6

Modeling Random Signals

6.1 WIDE-SENSE STATIONARY RANDOM PROCESSES

The need for models of random signals. Engineers deal with a wide variety of random signals. In Chapter 5 we studied the Poisson process, which consists of a series of events at random times. We introduced the concept of a random process to model such phenomena with our probability macromodel. We also used random processes in the section on reliability as a model to describe the life history of a large number of manufactured items. Random processes also were used in describing the behavior of queues.

In this chapter we deal with an important class of random processes called *wide-sense stationary* (WSS) random processes, which are used to model random signals such as

- *Information-bearing signals*. Engineers design, analyze, and use a variety of systems that generate, store, transmit, and process random signals. Models are required to establish design parameters such as the required bandwidth for such signals.
- *Noise*. One enemy of information is noise, which is also random. Although many sources of noise, such as man-made electrical noise on a power line, are not well modeled by WSS random processes, many natural forms of noise are. Examples are resistor noise, antenna noise from the atmosphere and extragalactic sources, and various forms of noise in electronic devices due to the discrete nature of electricity.

The definition of a random process. Although we focus in this chapter on WSS random processes, we begin by reviewing the definition of a random process generally and giving two examples. Definition: a *random process,* $X(t)$, is a family of functions that are associated with the outcomes of a chance experiment. A fuller notation would be $X(t, s_i)$, where t is an independent variable associated with time, and s_i represents the outcomes of a chance experiment.[1] The value of $X(t, s_i)$ can be discrete or continuous, and the time variable likewise can be discrete or continuous.

Example 6.1.1: A random process with two members

Let us assume a chance experiment with two outcomes, s_1 and s_2, where the probability of the elementary event $S_1 = \{s_1\}$ is $P[S_1] = p$, and $P[S_2] = 1 - p = q$. This defines the chance experiment. The random process is defined as $X(t, s_1) = 12\cos(12t)$ and $X(t, s_2) = 12\sin(12t)$. Figure 6.1.1 depicts this random process.

At any time, the random process is a random variable described by a PMF. For example, at $t = 0$, the PMF would be

$$P_{X(0)}(12) = p, \quad P_{X(0)}(0) = q, \quad \text{zow} \tag{6.1.1}$$

This random process is continuous in time but discrete in amplitude, since at any time only two values of $X(t, s_i)$ are possible.

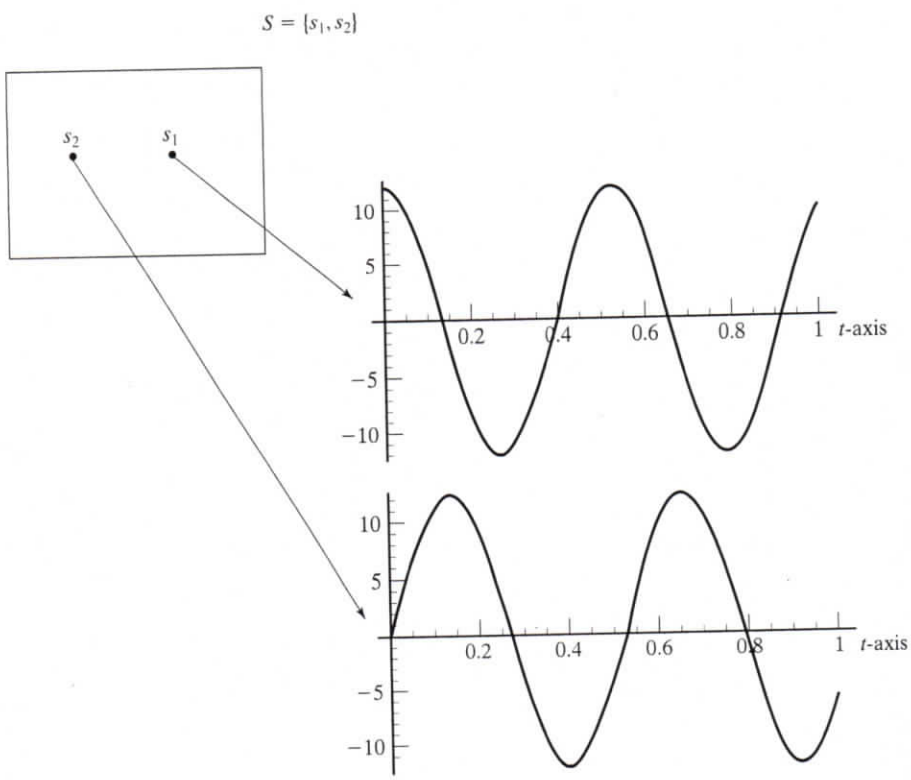


Figure 6.1.1 A random process with two members, a sine and a cosine function. The probability of having the cosine is p, and the probability of having the sine is q.

A random process with an infinite number of members. Consider a chance experiment that produces an exponential random variable, T, where $f_T(t) = 2e^{-2t}$, $t \geq 0$, zow. The random process is defined as

$$X(t) = 1 - \frac{t}{T}, \quad 0 < t \leq T, \text{ zow} \tag{6.1.2}$$

Two of the infinite number of members of this random process are shown in Fig. 6.1.2. The random process in Fig. 6.1.2 is continuous both in time and in amplitude.

Analog and digital signals. In Sec. 6.1 we present WSS models for random analog signals, synchronous and asynchronous digital signals, and the wideband noise inherent to many information systems. We begin with the asynchronous digital signal, which provides a good example of the basic concepts and methods for describing WSS random processes.

