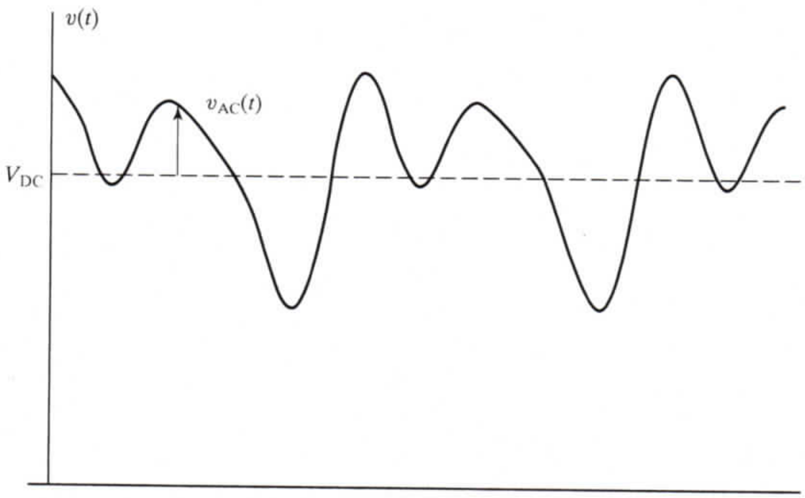

Figure 6.1.15 The definitions of the total signal, $v(t)$, the DC component of the signal, V_{DC}, and the AC component of the signal, $v_{AC}(t)$. Note that in this context "DC" does not mean "direct current" but rather "time-average value," and "AC" does not mean "alternating current" but rather "fluctuating."

For a periodic signal, such as in Fig. 6.1.15, the time average can be determined from one period,

$$\langle v(t)\rangle = \frac{1}{T}\int_0^T v(t)\,dt \tag{6.1.36}$$

but in general one has to average over all time:

$$\langle v(t)\rangle = \lim_{W\to\infty}\frac{1}{W}\int_{-\frac{W}{2}}^{+\frac{W}{2}} v(t)\,dt \tag{6.1.37}$$

where W is the width of a "window" centered on the origin, as shown in Fig. 6.1.16.

"Power" in volts squared. In signal analysis *power* means a measure of the square of the variable. In circuits, the true power would be the voltage squared divided by the impedance level of the circuit in ohms, all multiplied by the power factor if necessary. Here we will deal with power in voltage squared, with the understanding that for real power in watts the impedance level of the circuit must be considered.

The DC value and DC power. The DC value is the time average:

$$V_{DC} = \langle v(t)\rangle \tag{6.1.38}$$

and the DC power is the square of the DC value:

$$P_{DC} = V_{DC}^2 \tag{6.1.39}$$

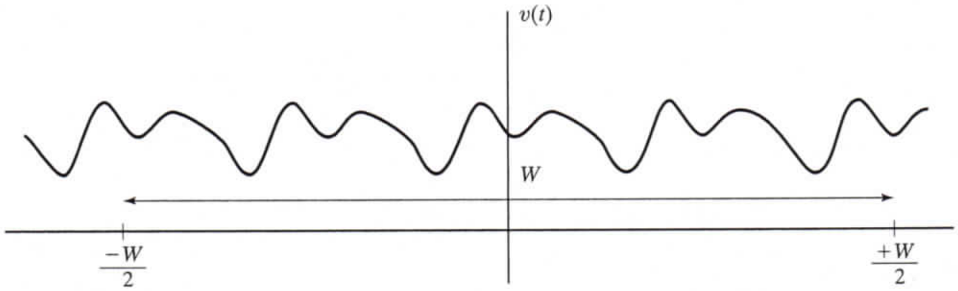

Figure 6.1.16 A "window" over which the function is averaged. The width of the window is allowed to go to infinity, $W \to \infty$, for the time average of the function. This definition of time average works for periodic and random functions.

The AC value and the AC power. The AC value is the total signal minus the DC value:

$$v_{AC}(t) = v(t) - V_{DC} \tag{6.1.40}$$

Note that the time average of the AC value is zero:

$$\langle v_{AC}(t)\rangle = \langle v(t) - V_{DC}\rangle = \langle v(t)\rangle - \langle V_{DC}\rangle = V_{DC} - V_{DC} = 0 \tag{6.1.41}$$

The time average distributes because averaging is a linear operation. The AC power is the time average of the square of the AC component of the signal:

$$P_{AC} = \langle v_{AC}^2(t)\rangle \tag{6.1.42}$$

The total power. The total power in volts squared is the time average of the square of the total voltage, which is also the sum of the DC and AC components, Eq. (6.1.40):

$$P_T = \langle v^2(t)\rangle = \langle (V_{DC} + v_{AC}(t))^2\rangle = \langle V_{DC}^2\rangle + \langle 2V_{DC}v_{AC}(t)\rangle + \langle v_{AC}^2(t)\rangle \tag{6.1.43}$$

The middle term in the expansion vanishes:

$$\langle 2V_{DC}v_{AC}(t)\rangle = 2V_{DC}\langle v_{AC}(t)\rangle = 2V_{DC} \times 0 = 0 \tag{6.1.44}$$

Thus Eq. (6.1.43) reduces to

$$P_T = P_{DC} + P_{AC} \tag{6.1.45}$$

Example 6.1.4: A square wave

Consider a periodic square wave having a period T and an amplitude V, as shown in endnote 6. Find the DC and AC power in this signal.

