We characterize each second-order filter section by the frequencies of the magnitude response extremes and by the 3-dB frequencies. The *3-dB frequencies*, denoted by $f_{3dB} = \theta_{3dB}/(2\pi)$, are frequencies at which the magnitude response $M(f) = |H(e^{j2\pi f})|$ is $\sqrt{2}$ times smaller than the magnitude response at some reference frequency f_r (3 dB only approximately corresponds to $\sqrt{2}$, strictly speaking it is $10^{3/20}$):

$$\frac{M(f_{3dB})}{M(f_r)} = \frac{|H(e^{j2\pi f_{3dB}})|}{|H(e^{j2\pi f_r})|} = \frac{1}{\sqrt{2}} \tag{8.19}$$

For example, for lowpass filters, the reference frequency is $f_r = 0$.

In filter design, we prefer to use the *normalized transfer function*, $H_n(z)$, defined by

$$H_n(z) = \frac{H(z)}{\max\limits_{f} M(f)} \tag{8.20}$$

The transfer function can be obtained by scaling the normalized transfer function by a constant:

$$H(z) = kH_n(z) \tag{8.21}$$

Quite generally, the normalization constant k can take any real value.

8.2.1 Second-Order Transfer Functions

In this section we analyze the properties of the basic second-order transfer functions. We examine the magnitude response $M(f) = |H(e^{j2\pi f})|$ for real positive digital angular frequencies $\theta = 2\pi f$. The angular frequencies of the magnitude response local extrema are designated by $\theta_e = 2\pi f_e$. We find it convenient to define the frequency f_j at which the frequency response becomes purely imaginary, $\mathrm{Re}(H(e^{j2\pi f_j})) = 0$.

Lowpass Transfer Function. The second-order *lowpass transfer function* is defined as

$$H_{LP}(z) = \frac{1+a+b}{4}\,\frac{(z+1)^2}{z^2+bz+a} = \frac{1+a+b}{4}\,\frac{(1+z^{-1})^2}{1+bz^{-1}+az^{-2}} \tag{8.22}$$

At higher frequencies ($f > f_j > f_e$), where

$$f_j = \frac{1}{2\pi}\cos^{-1}\frac{-b}{1+a} \tag{8.23}$$

the magnitude response $M(f) = |H_{LP}(e^{j2\pi f})|$ decreases and, thus, high-frequency sinusoidal sequences are rejected.

The key properties of the lowpass transfer function are summarized below:

$$\begin{aligned}
M(0) &= H_{LP}(1) = 1, & z &= 1, & f &= 0\\
M(0.5) &= H_{LP}(-1) = 0, & z &= -1, & f &= 0.5\\
M(f_j) &= |H_{LP}(e^{j2\pi f_j})| = \left|jQ_p\right| = Q_p, & z &= e^{j2\pi f_j}, & f &= f_j\\
M_e &= \max_{0\le f\le 0.5}(M(f)) = |H_{LP}(e^{j2\pi f_e})| = \frac{Q_p}{\sqrt{1-\dfrac{1}{4Q_p^2}}}, & z &= e^{j2\pi f_e}, & f &= f_e
\end{aligned} \tag{8.24}$$

where f_e is the frequency at which $M(f)$ has its maximal value M_e:

$$f_e = \frac{1}{2\pi}\cos^{-1}\frac{(1-a)^2 - b(1+a-b)}{4a - b - ab} \tag{8.25}$$

and the pole Q-factor is

$$Q_p = \frac{\sqrt{(1+a)^2 - b^2}}{2(1-a)} \tag{8.26}$$

The maximal value of the magnitude response is approximately equal to Q_p for $Q_p \gg 1$ (i.e., for $a \approx 1$), as shown in Fig. 8.8:

$$\max_{0\le f\le 0.5}\left|H_{LP}\left(e^{j2\pi f}\right)\right| = \left|H_{LP}\left(e^{j2\pi f_e}\right)\right| \approx Q_p, \quad f_e \approx f_j, \quad Q_p \gg 1 \tag{8.27}$$

This fact is very important for digital filters implemented in fixed point arithmetic. Suppose that the filter output sequence, y_k, must be bounded to $-1 \le y_k \le 1$. Then, the amplitude of the sinusoidal sequence of frequency f_e, at the input of the lowpass second-order filter, must be smaller than $1/Q_p$, so that, after filtering, the amplitude of the output sequence remains within the prescribed range, $-1 \le y_k \le 1$.