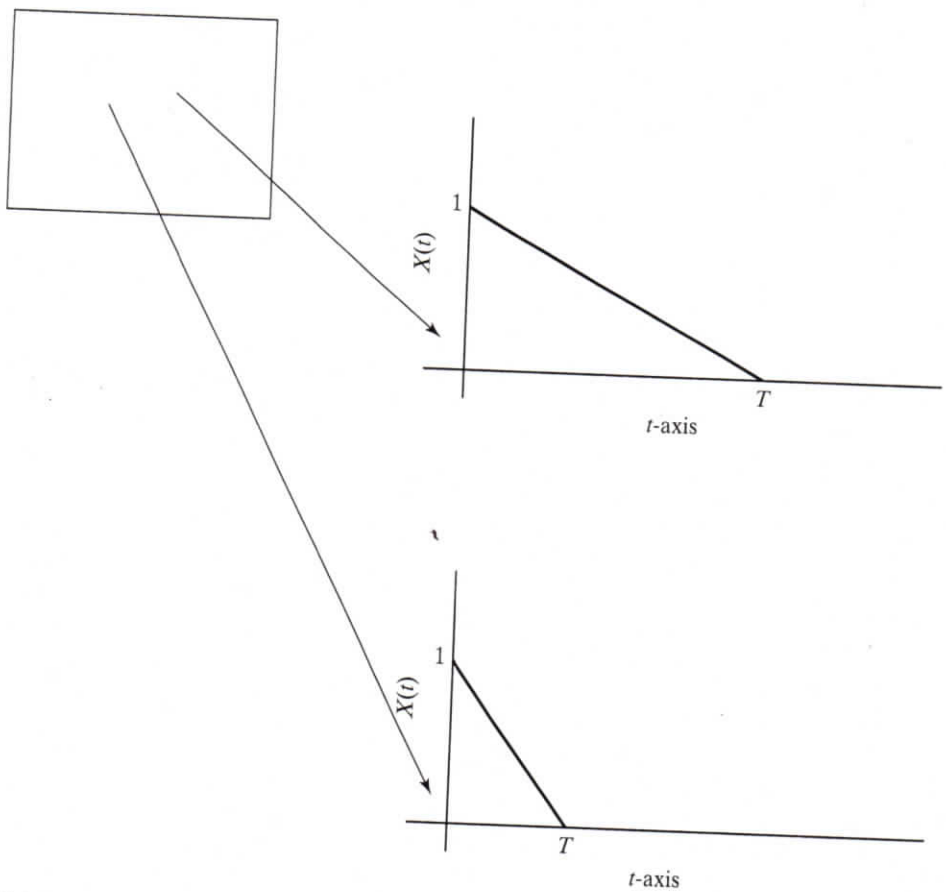


Figure 6.1.2 Two members of a random process that has an infinite number of members. The random variable, T, is exponentially distributed and thus can take any value between 0 and ∞.

6.1.1 Modeling an Asynchronous Digital Signal

Basic model. We will use a Poisson process as the underlying model for an asynchronous digital signal. The random-process model is based on the Poisson process shown in Fig. 6.1.3.

We model an asynchronous digital signal by using a Poisson point process as the clock input to a toggle flip-flop (FF). The output logic levels are assumed to be 0 and 1 V and the output is preset to 1 at $t = 0$. The circuit is shown in Fig. 6.1.4.

We call this a *semirandom flip-flop* because we have preset the output state to 1. Later, we will eliminate the initial condition to have a fully random FF. The output of the FF for the input sequence in Fig. 6.1.3 is shown in Fig. 6.1.5.

Describing the random processes with PMFs or PDFs. We now describe the state of the random process with a probability mass function, PMF. The output is discrete, $Q(t) = 0$ or 1, and hence a PMF is required. If the output were an analog signal, we would use a PDF.

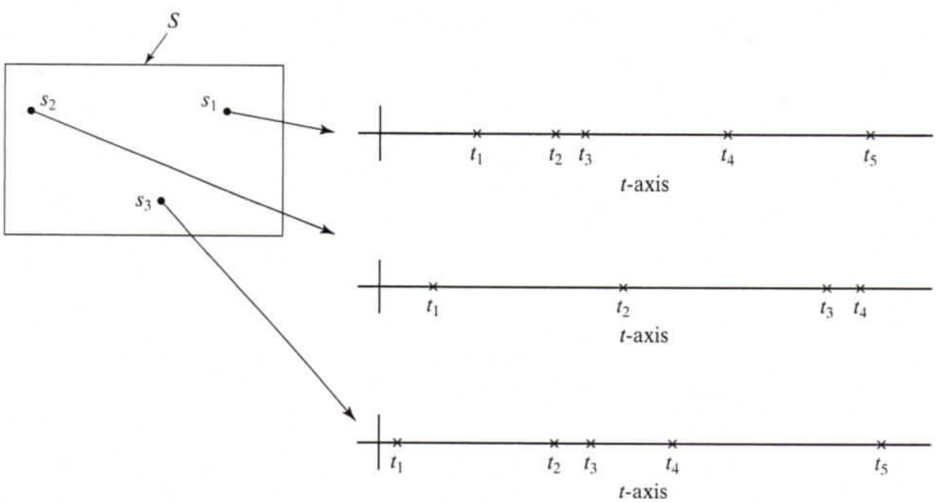


Figure 6.1.3 The Poisson point random process consists of a random sequence of events for each outcome of the chance experiment. We use this process to model an asynchronous digital signal.

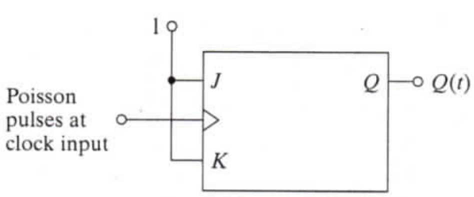
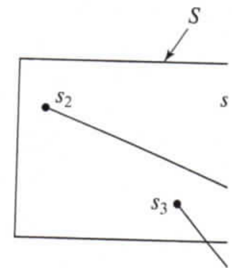


Figure 6.1.4 The output state of the toggle flip-flop will change at every Poisson event at the clock input. Our initial analysis assumes that $Q(0) = 1$ because the output is preset to 1.



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First-order PMF.