Solution The DC value is $\frac{V}{2}$, so the DC power is $P_{DC} = \left(\frac{V}{2}\right)^2 = \frac{V^2}{4}$. The AC signal, $v_{AC}(t) = v(t) - \frac{V}{2}$, is another square wave going from $+\frac{V}{2}$ to $-\frac{V}{2}$. Since $P_{AC} = \langle v_{AC}^2(t)\rangle = \left(\pm\frac{V}{2}\right)^2 = \frac{V^2}{4}$ also. Note that the total power, $P_T = \frac{V^2}{2}$, is the time average of the original square wave squared.

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Evaluate
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For the answer, see end

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You do it. If the square wave has the value V twice as long as it has the value 0, what now is the AC power? Let $V = 1$ V.

```
myanswer = ? ;
```

```
Evaluate
```

For the answer, see endnote 7.

The time-average autocorrelation function. Hitherto we have defined the autocorrelation function as a statistical average, an expectation, as follows:

$$R_V(\tau) = E[V(t)V(t+\tau)] \tag{6.1.46}$$

Thus the (statistical) autocorrelation function involves every member of the family of functions making up the random process. But a time-average autocorrelation function may be defined for a single function. This is true for all periodic deterministic functions, and it is true for each of the "random" functions making up a random process. We replace the expectation operator in Eq. (6.1.46) with the time-average operator for the time-average autocorrelation function:

$$\overline{R}_V(\tau) = \langle v(t)v(t+\tau)\rangle \tag{6.1.47}$$

In words, we average the function times itself shifted τ in the negative t direction. We used the overscore estimator symbol because for a WSS random process the time-average autocorrelation function serves as an estimator for the statistical autocorrelation function.

The time-average autocorrelation function is meaningful for both deterministic and random time functions. In endnote 6 we analyze the time-average autocorrelation function for the square wave used in Example 6.1.4.

Summary. In this section we gave definitions for the DC and AC components of a signal and of the total, DC, and AC power. These definitions apply to both deterministic and random signals. Power is expressed in volts squared.

The ergodic concept. We now have in mind two types of averaging: statistical averaging for the mean and autocorrelation function, and time averaging as described in the previous paragraphs and suggested in Fig. 6.1.17. The ergodic concept addresses the relationship between these two types of averages. In general, a WSS ergodic random process, $X(t)$, has the property that statistical averages are equal to time averages. For the mean

$$\mu_{X(t)} = E[X(t)] \quad \text{and} \quad X_{DC} = \langle X(t)\rangle, \text{ but } E[X(t)] = \langle X(t)\rangle \tag{6.1.48}$$

For the autocorrelation functions, we have the corresponding relation:

$$R_X(\tau) = E[X(t)X(t+\tau)] \quad \text{and} \quad \overline{R}_X(\tau) = \langle X(t)X(t+\tau)\rangle, \text{ but } R_X(\tau) = \overline{R}_X(\tau) \tag{6.1.49}$$

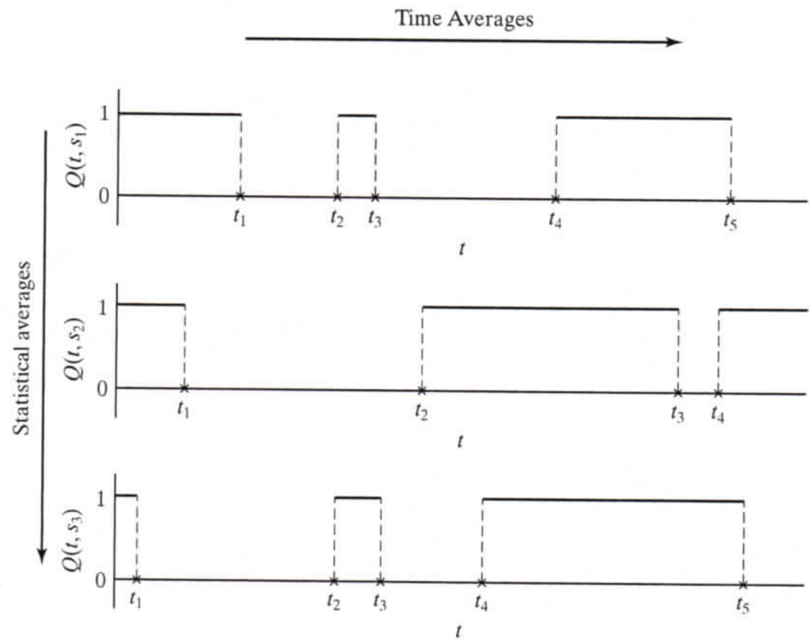

Figure 6.1.17 Time averages must be performed on a single member function of the random process. Statistical averages, the expectation, are performed on the entire random process. The ergodic property requires that the two averages be the same.

