \\ &= \mu_X H(0), \end{aligned} \tag{7.5-2}$$

where $H(\omega)$ is the system's frequency response.

We thus see that the mean of the output is constant and equals the mean of the input times the system function evaluated at $\omega = 0$, the so-called "dc gain" of the system. If we compute the cross-correlation function between the input process and the output process we find that

$$\begin{aligned} R_{YX}(\tau) &= E[Y(t+\tau)X^*(t)] \\ &= E[Y(t)X^*(t-\tau)] \quad \text{by substituting } t-\tau \text{ for } t, \\ &= \int_{-\infty}^{+\infty} h(\alpha)E[X(t-\alpha)X^*(t-\tau)]d\alpha, \end{aligned}$$

and bringing the operator E inside the integral by Theorem 7.3-2,

$$= \int_{-\infty}^{+\infty} h(\alpha)R_{XX}(\tau-\alpha)d\alpha$$

which can be rewritten as

$$R_{YX}(\tau) = h(\tau) * R_{XX}(\tau). \tag{7.5-3}$$

Thus the cross-correlation R_{YX} equals h convolved with the autocorrelation R_{XX}. This fact can be used to identify unknown systems (see Problem 7.28).

The output autocorrelation function $R_{YY}(\tau)$ can now be obtained from $R_{YX}(\tau)$ as follows:

$$\begin{aligned} R_{YY}(\tau) &= E[Y(t+\tau)Y^*(t)] \\ &= E[Y(t)Y^*(t-\tau)] \quad \text{by substituting } t \text{ for } t-\tau, \\ &= \int_{-\infty}^{+\infty} h^*(\alpha)E[Y(t)X^*(t-\tau-\alpha)]d\alpha \\ &= \int_{-\infty}^{+\infty} h^*(\alpha)E[Y(t)X^*(t-(\tau+\alpha))]d\alpha \\ &= \int_{-\infty}^{+\infty} h^*(\alpha)R_{YX}(\tau+\alpha)d\alpha \\ &= \int_{-\infty}^{+\infty} h^*(-\alpha)R_{YX}(\tau-\alpha)d\alpha \\ &= h^*(-\tau) * R_{YX}(\tau). \end{aligned}$$

Combining both equations, we get

$$R_{YY}(\tau) = h(\tau) * h^*(-\tau) * R_{XX}(\tau). \tag{7.5-4}$$

We observe that when $R_{XX}(\tau) = \delta(\tau)$, then the output correlation function is $R_{YY}(\tau) = h(\tau) * h^*(-\tau)$, which is sometimes called the *autocorrelation impulse response* denoted as $g(\tau) = h(\tau) * h^*(-\tau)$. Note that $g(\tau)$ must be positive semidefinite, and indeed $FT\{g(\tau)\} = |H(\omega)|^2 \geq 0$.

Similarly, we also find (proof left as an exercise for the reader)

$$\begin{aligned} R_{XY}(\tau) &= \int_{-\infty}^{+\infty} h^*(-\alpha) R_{XX}(\tau - \alpha) d\alpha \\ &= h^*(-\tau) * R_{XX}(\tau), \end{aligned} \tag{7.5-5}$$

and

$$\begin{aligned} R_{YY}(\tau) &= \int_{-\infty}^{+\infty} h(\alpha) R_{XY}(\tau - \alpha) d\alpha. \\ &= h(\tau) * R_{XY}(\tau) \\ &= h(\tau) * h^*(-\tau) * R_{XX}(\tau) \\ &= g(\tau) * R_{XX}(\tau). \end{aligned}$$

Example 7.5-1

(Derivative of wide-sense stationary process.) Let the second-order random process $X(t)$ be stationary with one parameter correlation function $R_X(\tau)$ and constant mean function $\mu_X(t) = \mu_X$. Consider the system consisting of a derivative operator, i.e.,

$$Y(t) = \frac{dX(t)}{dt}.$$

Using the above equations we find $\mu_Y(t) = \frac{d\mu_X(t)}{dt} = 0$ and cross-correlation function

$$\begin{aligned} R_{XY}(\tau) &= \delta_1^*(-\tau) * R_{XX}(\tau) \\ &= -\frac{dR_{XX}(\tau)}{d\tau}, \end{aligned}$$

since the impulse response of the derivative operator is $h(t) = \delta_1(t)$, the (formal) derivative of the Dirac delta function or impulse $\delta(t)$.

$$\begin{aligned} R_{YY}(\tau) &= \delta_1(\tau) * R_{XY}(\tau) \\ &= \frac{dR_{XY}(\tau)}{d\tau} \\ &= -\frac{d^2 R_{XX}(\tau)}{d\tau^2}. \end{aligned}$$

Notice the autocorrelation impulse response here is $g(\tau) = -\delta_2(\tau)$, or minus the second (formal) derivative of $\delta(\tau)$.

Power Spectral Density

For wide-sense stationary, and hence for stationary processes, we can define a useful density for average power versus frequency, called the *power spectral density* (psd).

Definition 7.5-1 Let $R_{XX}(\tau)$ be an autocorrelation function. Then we define the *power spectral density* $S_{XX}(\omega)$ to be its Fourier transform (if it exists), that is,

$$S_{XX}(\omega) \triangleq \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau. \quad \blacksquare \tag{7.5-6}$$

Under quite general conditions one can define the inverse Fourier transform, which equals $R_{XX}(\tau)$ at all points of continuity,

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{+j\omega\tau}\, d\omega. \tag{7.5-7}$$

In operator notation we have,

$$S_{XX} = FT\{R_{XX}\}$$

and

$$R_{XX} = IFT\{S_{XX}\}$$

where FT and IFT stand for the respective Fourier operators.