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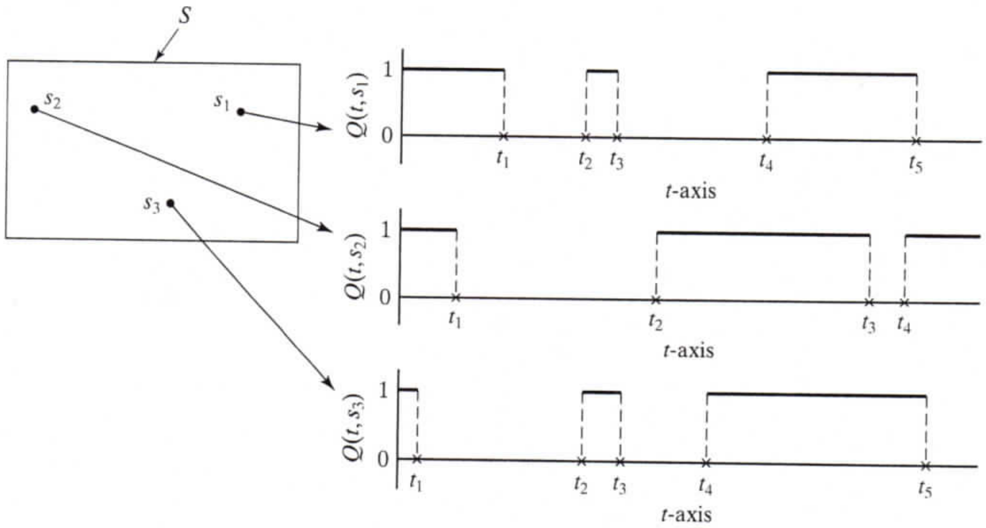

Figure 6.1.5 The output random process is generated by the input random process of Poisson points in Fig. 6.1.3. Note that all output signals begin with $Q(0) = 1$ because we preset the output to 1.

First-order PMF. The first-order PMF gives the output PMF at some time t. By definition,

$$P_{Q(t)}(q) = P[Q(t) = q] \tag{6.1.3}$$

which takes on values at $q = 0$ and $q = 1$ only. Thus $Q(t)$ is a Bernoulli random variable [(Fig. 2.2.6)]. The condition for $Q(t) = 1$ is that there is an even number of clock events between 0 and t. The probability of this event can be calculated from the Poisson process with an average rate of λ as follows:

$$P_{Q(t)}(1) = P[K \text{ even}] = P[K = 0] + P[K = 2] + P[K = 4] + \cdots \tag{6.1.4}$$

We now substitute the Poisson probabilities of Eq. (5.1.12) and obtain the following series:

$$P_{Q(t)}(1) = e^{-\lambda t} + \frac{(\lambda t)^2}{2!}e^{-\lambda t} + \frac{(\lambda t)^4}{4!}e^{-\lambda t} + \cdots = e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] \tag{6.1.5}$$

The series is unfamiliar, but from the standard power-series expansion of the exponential, given in Eq. (5.1.17), we derive the identity

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \frac{e^{+x} + e^{-x}}{2} \tag{6.1.6}$$

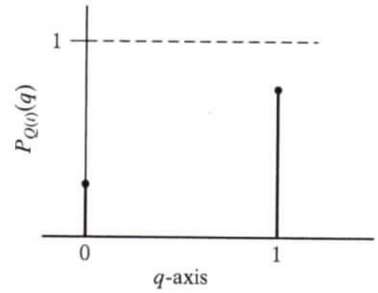

Figure 6.1.6 The PMF of $Q(t)$ at a time when 1 is still favored over 0. As time increases the probabilities become equal.

Using this identity in Eq. (6.1.5) we obtain

$$P_{Q(t)}(1) = e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right] = e^{-\lambda t}\left[\frac{e^{+\lambda t} + e^{-\lambda t}}{2}\right] = \frac{1}{2}[1 + e^{-2\lambda t}] \qquad (6.1.7)$$

Equation (6.1.7) gives the result we seek and is easily interpreted in terms of the FF output. Note that for $t = 0$, we have $P_{Q(t)}(1) = 1$, which must be true because we preset the FF to 1. With time increasing, the probability that the output is in the 1 state approaches $P_{Q(t)}(1) \to \frac{1}{2}$, which must be true, since the clock events are random and eventually randomize the output.

The derivation of $P_{Q(t)}(0)$ is similar and leads to a similar result:

$$P_{Q(t)}(0) = \tfrac{1}{2}[1 - e^{-2\lambda t}] \qquad (6.1.8)$$

Note that $P_{Q(t)}(1) + P_{Q(t)}(0) = 1$. As stated earlier, $Q(t)$ is a Bernoulli random variable, so the PMF is that shown in Fig. 6.1.6. Figure 6.1.7 shows the PMF of the output state as a function of time.

Figure 6.1.7 confirms what we would expect from the way $Q(t)$ is generated. Initially, the output is preset to 1 on all members of the random process. As the Poisson pulses arrive at the clock input, either earlier or later in the various realizations of the random function, more and more of the outputs change states, and with time the output states become fully randomized.

Second-order PMF. We use the results obtained in Eq. (6.1.7) to derive the second-order PMF of $Q(t)$. By definition, this is

$$P_{Q(t_1)Q(t_2)}(q_1, q_2) = P[(Q(t_1) = q_1) \cap (Q(t_2) = q_2)] \qquad (6.1.9)$$

Because $Q(t_1)$ and $Q(t_2)$ are not independent, we consider the four possibilities for $q_1q_2 = 00, 01, 10, 11$. We will work out the last case because we need it later. The other cases are similar. We pick two times, t_1 and t_2, with $t_2 > t_1$, and determine the probability that the output state is 1 at both times, as shown in Fig. 6.1.8.