Equations (6.1.48) and (6.1.49) are a bit odd in that the statistical averages, $E[X(t)]$ and $E[X(t)X(t+\tau)]$, involve every member function in the random process, whereas the time average, $\langle X(t)\rangle$ and $\langle X(t)X(t+\tau)\rangle$, must of necessity be performed on only one of the member functions of the random process. For Eqs. (6.1.48) and (6.1.49) to be true, the randomness inherent in the entire random process must be present in, and fully represented by, every member function of the random process. The implications of this will be explored after we examine the ergodic property of the fully random sinusoid.

The time-average value and the time-average autocorrelation function of the fully random sinusoid. Because the random sinusoid is a periodic function, we may compute time averages by averaging over one period. Thus the DC value is

$$V_{\text{DC}} = \langle V(t,\theta_i)\rangle = \frac{1}{T}\int_0^T V_p \cos(\omega t + \theta_i)\, dt = 0 \tag{6.1.50}$$

Note in Eq. (6.1.50) that we have explicitly stated that the random variable, Θ, takes on a specific value, here called θ_i, for the time average. The time average in Eq. (6.1.50) is similar to the statistical average in Eq. (6.1.31), except that we are now averaging in time. Similarly, the result is zero because we are averaging a sinusoid over one full period:

$$V_{\mathrm{DC}} = 0 \tag{6.1.51}$$

The time-average autocorrelation function is

$$\overline{R}_V(\tau) = \langle V(t, \theta_i) V(t + \tau, \theta_i) \rangle = \frac{1}{T} \int_0^T V_p \cos(\omega t + \theta_i) V_p \cos(\omega(t + \tau) + \theta_i)\, dt \tag{6.1.52}$$

Again we use the trig identity for $\cos A \cos B$ (see endnote 4) with the resulting form

$$\overline{R}_V(\tau) = \frac{V_p^2}{2T} \int_0^T [\cos \omega\tau + \cos(\omega(2t + \tau) + 2\theta_i)]\, dt \tag{6.1.53}$$

As before, the second term integrates to zero, and the first is constant with respect to the variable of integration. The result is

$$\overline{R}_V(\tau) = \frac{V_p^2}{2T} \cos \omega\tau \int_0^T dt = \frac{V_p^2}{2} \cos \omega\tau \tag{6.1.54}$$

Comparing time and statistical averages. The time-average value in Eq. (6.1.51) is the same as the statistical expectation in Eq. (6.1.31), and the time-average autocorrelation function in Eq. (6.1.54) is the same as the statistical expectation in Eq. (6.1.35). We therefore conclude that the fully random sinusoid, which is our model for a random analog signal, is not only WSS but is also ergodic.

Why the ergodic property is important. The ergodic property is important to the engineering applications of random processes. Consider the following scenario. You are required to design a system to process a certain type of signal. The signal is an ongoing random signal that looks well behaved, so you take a lot of data and study their properties such as the mean and the autocorrelation function. To design your system, you need to generalize that model into a random process. In that generalization you are assuming the ergodic property. Once your system is designed, it needs to work on specific time signals. In effect, you again needed the ergodic property.

Proving that a random process has the ergodic property. The fully random sinusoid is a random process that is easily shown to be ergodic in mean and autocorrelation function. There are but a few random processes like that. For the rest, the ergodic property is assumed as a matter of convenience, or perhaps necessity. Ultimately, the justification for using the ergodic property is a working system, that is, a good piece of engineering.

Why the autocorrelation function is so important. When we spoke in Sec. 3.5 about correlation between random variables, we criticized the correlation and the covariance as difficult to interpret, and we focused on the correlation coefficient as the major way that the linear relationship between random variables is described. Yet, in dealing with random processes, we focus on the autocorrelation function, which is the correlation of the random process at one time with itself at another time. In this section we show why the autocorrelation function of a WSS random process contains important information about the random process in a concise form. Our purpose here is to show how to read this information from the autocorrelation function.