The name power spectral density (psd) will be justified later. All that we have done thus far is define it as the Fourier transform of $R_{XX}(\tau)$. We can also define the Fourier transform of the cross-correlation function $R_{XY}(\tau)$ to obtain a frequency function called the *cross-power spectral density*,

$$S_{XY}(\omega) \triangleq \int_{-\infty}^{+\infty} R_{XY}(\tau)e^{-j\omega\tau}\, d\tau. \tag{7.5-8}$$

We will see later that the power spectral density or psd, $S_{XX}(\omega)$, is real and everywhere nonnegative and in fact, as the name implies, has the interpretation of a density function for average power versus frequency. By contrast, the cross-power spectral density has no such interpretation and is generally complex valued.

We next list some properties of the psd $S_{XX}(\omega)$:

1. $S_{XX}(\omega)$ is real-valued since $R_{XX}(\tau)$ is conjugate symmetric.
2. If $X(t)$ is a real-valued WSS process, then $S_{XX}(\omega)$ is an even function since $R_{XX}(\tau)$ is real and even. Otherwise $S_{XX}(\omega)$ may not be an even function of ω.
3. $S_{XX}(\omega) \geq 0$ (to be shown in Section in Theorem 7.5-1).

Additional properties of the psd are shown in Table 7.5-1. One could continue with such a table, but it will suit our purposes to stop at this point. One comment is in order: We note the simplicity of these operations in the frequency domain. This suggests that for LSI systems and stationary or WSS random processes, we should solve for output correlation functions by first transforming the input correlation function into the frequency domain, carry out the indicated operations, and then transform back to the correlation domain.

Table 7.5-1 Correlation Function Properties of Corresponding Power Spectral Densities

Random Process	Correlation Function	Power Spectral Density
$X(t)$	$R_{XX}(\tau)$	$S_{XX}(\omega)$
$aX(t)$	$\|a\|^2 R_{XX}(\tau)$	$\|a\|^2 S_{XX}(\omega)$
$X_1(t) + X_2(t)$ with X_1 and X_2 orthogonal	$R_{X_1X_1}(\tau) + R_{X_2X_2}(\tau)$	$S_{X_1X_1}(\omega) + S_{X_2X_2}(\omega)$
$X'(t)$	$-d^2 R_{XX}(\tau)/d\tau^2$	$\omega^2 S_{XX}(\omega)$
$X^{(n)}(t)$	$(-1)^n d^{2n} R_{XX}(\tau)/d\tau^{2n}$	$\omega^{2n} S_{XX}(\omega)$
$X(t)\exp(j\omega_0 t)$	$\exp(j\omega_0\tau)R_{XX}(\tau)$	$S_{XX}(\omega - \omega_0)$
$X(t)\cos(\omega_0 t + \Theta)$ with independent Θ uniform on $[-\pi, +\pi]$	$\frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau)$	$\frac{1}{4}[S_{XX}(\omega+\omega_0) + S_{XX}(\omega-\omega_0)]$
$X(t) + b \quad (\bar{X} = 0)$	$R_{XX}(\tau) + \|b\|^2$	$S_{XX}(\omega) + 2\pi\|b\|^2\delta(\omega)$

This is completely analogous to the situation in deterministic linear system theory for shift-invariant systems.

Another comment would be that if the interpretation of $S_{XX}(\omega)$ as a density of average power is correct, then the constant or mean component has all its average power concentrated at $\omega = 0$ by the last entry in the table. Also by the next-to-last two entries in the table, modulation by the frequency ω_0 shifts the distribution of average power up in frequency by ω_0. Both of these results should be quite intuitive.

Example 7.5-2

(*Power spectral density of white noise.*) The correlation function of a white noise process $W(t)$ with parameter σ^2 is given by $R_{WW}(\tau) = \sigma^2\delta(\tau)$. Hence the power spectral density (psd), its Fourier transform, is just

$$S_{WW}(\omega) = \sigma^2, \quad -\infty < \omega < +\infty.$$

The psd is thus flat, and hence the name, *white noise*, by analogy to white light, which contains equal power at every wavelength. Just like white light, white noise is an idealization that cannot really occur, since as we have seen earlier $R_{WW}(0) = \infty$, necessitating infinite power.

An Interpretation of the psd

Given a WSS process $X(t)$, consider the finite support segment,

$$X_T(t) \triangleq X(t) I_{[-T,+T]}(t),$$

where $I_{[-T,+T]}$ is an indicator function equal to 1 if $-T \le t \le +T$ and equal to 0 otherwise, and $T > 0$. We can compute the Fourier transform of X_T by the integral

$$FT\{X_T(t)\} = \int_{-T}^{+T} X(t)e^{-j\omega t}\, dt.$$

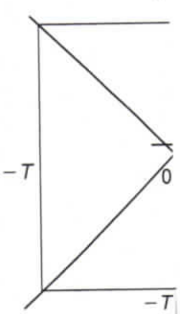

Figure 7.5-1 a) Square
the transformation $s = t_1$

The magnitude square

$$|FT\{X$$

Dividing by $2T$ and ta

$$\frac{1}{2T}E\left[|FT\{X\right.$$

To evaluate the doubl $t_1 + t_2, \tau = t_1 - t_2$. T is shown in Figure 7.5 the region of integratio Figure 7.5-1a rotated c double integral in Equ

$$\frac{1}{4T}\iint_{\wp} R_{XX}(\tau)e^{-j\omega\tau}$$

$$= \frac{1}{4T}\left\{\int_{-2T}^{0} R_{XX}\right.$$

$$+\frac{1}{4T}\left\{\int_{0}^{2T} R_{XX}\right.$$

In the limit as T – thus

so that $S_{XX}(\omega)$ is real