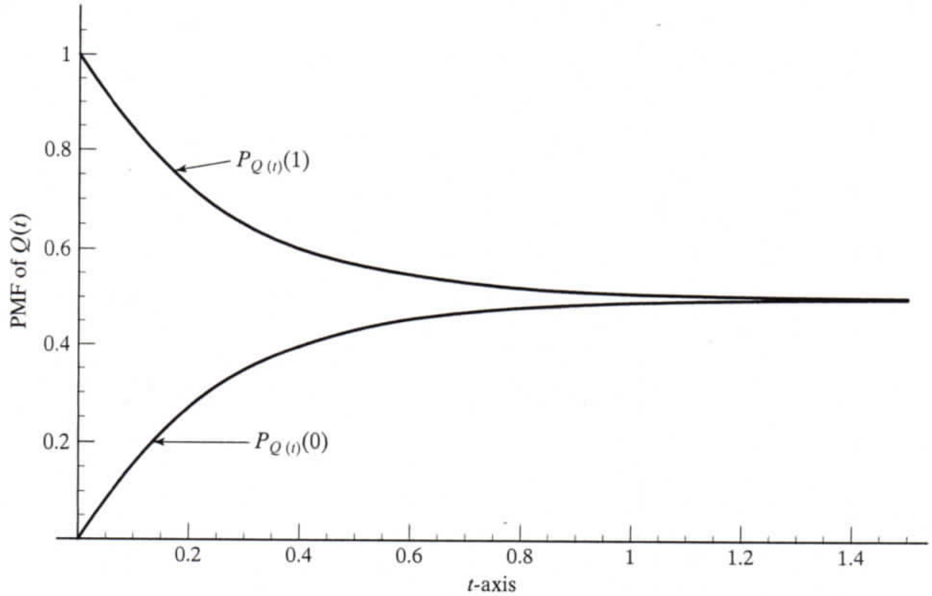

Figure 6.1.7 The PMF of the FF output, $Q(t)$, as a function of time. Initially the output is sure to be 1, but with time the output is randomized by the random input pulses, and eventually 1 and 0 become equally likely. For this part, $\lambda = 2$.

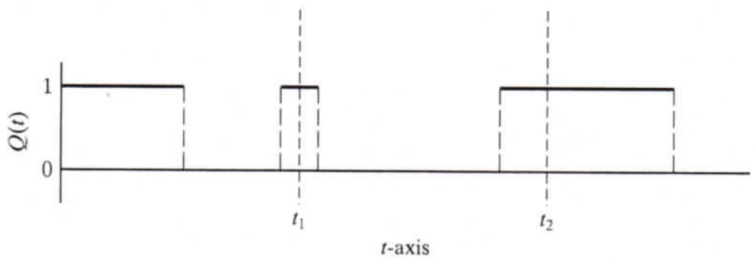

Figure 6.1.8 We pick two times and calculate the probability that $Q(t)$ is 1 at both times, as shown here. The required condition is that there be an even number of transitions between the origin and t_1 and also an even number of transitions between t_1 and t_2.

The probability we will calculate is expressed in Eq. (6.1.10) in terms of a conditional probability:

$$P_{Q(t_1)Q(t_2)}(1, 1) = P[(Q(t_1) = 1) \cap (Q(t_2) = 1)] = P[Q(t_2) = 1 | Q(t_1) = 1] \times P[Q(t_1) = 1] \tag{6.1.10}$$

The conditional probability in Eq. (6.1.10) is the probability of an even number of transitions between t_1 and t_2, which is essentially what we derived in Eqs. (6.1.4)–(6.1.7). Adapting

Eq. (6.1.7), we have

$$P[Q(t_2) = 1 | Q(t_1) = 1] = \tfrac{1}{2}[1 + e^{-2\lambda(t_2 - t_1)}] \tag{6.1.11}$$

The second term in the second form of Eq. (6.1.10) is exactly what we derived in Eq. (6.1.7), so we obtain

$$P_{Q(t_1)Q(t_2)}(1, 1) = \tfrac{1}{2}[1 + e^{-2\lambda(t_2 - t_1)}] \times \tfrac{1}{2}[1 + e^{-2\lambda t_1}] \tag{6.1.12}$$

We have thus determined the second-order PMF of the output of the semirandom flip-flop for one of four possible states. If needed, the others can be derived similarly.

PMFs or expectations? The ultimate and complete description of this random process would be the PMFs to any order desired. Such detail is not required for many basic theoretical and practical applications. Expectations, which represent averages, give much less information about a random process, but the information given is often adequate to design systems for processing a signal. We turn, therefore, to the first- and second-order expectations, which are the mean and the autocorrelation function of $Q(t)$.

The mean of *Q*(*t*). By mean, we do not mean the time average but the statistical mean. Look back at Fig. 6.1.5, and consider that you have a vertical line at some time t. The intersection of that line and the random process $Q(t)$ is either 1 or 0 for the individual functions in the random process. The average of those 1s and 0s would be the statistical mean at that time. The mean of a Bernoulli random variable is easily calculated from Eqs. (6.1.7) and (6.1.8) as

$$\begin{aligned}\mu_{Q(t)} &= E[Q(t)] = 0 \times P_{Q(t)}(0) + 1 \times P_{Q(t)}(1) = 0 \times \tfrac{1}{2}[1 - e^{-2\lambda t}] + 1 \times \tfrac{1}{2}[1 + e^{-2\lambda t}] \\ &= \tfrac{1}{2}[1 + e^{-2\lambda t}]\end{aligned} \tag{6.1.13}$$

This result looks like the top curve in Fig. 6.1.7, and this makes sense. Because the FF was preset to 1, the mean should start out at 1, but with time the mean should approach $\frac{1}{2}$ because the output becomes randomized by the Poisson clock pulses.

The autocorrelation function. The autocorrelation function is defined as

$$R_Q(t_1, t_2) = E[Q(t_1)Q(t_2)] \tag{6.1.14}$$

For discrete bivariate random variables, this function is

$$\begin{aligned}R_Q(t_1, t_2) = E[Q(t_1)Q(t_2)] &= \sum_{\text{all states}}\sum q_1 q_2 P_{Q(t_1)Q(t_2)}(q_1, q_2) \\ &= 0 \times 0 \times P_{Q(t_1)Q(t_2)}(0, 0) \\ &\quad + 0 \times 1 \times P_{Q(t_1)Q(t_2)}(0, 1) \\ &\quad + 1 \times 0 \times P_{Q(t_1)Q(t_2)}(1, 0) \\ &\quad + 1 \times 1 \times P_{Q(t_1)Q(t_2)}(1, 1)\end{aligned} \tag{6.1.15}$$