The definition of the autocorrelation function of a WSS random process, $X(t)$, is

$$R_X(\tau) = E[X(t)X(t+\tau)] \tag{6.1.55}$$

When the WSS random process is also ergodic, then the time-average autocorrelation function is the same:

$$R_X(\tau) = E[X(t)X(t+\tau)] = \langle x(t)x(t+\tau)\rangle \tag{6.1.56}$$

where $x(t)$ is a member function of the random process.

When $\tau = 0$. Setting $\tau = 0$ gives information about the total power in the random process. From Eq. (6.1.56) at $\tau = 0$, we have

$$R_X(0) = E[X^2(t)] = \langle x^2(t)\rangle = P_T \tag{6.1.57}$$

so the autocorrelation at $\tau = 0$ is the total power in the signal. Because $R_X(0)$ also is the mean-square value of the random process, and since the total power is the sum of the AC and DC powers, it follows that

$$R_X(0) = \sigma_X^2 + \mu_X^2 = P_{\text{AC}} + P_{\text{DC}} \tag{6.1.58}$$

But for the ergodic random process, the mean is the same as the time average, or DC value, of the member functions, so we assume for all WSS random process that the variance is the AC power and the square of the mean is the DC power:

$$\sigma_X^2 = P_{\text{AC}} \text{ and } \mu_X^2 = P_{\text{DC}} \tag{6.1.59}$$

When τ gets large. We consider now random processes that lose coherence as τ gets large. This would be true of any random process modeling a physical process without long-term memory. It is true, for example, in our models for synchronous and asynchronous digital signals; however, it is not true for the random sinusoid because this random process never becomes randomized in time. When we let τ get large in Eq. (6.1.55), $X(t)$ and $X(t+\tau)$ become uncorrelated, and the autocorrelation function must approach the square of the mean, as shown by Eq. (3.5.15), when the covariance is zero:[8]

$$\underset{\tau\to\infty}{R_X(\tau)\to} E[X(t)] \times E[X(t+\tau)] = \mu_X^2 = P_{\text{DC}} \tag{6.1.60}$$

From Eq. (6.1.60) it follows that the difference between the autocorrelation at $\tau = 0$ and at $\tau \to \infty$ is the variance, or the AC power:

$$R_X(0) - R_X(\infty) = \sigma_X^2 = P_{\text{AC}} \tag{6.1.61}$$

Thus we can readily derive the total power, the AC power, and the DC power from the autocorrelation function of a WSS ergodic random process.

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The coherence function. The correlation coefficient between two random variables, X and Y, is defined in Eq. (3.5.17) as

$$\rho_{XY} = \frac{E[XY] - \mu_X \mu_Y}{\sigma_X \sigma_Y} \tag{6.1.62}$$

Consider now the random process $X(t)$. If we let $X \rightarrow X(t)$ and $Y \rightarrow X(t+\tau)$, the result is

$$\rho_X(\tau) = \frac{E[X(t)X(t+\tau)] - \mu_X \mu_X}{\sigma_X \sigma_X} = \frac{R_X(\tau) - R_X(\infty)}{R_X(0) - R_X(\infty)} \tag{6.1.63}$$

The function $\rho_X(\tau)$ we will call the *coherence function* because it gives the correlation coefficient between the WSS random process at some time and itself at some increment of time τ later or earlier. Note that $\rho_X(0) = 1$.

The coherence time. A WSS random process that decorrelates as τ increases has a characteristic time during which it maintains a degree of coherence; or one could also say that after a period of time it decorrelates. We define a coherence time as that time in τ beyond which the correlation coefficient remains below 0.1. Thus the coherence time, τ_c, is defined as

$$|\rho_X(\tau)| \leq 0.1 \text{ for all } \tau > \tau_c \tag{6.1.64}$$

We could also call the coherence time the *decorrelation time,* since this is the time when the random process loses most of its self-correlation, or even *correlation time,* since this is the time period over which the random process retains a measure of correlation. We give examples later.

A standard form for the autocorrelation function. If we solve Eq. (6.1.63) for $R_X(\tau)$ and use Eqs. (6.1.61) and (6.1.60) we obtain the form

$$R_X(\tau) = \sigma_X^2 \rho_X(\tau) + \mu_X^2 = P_{\text{AC}} \rho_X(\tau) + P_{\text{DC}} \tag{6.1.65}$$

Thus the autocorrelation function contains information about the total power, the AC power, the DC power, and the coherence properties of the random process. We now apply Eq. (6.1.65) to the three random process models that we have analyzed in this section.