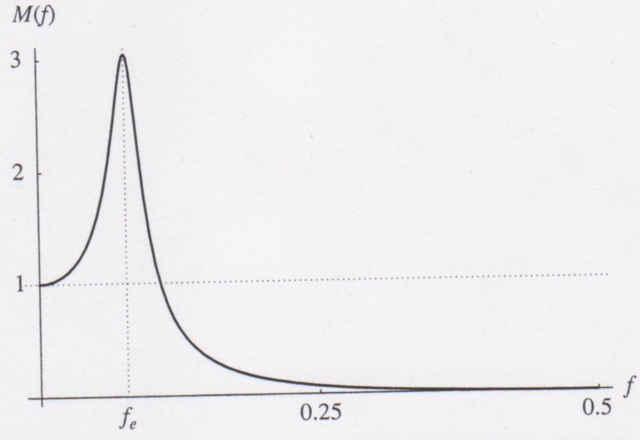

Figure 8.8 Magnitude of second-order lowpass transfer function: $Q_p = 3$, $a = 0.85117$, and $b = -1.621545$.

The maximal value of the magnitude response, M_e, can be expressed in terms of the coefficients a and b:

$$M_e = \max_{0 \le f \le 0.5} (M(f)) = \frac{(1+a)^2 - b^2}{2\,(1-a)\,\sqrt{4a - b^2}} \tag{8.28}$$

The magnitude of the normalized transfer function, $H_{LPn}(z)$, is shown in Fig. 8.9. This normalized transfer function is defined as

$$H_{LPn}(z) = \frac{H_{LP}(z)}{M_e} = \frac{(1-a)\,\sqrt{4a-b^2}}{2\,(1+a-b)}\;\frac{\left(z^{-1}+1\right)^2}{1+bz^{-1}+az^{-2}} \tag{8.29}$$

and it has the maximal magnitude, equal to 1, at the frequency f_e.

For $Q_p \le 1/\sqrt{2}$ we find $f_e = 0$ as shown in Fig. 8.10. Therefore, for $1/2 < Q_p \le 1/\sqrt{2}$ the maximal magnitude function is at frequency $f_e = 0$.

The maximal value of the magnitude of second-order lowpass transfer functions, M_e, in terms of the coefficients is shown in Fig. 8.11. M_e dramatically increases when a approaches to 1, while the influence of b is negligible.

Highpass Transfer Function. The second-order *highpass transfer function* is defined as

$$H_{HP}(z) = \frac{1+a-b}{4}\;\frac{(z-1)^2}{z^2+bz+a} = \frac{1+a-b}{4}\;\frac{\left(z^{-1}-1\right)^2}{1+bz^{-1}+az^{-2}} \tag{8.30}$$

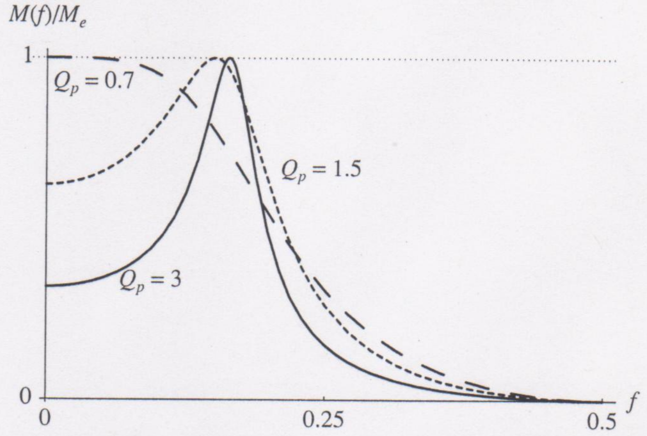

Figure 8.9 Magnitude of second-order normalized lowpass transfer functions $M(f)/M_e$: $Q_p = 3,\ 1.5,\ 0.7$.