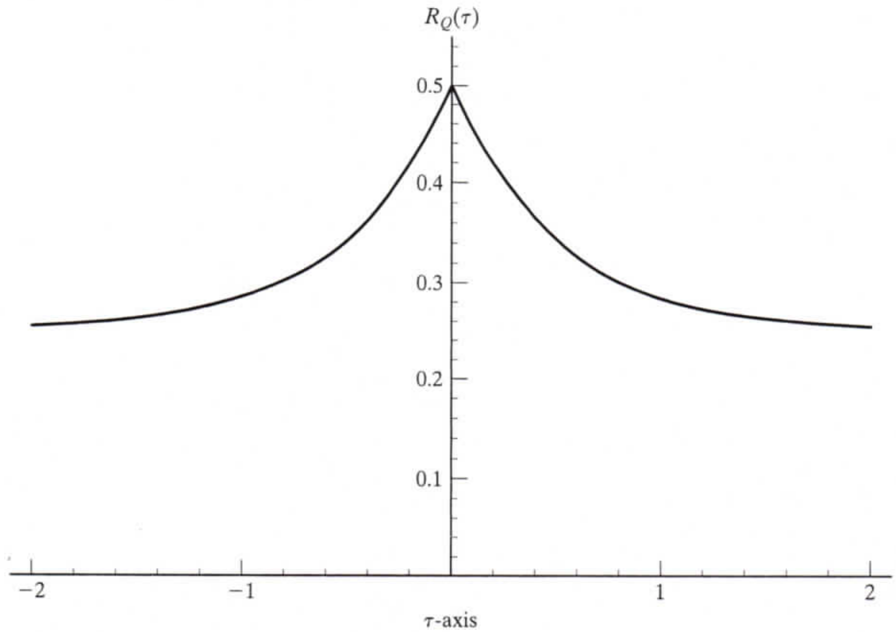

Figure 6.1.9 The autocorrelation function of the output state of the random flip-flop.

Clearly, the only term in the sum that contributes is the last, which requires the probability we calculated in Eq. (6.1.12). The autocorrelation function is

$$R_Q(t_1, t_2) = 1 \times 1 \times \tfrac{1}{2}[1 + e^{-2\lambda(t_2 - t_1)}] \times \tfrac{1}{2}[1 + e^{-2\lambda t_1}] \tag{6.1.16}$$

The result in Eq. (6.1.16) is hard to interpret because it has an early period before the system randomizes. That is why we call this the "semirandom" FF case. We now remove that preset condition to have a fully random FF. We may do this in two ways: we can randomize the initial output state, or we can examine the results far away from the initial time. The latter is easier, so we make the following changes in Eq. (6.1.16)

- Let $t_1 \to$ large.
- Keep $t_2 - t_1 = \text{constant} = \tau$.
- Realize that the expectation is independent of the sign of $t_2 - t_1 = \tau$.

With these changes Eq. (6.1.16) becomes

$$R_Q(\tau) = E[Q(t)Q(t + \tau)] = \tfrac{1}{4}[1 + e^{-2\lambda|\tau|}] \tag{6.1.17}$$

A plot of Eq. (6.1.17) is shown in Fig. 6.1.9, for $\lambda = 1$.

The autocorrelation is now a function only of time difference, τ. Absolute time does not matter because we have moved far from the time origin, where we preset to 1. Note that under these conditions, the mean is a constant, Eq. (6.1.13).

Example 6.1.2: Micrometeorite counter

A satellite is equipped with a micrometeorite counter. The counter records, on average, 3000 events per day. The input stage of the digital counter changes state for each input. Assuming voltage levels of 0 and 2 voltage, what is the value of the autocorrelation function for that output for a time difference of 15 seconds.

Solution The average rate would be $\lambda = 3000 \times \frac{1}{24} \times \frac{1}{60} = 2.083 \ \frac{\text{events}}{\text{minute}}$. The autocorrelation function for the output is given in Eq. (6.1.17) for logic levels of 0 and 1 V. For logic levels of 0 and 2 V, the autocorrelation function will increase by a factor of 2^2, since $Q(t)$ is multiplied by itself shifted in time. Thus at $\tau = 0.25$ minute, the autocorrelation function has the value $R_Q(0.25) = 2^2 \times \frac{1}{4}(1 + e^{-2\times 2.083\times 0.25}) = 1.353\ \text{V}^2$.

Definition of wide-sense stationary (WSS) random processes. We now are in a position to define WSS random processes. Let $X(t)$ represent a random process. The definition of WSS is that $X(t)$ satisfies two criteria:

1. The mean is constant, $\mu_X = E[X(t)] = \text{constant}$.
2. The autocorrelation is a function of magnitude of time difference only:[2] $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = f(|t_2 - t_1|)$. Another notation for this expression is that $R_X(\tau) = E[X(t)X(t + \tau)] = f(|\tau|)$, where $f(\cdot)$ is an appropriate function.

Interpretation of WSS random processes. We may interpret WSS random processes as those random processes that look statistically the same at all times, at least as concerns the first- and second-order effects. As we shall see, this means that the total power in the process is constant, and the split between DC and AC power is constant. Thus the DC power and the AC power are constant. This "power" interpretation will be explored later.

The fully random FF random process is WSS. Its mean, Eq. (6.1.13), is constant, $\mu_{Q(t)} \to \frac{1}{2}$ as $\tau \to$ large, and its autocorrelation function is a function of $|\tau|$ only, Eq. (6.1.17). Once we get far away from the time of initialization all that matters is differences in time; absolute time does not matter.

Summary and look ahead. We now have investigated a model for an asynchronous digital signal. Our model is a digital signal that changes states randomly, in accordance with a Poisson process. Some physical realizations of such a signal would be the first stage of a counter monitoring radioactive decay events or monitoring the passage of automobiles on a highway. Our results confirm the WSS nature of the signal. We next develop a model for a synchronous, or clocked, digital signal, then for a random analog signal, and finally for a random noise signal. These remaining sections will be much briefer.

6.1.2 Modeling a Synchronous Digital Signal

The model. Our probability model for a synchronous (clocked) digital signal is shown in Fig. 6.1.10. The probability model shown in Fig. 6.1.10 is random in two ways. The signal is 1 or 0 V in each clock period, with equal probability. The other random aspect is the delay to the beginning of the first full clock period from the arbitrary time origin. This we represent as D in Fig. 6.1.11.