The asynchronous digital signal model. The model for the asynchronous random process, the fully random flip-flop, has an autocorrelation function of Eq. (6.1.66), repeated from Eq. (6.1.17),

$$R_Q(\tau) = \tfrac{1}{4}[1 + e^{-2\lambda|\tau|}] = \underbrace{\tfrac{1}{4}}_{P_{\text{AC}}} \underbrace{e^{-2\lambda|\tau|}}_{\rho_Q(\tau)} + \underbrace{\tfrac{1}{4}}_{P_{\text{DC}}} \tag{6.1.66}$$

lere the autocorrelation function is readily placed into the form of Eq. (6.1.65), and the various omponents are evident. Because $Q(t)$ varies randomly between 1 and 0, with equal probability, ie DC value is $\frac{1}{2}$, and hence the DC power is $\frac{1}{4}$. Because 1^2 is also 1, the total power also must e $\frac{1}{2}$, and hence the AC power must be $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. These values are identified in Eq. (6.1.66). he coherence function shows a steady loss of coherence due to the random clock pulses. The

2. *The noise is WSS.* This will be true provided the noise source is stable.
3. *The noise is ergodic.* The random properties of the random process are fully contained in any of the member functions.

The autocorrelation model. We assume an autocorrelation model in the form of Eq. (6.1.65)

$$R_X(\tau) = \sigma_X^2 \rho_X(\tau) + \mu_X^2 \tag{6.1.74}$$

In general, we have AC and DC power, represented by σ_X^2 and μ_X^2, respectively, and the coherence function, $\rho_X(\tau)$, is determined by the physical system producing the noise. These three components of the noise could be determined from a time-average autocorrelation function performed on one piece of data. We will give an example in Sec. 6.3.

The PDFs. Because the random process is Gaussian, we can state $X(t) = N(\mu_X, \sigma_X^2)$ at any time. Thus we can establish the probability that the signal exceeds a certain level at any time:

$$f_{X(t)}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] \quad \text{and} \quad P[X(t) > x] = 1 - \Phi\left(\frac{x-\mu_X}{\sigma_X}\right) \tag{6.1.75}$$

From the ergodic property we can also say these properties will be reflected in the time behavior of any member function. For example, the fraction of the time that $X(t)$ spends above x will also be $1 - \Phi\left(\frac{x-\mu_X}{\sigma_X}\right)$. Such analysis is useful in the design of signal-processing devices. Second Eq. (3.5.34), and higher-order PDFs, can also be written, with the coherence function playing the role of the correlation coefficient.

6.1.6 Relationships Between Two WSS Random Processes

In many contexts, two WSS random processes much be considered in the analysis of a random system. In a communication system, we may be dealing with one random process representing the signal and another representing the noise; or in a signal-processing system we may be dealing with one random process representing the input signal and another representing the output signal. In this section we discuss means for relating two such random processes, call them $X(t)$ and $Y(t)$. We assume them to be WSS random processes and jointly WSS, meaning that any relationship between them is also independent of absolute time in first- and second-order matters such as mean and power relationships.

Cross-correlation. The cross-correlations between two jointly WSS random processes, $X(t)$ and $Y(t)$, are defined as

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \quad \text{and} \quad R_{YX}(\tau) = E[Y(t)X(t+\tau)] \tag{6.1.76}$$

From the definitions in Eq. (6.1.76) and the time-invariant nature of the WSS random processes, the symmetry property is easily shown to be $R_{XY}(-\tau) = R_{YX}(\tau)$. If $X(t)$ and $Y(t)$ are independent random processes, then the cross-correlations reduce to the product of the means:

$$R_{XY}(\tau) = R_{YX}(\tau) = \mu_X \mu_Y, \text{ if } X(t) \text{ and } Y(t) \text{ are independent.} \tag{6.1.77}$$

A cross-covariance function between $X(t)$ and $Y(t)$ can be defined similar to Eq. (3.5.14). When $X(t)$ and $Y(t)$ are independent, the cross-covariance function is zero for all τ.

An application of cross-correlation as a signal-detection process was presented in the Venus radar experiment described in Chapter 1. There, you may recall, a random sequence of radar pulses was bounced off Venus, and the return signal was compared through cross-correlation with the transmitted sequence. The value of τ that gave the maximum return gave information about the round-trip travel time and hence on the distance to Venus. We will use the cross-correlation function in the analysis of linear systems in Sec. 6.3.

The sum of two jointly WSS random processes. Let $Z(t) = X(t) + Y(t)$ represent the sum of two random processes. The mean of $Z(t)$ is simply $\mu_Z = \mu_X + \mu_Y$, regardless of any relationship between $X(t)$ and $Y(t)$. The autocorrelation function of $Z(t)$ is

$$\begin{aligned} R_Z(\tau) &= E[(X(t) + Y(t))(X(t+\tau) + Y(t+\tau))] \\ &= E[X(t)X(t+\tau)] + E[X(t)Y(t+\tau)] + E[Y(t)X(t+\tau)] + E[Y(t)Y(t+\tau)] \quad (6.1.78) \\ &= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \end{aligned}$$

Thus the sum of $X(t)$ and $Y(t)$ is WSS and can be expressed in terms of the auto- and cross-correlation functions of $X(t)$ and $Y(t)$. If $X(t)$ and $Y(t)$ are independent, Eq. (6.1.78) reduces to

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) + 2\mu_X\mu_Y \quad (6.1.79)$$

which says that if either random process has zero mean, the autocorrelation function of the sum is the sum of the autocorrelation functions.

Example 6.1.5: Analog signal plus noise

Consider the sum of the random analog signal and zero-mean broadband Gaussian noise

$$Z(t) = V_p \cos(\omega_1 t + \Theta) + N(t) \quad (6.1.80)$$

where Θ is a random phase uniformly distributed between 0 and 2π, and $R_N(\tau) = \sigma_N^2 \rho_N(\tau)$. Find the autocorrelation function of $Z(t)$.

Solution According to Eq. (6.1.79), the autocorrelation of such a sum is the sum of the autocorrelation functions, plus a constant. The autocorrelation function of the random sinusoid is given in Eq. (6.1.35), and the autocorrelation function of the noise is given in the problem statement. The autocorrelation function of the sum is

$$R_Z(t) = \frac{V_p^2}{2}\cos(\omega_1 \tau) + \sigma_N^2 \rho_N(\tau) \quad (6.1.81)$$

In this case the constant is zero, since both random processes have zero mean. For $\tau \to$ large, $\rho_N(\tau) \to 0$, and the periodic analog signal will increasingly dominate the sum. Thus periodic signals can be detected in broadband noise by autocorrelation. Indeed, the cross-correlation between the random analog signal and $Z(t)$ would give the first term in Eq. (6.1.81) directly without the noise term.

For a sinusoid: right-hand path. The time-average autocorrelation function of a sinusoid is performed with the aid of the familiar trig identity (see endnote 4).

$$\begin{aligned}\overline{R}_V(\tau) &= \langle V_p \cos(\omega_1 t + \theta) V_p \cos(\omega_1 (t+\tau) + \theta)\rangle \\ &= \frac{V_p^2}{2}\langle \cos(\omega_1 \tau) + \cos(\omega_1(2\,t + \tau) + 2\,\theta)\rangle\end{aligned} \tag{6.2.30}$$

The first term is constant with respect to time and thus averages to itself. The second time is a pure sinusoid and averages to zero. The result is

$$\overline{R}_V(\tau) = \frac{V_p^2}{2}\cos(\omega_1 \tau) \tag{6.2.31}$$

This is the first box on the right in Fig. 6.2.17. We now take the Fourier transform of the autocorrelation function[13]:

$$S_V(f) = \int_{-\infty}^{+\infty} \frac{V_p^2}{2}\cos(\omega_1 \tau) e^{-j\omega\tau}\, d\tau = \frac{V_p^2}{4}\delta(f + f_1) + \frac{V_p^2}{4}\delta(f - f_1) \text{ volts}^2/\text{Hz} \tag{6.2.32}$$

Because we see that Eq. (6.2.32) agrees with (6.2.29), we conclude that *for a simple sinusoid* both paths lead to the power spectrum.

Generalizing. Because we are dealing with deterministic signals, we recall that we can express all deterministic signals as the sum or the integral of sinusoids. We reason, therefore, that if both paths work for a sinusoid, they must work for all deterministic signals. We work out the details for the square wave in endnote 15.

Back to the main plot. We began this section with a review of spectral analysis to fix concepts, refresh memories, and standardize vocabulary and notation. We return now to the announced subject of this section, the spectral analysis of random signals.

6.2.3 The Spectral Analysis of WSS Random Processes

No voltage spectra. Random processes do not have voltage spectra. Consider the following thought experiment. You have a source of a random signal. On Monday, you take a lot of data and analyze the spectrum in a high-performance computer. After considerable number crunching, the computer outputs a spectrum, both amplitude and phase. On Tuesday you do the same. When you compare the two spectra, you see that the amplitude spectrum is very similar, but the phase of the spectrum bears no resemblance to the previous one. Perhaps the computer had problems, so you repeat the procedure on Wednesday. Again, the amplitude spectrum is the same, but the phase of the spectrum is different from the first two.

What the computer is telling you is that the random signal has an amplitude spectrum but no stable phase spectrum. In effect, it is telling you to forget about voltage spectra and focus on power spectra. Thus the left-hand path from the signal to the power spectrum does not work for a random process, as pictured in Fig. 6.2.18.