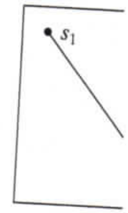

Figure 6.1.10 A probab each period contains a 1 lead to all such sequence but have no synchronism

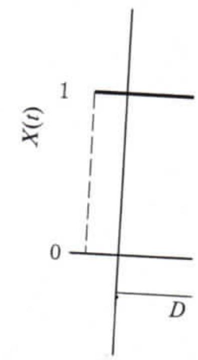

Figure 6.1.11 Definition c clock period, *D*, from the a equal probability.

The period, T, in Fig. 6 is uniformly distributed bet

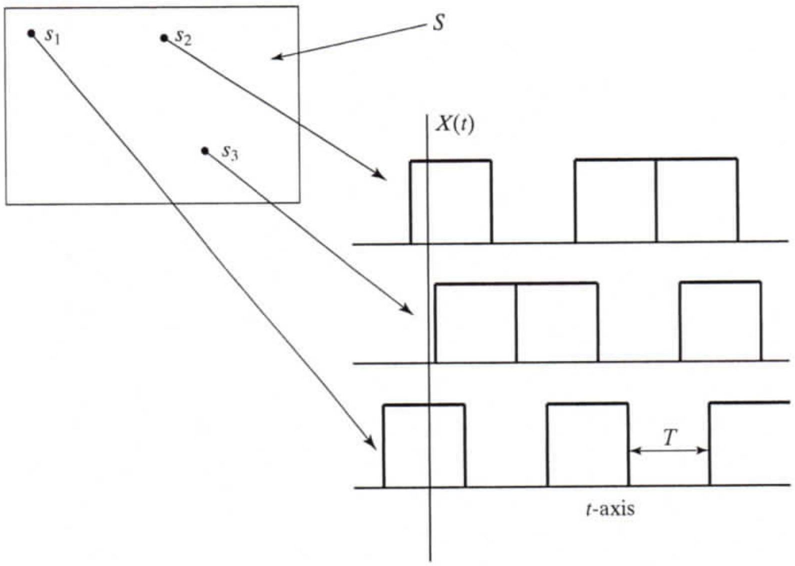

Figure 6.1.10 A probability model for a synchronous digital signal. The clock period is *T*, and each period contains a 1 or 0 with equal probability. The outcomes of the chance experiment lead to all such sequences. Note that the digital functions are synchronous with themselves but have no synchronism with each other. There is therefore no absolute time origin.

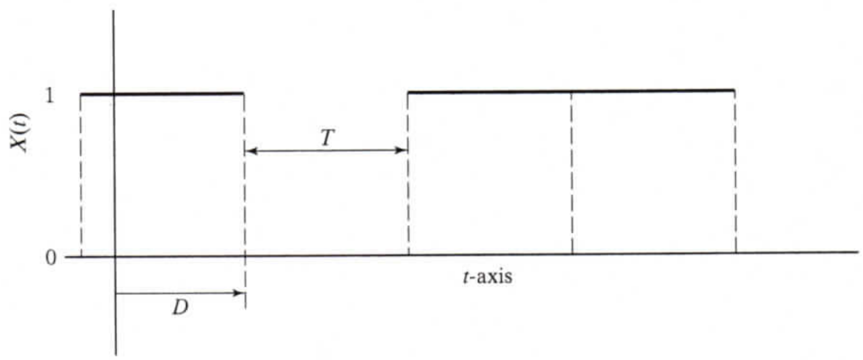

Figure 6.1.11 Definition of the clock period, *T*, and the delay to the beginning of the first full clock period, *D*, from the arbitrary time origin. In each clock period, the signal is 1 or 0 with equal probability.

The period, T, in Fig. 6.1.11 is a known constant, but the delay, D, is a random variable that is uniformly distributed between 0 and T:[3]

$$f_D(d) = \frac{1}{T}, \;\; 0 < d \le T, \;\; \text{zow} \tag{6.1.18}$$

The mean. Again, the random process is 1 or 0 at any time, and we can easily calculate the mean:

$$\mu_{X(t)} = E[X(t)] = 0 \times P[X(t)=0] + 1 \times \underbrace{P[X(t)=1]}_{1/2} = \tfrac{1}{2} \text{ volt} \tag{6.1.19}$$

Thus the mean is $\frac{1}{2}$ V at all times because the signal is equally likely to be 1 or 0.

The autocorrelation function. The autocorrelation function is defined as

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] \tag{6.1.20}$$

Because $X(t)$ is binary in nature the expectation is similar to Eq. (6.1.15), and the only term that contributes is

$$R_X(t_1, t_2) = 1 \times 1 \times P[(X(t_1)=1) \cap (X(t_2)=1)] \tag{6.1.21}$$

There are two cases to consider.

Case 1: If $t_2 - t_1 > T$, then at least one clock transition between t_1 and t_2 is sure to occur, and the values at $X(t_1)$ and $X(t_2)$ are independent and equally likely to be 1 or 0. In this case Eq. (6.1.21) becomes

$$R_X(t_1, t_2) = 1 \times 1 \times P[(X(t_1)=1) \cap (X(t_2)=1)] = 1 \times 1 \times \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4} \tag{6.1.22}$$

Case 2: If $t_2 - t_1 < T$, then a clock transition may or may not occur between t_1 and t_2. We denote the event of a clock transition occurring in this period as $CT = \{t_1 < \text{clock transition} < t_2\}$. The probability of this event and its complement are

$$P[CT] = \int_{t_1}^{t_2} f_D(d)d(d) = \frac{t_2 - t_1}{T} \quad \text{and} \quad P[\overline{CT}] = 1 - P[CT] = 1 - \frac{t_2 - t_1}{T}, 0 < t_2 - t_1 < T \tag{6.1.23}$$

In the case where $t_2 - t_1 < T$, we may express the autocorrelation function using the law of total probability [Eq. (1.5.7)] as

$$\begin{aligned} R_X(t_1, t_2) &= 1 \times 1 \times P[(X(t_1)=1) \cap (X(t_2)=1)] \\ &= P[(X(t_1)=1) \cap (X(t_2)=1) | CT] \times P[CT] \\ &\quad + P[(X(t_1)=1) \cap (X(t_2)=1) | \overline{CT}] \times P[\overline{CT}] \end{aligned} \tag{6.1.24}$$

In the first term, in which a clock transition occurs between t_1 and t_2, $X(t_1)$ and $X(t_2)$ are independent and equally likely to be 1 or 0, and it follows that

$$P[(X(t_1)=1) \cap (X(t_2)=1) | CT] \times P[CT] = \frac{1}{2} \times \frac{1}{2} \times \frac{t_2 - t_1}{T} \tag{6.1.25}$$

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We now combine Eqs. (6
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and thus $|\tau|$ is the true

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Figure 6.1.12 Autoco

where Eq. (6.1.23) was used. In the second term in Eq. (6.1.24), in which a clock transition does not occur between t_1 and t_2, $X(t_1)$ and $X(t_2)$ are the same, with 1 and 0 equally probable, and it follows that

$$P[(X(t_1) = 1) \cap (X(t_2) = 1)|\overline{CT}] \times P[\overline{CT}] = \frac{1}{2} \times \left(1 - \frac{t_2 - t_1}{T}\right) \tag{6.1.26}$$

where again Eq. (6.1.23) was used. Substituting the results of Eqs. (6.1.25) and (6.1.26) into Eq. (6.1.24) and combining terms, we have the result in Eq. (6.1.27) for case 2, $t_2 - t_1 < T$:

$$R_X(t_1, t_2) = \frac{1}{4} \times \frac{t_2 - t_1}{T} + \frac{1}{2} \times \left(1 - \frac{t_2 - t_1}{T}\right) = \frac{1}{2} - \frac{1}{4}\left(\frac{t_2 - t_1}{T}\right) \tag{6.1.27}$$

We now combine Eqs. (6.1.27) and (6.1.22) to get the final result, with the following two additional changes. We note that only the difference between t_2 and t_1 matters; thus we substitute $\tau = t_2 - t_1$. Finally, it does not matter which is greater, t_1 or t_2, because the same probabilities will apply, and thus $|\tau|$ is the true variable of the autocorrelation. The final result is

$$\begin{aligned} R_X(\tau) &= \frac{1}{2} - \frac{1}{4}\frac{|\tau|}{T}, \quad 0 < |\tau| < T \\ &= \frac{1}{4}, \quad |\tau| > T \end{aligned} \tag{6.1.28}$$

This autocorrelation function is plotted in Fig. 6.1.12.

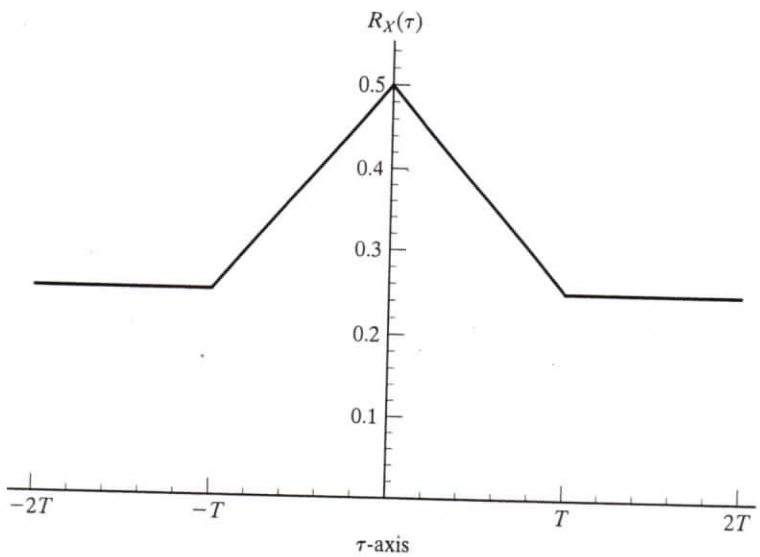

Figure 6.1.12 Autocorrelation function of the model of a synchronous digital signal.

Model is WSS. The model for the synchronous digital signal is WSS, since its mean is constant, Eq. (6.1.19), and its autocorrelation function depends on time differences alone, Eq. (6.1.28). This is a consequence of the stable nature of the statistical properties of the model. The randomization of the synchronous digital signal over the clock period is clearly shown in Fig. 6.1.12.

6.1.3 Modeling a Random Analog Signal

The random sinusoid. In the study of linear systems, we rightly focus much effort in solving problems with sinusoidal waveforms. Sinusoids are used in signal analysis as a basic building block with which more complex waveforms can be analyzed. Sinusoids are also used as carriers for communication signals.

In AC circuit problems, for example, we deal with circuits with known sources, including the amplitude, frequency, and phase of the sinusoidal sources. In contrast, in the real situation that we deal with in a power system we know the frequency quite well, 60 Hz in the United States, and we know the amplitude of the voltage within reasonable bounds, from about $110\sqrt{2}$ to about $125\sqrt{2}$ V, but when we turn on a switch, say to start a dishwasher, we engage the switch at a random time. This is equivalent to turning on the voltage at a random phase. This equivalence is suggested in Fig. 6.1.13.

Semi- and fully random sinusoids. A semirandom sinusoid is a sinusoid turned on at a random time. This model is useful in studying turn-on characteristics of electrical equipment. Here we will study the fully random sinusoid, by which we mean a sinusoid of random phase that exists for all time. Figure 6.1.14 gives the results of a chance experiment that generates six random phases and plots the associated sinusoids.

Properties of the random process. Our main concern is WSS random processes, which require the analysis of the mean and the autocorrelation function. These we can obtain without the PDFs of the random process because we can express this random process as a function of the random variable, Θ, which is the phase of the various sinusoids. Thus we may represent this

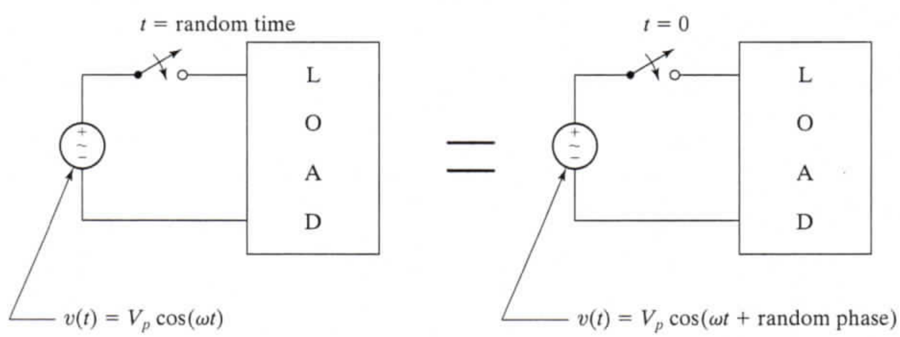

Figure 6.1.13 Turning on a sinusoid at a random time is equivalent to turning on a sinusoid of random phase. The latter would be a semirandom sinusoid. The fully random sinusoid has random phase but no turn-on. This will be our model for a random analog signal.