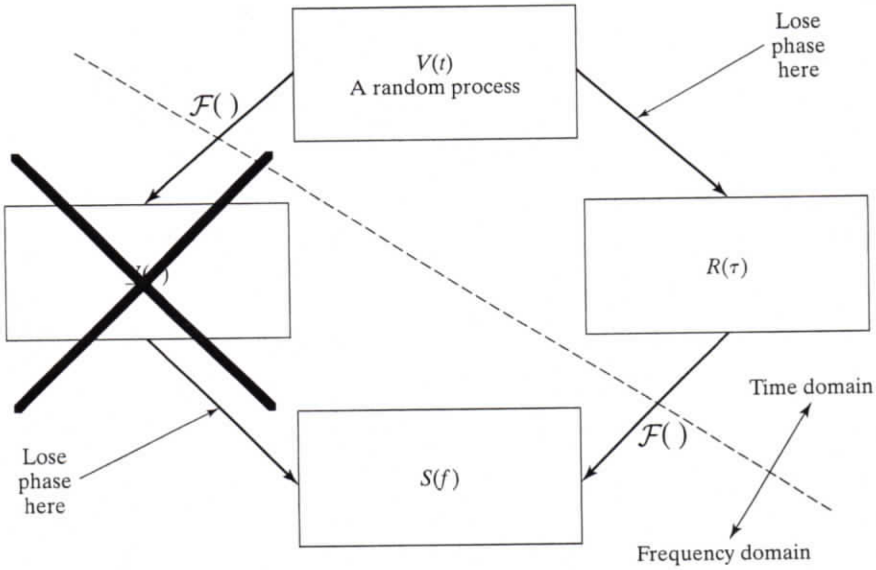

Figure 6.2.18 A random process has no voltage spectrum but does have a power spectrum. The only way to get the power spectrum is through the autocorrelation function.

The definition of power spectral density, PSD. Following the right-hand path, we define the PSD of a WSS random process as

$$S_V(\omega) = \int_{-\infty}^{+\infty} R_V(\tau) e^{-j\omega\tau}\, d\tau \ \text{volts}^2/\text{Hz} \tag{6.2.33}$$

where $R_V(\tau) = E[V(t)V(t+\tau)]$, $V(t)$ a WSS random process. The remainder of this section addresses the question, Does the definition of PSD in Eq. (6.2.33) make sense? We have led you to this definition by showing it to be plausible. We will continue to investigate its plausibility through a number of means, to increase your understanding of and intuition about the spectral analysis of random signals.

The inverse transform. Because Fourier transforms come in pairs, we get the inverse transform of Eq. (6.2.33) for free:

$$R_V(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_V(\omega) e^{+j\omega\tau}\, d\omega \ \text{volts}^2 \tag{6.2.34}$$

Thus if we know the PSD of a random process, we can derive the autocorrelation function by Eq. (6.2.34).

Ergodic random processes. Recall that an ergodic random process is one in which statistical averages are equal to time averages. Because the right-hand path in Fig. 6.2.18 works for all time functions, it must therefore work for every member of an ergodic random process.

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Properties of the P

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Therefore it works for the random process itself, and we conclude that the definition of PSD for an ergodic random process is valid; however, since the ergodic property is difficult to prove, but is usually assumed for convenience, we continue to investigate Eq. (6.2.33) for all WSS random processes.

Properties of the PSD

1. *Total power.* If we set $\tau = 0$ in Eq. (6.2.34), we have

$$R_V(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_V(\omega)\,d\omega = \int_{-\infty}^{+\infty} S(f)\,df \text{ volts}^2 \tag{6.2.35}$$

But $R_V(0) = E\left[V^2(t)\right] = \sigma_V^2 + \mu_V^2$, which is associated with the total power in the random process. For an ergodic random process, this *is* the total time-average power in every member function in the random process and, according to Eq. (6.2.35), is equal to the integral of the two-sided power spectral density over all frequencies. This result is equivalent to Parseval's theorem, which expresses that power is conserved between the time and frequency domains.

2. *Even symmetry.* Using the Euler identity, $e^{-j\omega\tau} = \cos\omega\tau - j\sin\omega\tau$, we can express Eq. (6.2.33) as

$$S_V(\omega) = \int_{-\infty}^{+\infty} R_V(\tau)(\cos\omega\tau - j\sin\omega\tau)d\tau = 2\int_{0}^{+\infty} R_V(\tau)\cos\omega\tau\,d\tau \text{ volts}^2/\text{Hz} \tag{6.2.36}$$

Recalling that the autocorrelation function, $R_V(\tau)$, is an even function of τ, we see that the $R_V(\tau)\cos\omega\tau$ term in Eq. (6.2.36) is even, and the $R_V(\tau)\sin\omega\tau$ term is odd. The odd term drops out when integrated from $-\infty$ to $+\infty$. Also, we integrate the even term in the second form of Eq. (6.2.36) from 0 to $+\infty$ and double the result. From the second form of Eq. (6.2.36) we see that the PSD is an even function of ω: $S_V(-\omega) = S_V(+\omega)$.

3. *Nonnegative.* It can be shown that $S_V(\omega) \geq 0$ for all ω. This mathematical property accords with our physical interpretation of $S_V(\omega)$ as a power spectrum. We conclude that not all mathematical functions can be autocorrelation functions; only even functions that have Fourier transforms that are nonnegative everywhere can be autocorrelation functions.