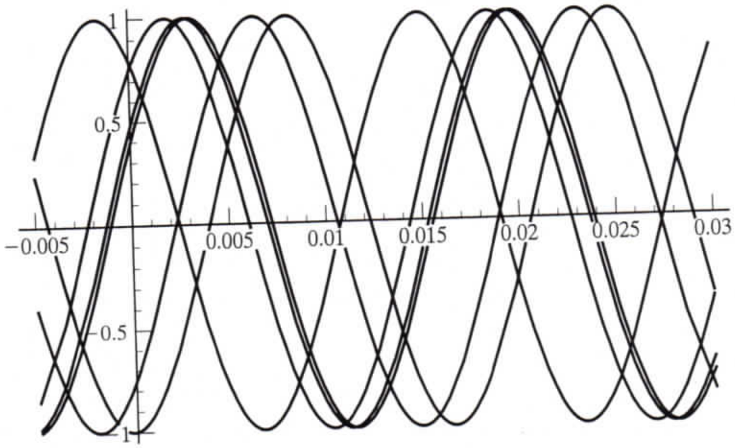

Figure 6.1.14 Six members of the random process modeling a random analog signal. Here we show six sinusoids of random phase. The fully random process treats the phase as a uniformly distributed random variable.

random process simply:

$$V(t, \Theta) = V_p \cos(\omega_1 t + \Theta) \text{ volts} \tag{6.1.29}$$

where V_p is the peak value, ω_1 is the frequency in radians per second (rad/s) and Θ is a uniformly distributed random variable with the PDF

$$f_\Theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta \le 2\pi, \text{ zow} \tag{6.1.30}$$

The mean of $V(t)$. We now can calculate the mean and autocorrelation function of the random process by averaging with respect to Θ, as presented in Eq. (3.1.10). The mean is

$$\mu_{V(t)} = E[V(t, \Theta)] = \int_{-\infty}^{+\infty} V(t, \theta) f_\Theta(\theta)\, d\theta = \int_0^{2\pi} V_p \cos(\omega t + \theta) \frac{1}{2\pi}\, d\theta = 0 \tag{6.1.31}$$

Note that the integration in Eq. (6.1.31) is with respect to θ, not time. The results are zero because we are averaging a sinusoid over a full cycle in θ with ωt fixed.

The autocorrelation function. The autocorrelation function can be determined by the same process:

$$R_V(t_1, t_2) = E[V(t_1, \Theta) V(t_2, \Theta)] = \int_0^{2\pi} V_p \cos(\omega_1 t_1 + \theta) V_p \cos(\omega_1 t_2 + \theta) \frac{1}{2\pi}\, d\theta \tag{6.1.32}$$

The integration in Eq. (6.1.32) can be performed with the aid of a trig identity,[4] which gives the form

$$R_V(t_1, t_2) = \frac{V_p^2}{2} \int_0^{2\pi} [\cos(\omega_1 (t_2 - t_1)) + \cos(\omega_1 (t_2 + t_1) + 2\theta)] \frac{1}{2\pi}\, d\theta \tag{6.1.33}$$

The second term in Eq. (6.1.33) integrates to zero because the average is performed over two cycles of the sinusoid. The first term in Eq. (6.1.33) has no θ dependence; hence the autocorrelation function is

$$R_V(t_1, t_2) = \frac{V_p^2}{2}\cos(\omega_1(t_2 - t_1))\int_0^{2\pi}\frac{1}{2\pi}\,d\theta = \frac{V_p^2}{2}\cos(\omega_1(t_2 - t_1)) \qquad (6.1.34)$$

If again we let $t_2 - t_1 = \tau$, we have

$$R_V(\tau) = \frac{V_p^2}{2}\cos(\omega_1\tau) \quad \text{volts}^2 \qquad (6.1.35)$$

The unit of volts squared relates to power and will be discussed presently.

The fully random sinusoid is WSS. Note that the mean of the fully random sinusoid is zero, which is a constant, and the autocorrelation function is an even function of $t_2 - t_1 = \tau$. Thus the fully random sinusoid is WSS.

Example 6.1.3: The power line
The power input for domestic appliances is 120 V, rms, and a frequency of 60 hertz (Hz), but phase is arbitrary relative to the clocks in your house. What would be the autocorrelation function of the voltage of an appliance output in your house?

Solution The peak voltage would be $120\sqrt{2} = 169.7$ V. The frequency would be $\omega_1 = 2\pi \times 60 = 377.0$ rad/s. Using Eq. (6.1.35) we find the autocorrelation function to be $R_V(\tau) = \frac{(120\sqrt{2})^2}{2}\cos(377\tau)$ volts^2.

You do it. Assume you have an analog clock in your house that has a minute hand 4 in. long. Let $X(t)$ = the horizontal projection of the tip of the minute hand relative to the axis of rotation. Let t be time in minutes from the instant of your birth. Find the autocorrelation of X in feet squared and evaluate at $\tau = 10$ minutes.

```
myanswer = ? ;
```

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Evaluate
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For the answer, see endnote 5.

6.1.4 Ergodic Random Processes

Time averages. The following material relates to WSS random processes and to deterministic as well as to random signals. Figure 6.1.15 shows the definitions of the total signal, $v(t)$, the DC component of the signal, V_{DC}, and the AC component of the signal, $v_{AC}(t)$.

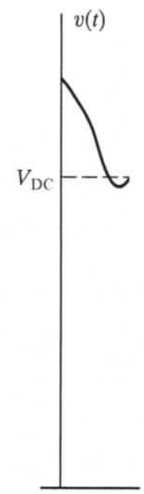

Figure 6.1.15 The defi
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