Cross-power spectra. We defined the cross-correlation function between two jointly WSS random processes, $X(t)$ and $Y(t)$, as $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$ and $R_{YX}(\tau) = E[Y(t)X(t+\tau)]$. The cross-power spectra are defined as

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)e^{-j\omega\tau}\,d\tau \quad \text{and} \quad S_{YX}(\omega) = \int_{-\infty}^{+\infty} R_{YX}(\tau)e^{-j\omega\tau}\,d\tau \tag{6.2.37}$$

The symmetry properties of the cross-power spectra are readily shown to be

$$S_{XY}(-\omega) = S_{XY}^*(+\omega) = S_{YX}(+\omega) \tag{6.2.38}$$

Cross-power spectra can provide a means for measuring the frequency response of a system that does not admit of direct measurement. For example, the frequency response of the ionosphere/atmosphere system can be measured using naturally occurring radiation as a signal.

In Sec. 6.1 we presented and investigated four WSS random processes that model four important random signals. We now extend that presentation to include the frequency domain. Our purpose is to demonstrate the plausibility of the PSD concept and to give insight into the properties of these models.

Model of a random asynchronous digital signal

The model. We modeled a random asynchronous digital signal (Sec. 6.1.1) by triggering a toggle flip-flop with a Poisson process of average rate λ. The results were that the model was WSS and that the autocorrelation function was of the form Eq. (6.1.17),

$$R_Q(\tau) = \tfrac{1}{4}e^{-2\lambda|\tau|} + \tfrac{1}{4} \qquad (6.2.39)$$

Although we did not show that this random process is ergodic, we pointed out that the total power, AC power, and DC power were reasonable for a digital signal that spends half its time in the 1 state and half in the 0 state.

The PSD. Using a common Fourier transform pair,[13] we find the PSD of the model to be

$$S(f) = \frac{1}{4}\delta(f) + \frac{\lambda}{(2\lambda)^2 + (2\pi f)^2} \text{ volts}^2/\text{Hz} \qquad (6.2.40)$$

This spectrum is plotted in Fig. 6.2.19.

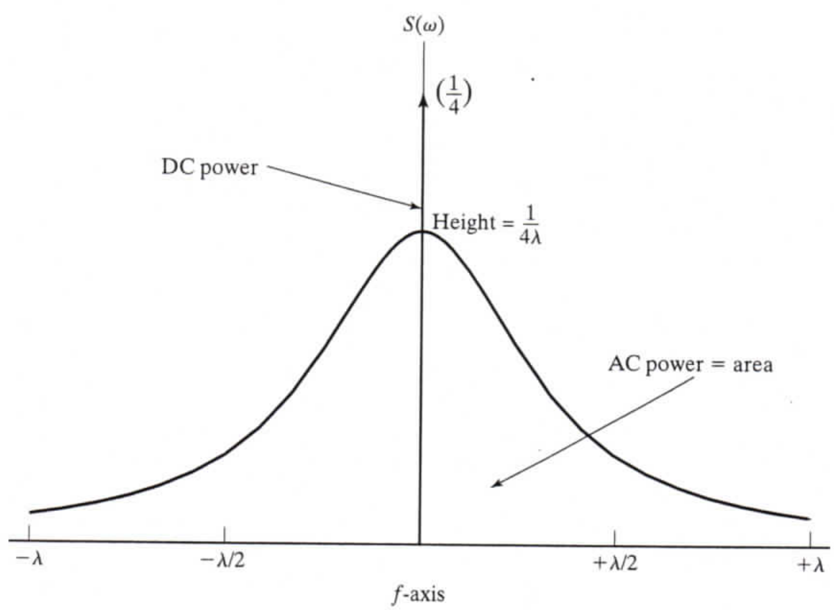

Figure 6.2.19 The spectrum of the asynchronous digital signal. The DC power is represented by an impulse at $f = 0$. The AC power is spread over a continuous spectrum.

The spectrum shows of frequencies.

The bandwidth. arbitrarily we define the 90% of the AC power. T

power in B Hz = 2

We were able to integrate integrand. We also chang that as $B \to \infty$, the total power in the time domair

Let us put in some numb the input to a toggle flip-f width of 2 milliseconds a need 2000 Hz to pass 90% frequency are therefore n are some very short pulse

Bandwidth and co relationship between the b

As expected, the usual in usual. This occurs becaus bandwidth, and the long p

Example 6.2.1: Photod A system is designed to c photon and has an averag is required at the input an

Solution For the input $\frac{1}{0.1\times10^{-6}}$ = 10 MHz. For 100 kHz.

You do it. For suc response time of the photo to the recorded count? Ent