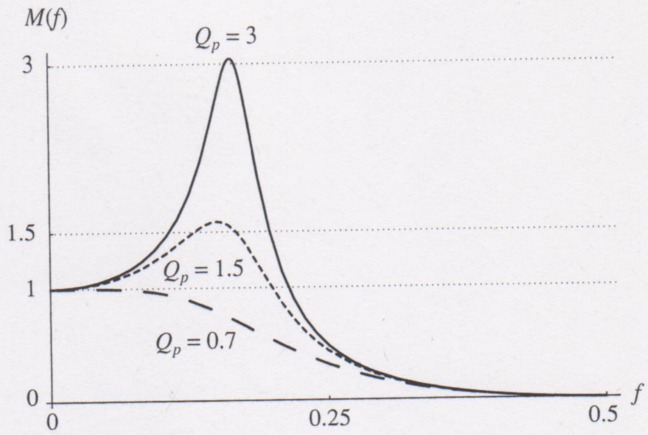

Figure 8.10 Magnitude of second-order lowpass transfer functions: $Q_p = 3$, 1.5, 0.7.

At lower frequencies, $f < f_j < f_e$, where

$$f_j = \frac{1}{2\pi} \cos^{-1} \frac{-b}{1+a} \tag{8.31}$$

the magnitude response $M(f) = |H_{HP}(e^{j2\pi f})|$ decreases when f approaches zero and, thus, low-frequency sinusoidal sequences are rejected, while the high-frequency sinusoidal sequences, $f \geq f_p$, pass without attenuation.

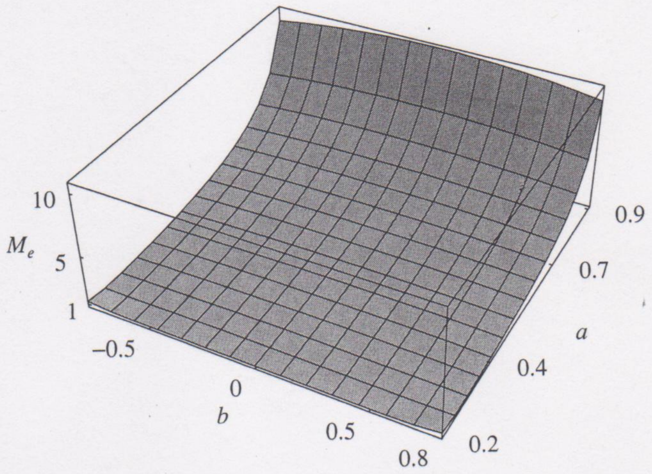

Figure 8.11 Maximal value of magnitude of second-order lowpass transfer functions in terms of filter coefficients.

The key properties of the highpass transfer function are summarized below:

$$
\begin{aligned}
M(0) &= H_{HP}(1) = 0, && z = 1, && f = 0 \\
M(0.5) &= H_{HP}(-1) = 1, && z = -1, && f = 0.5 \\
M(f_j) &= |H_{HP}(e^{j2\pi f_j})| = |jQ_p| = Q_p, && z = e^{j2\pi f_j}, && f = f_j \\
M_e &= \max_{0 \le f \le 0.5}(M(f)) = |H_{HP}(e^{j2\pi f_e})| = \frac{Q_p}{\sqrt{1 - \dfrac{1}{4Q_p^2}}}, && z = e^{j2\pi f_e}, && f = f_e
\end{aligned}
\tag{8.32}
$$

where f_e is the frequency at which $M(f)$ has its maximal value M_e

$$
f_e = -\frac{1}{2\pi}\cos^{-1} - \frac{(1-a)^2 + b(1+a+b)}{4a + b + ab}
\tag{8.33}
$$

and the pole Q-factor is

$$
Q_p = \frac{\sqrt{(1+a)^2 - b^2}}{2(1-a)}
\tag{8.34}
$$

The maximal value of the magnitude response is approximately equal to Q_p for $Q_p \gg 1$ (i.e., for $a \approx 1$), as shown in Fig. 8.12:

$$
\max_{0 \le f \le 0.5} \left|H_{HP}\left(e^{j2\pi f}\right)\right| = \left|H_{HP}\left(e^{j2\pi f_e}\right)\right| \approx Q_p, \quad f_e \approx f_j, \quad Q_p \gg 1
\tag{8.35}
$$

As in the case of the lowpass transfer function, the maximal value of the magnitude response is approximately equal to Q_p, as shown in Fig. 8.12. For $Q_p \le 1/\sqrt{2}$ we find that $f_e = 0.5$, as shown in Fig. 8.13.

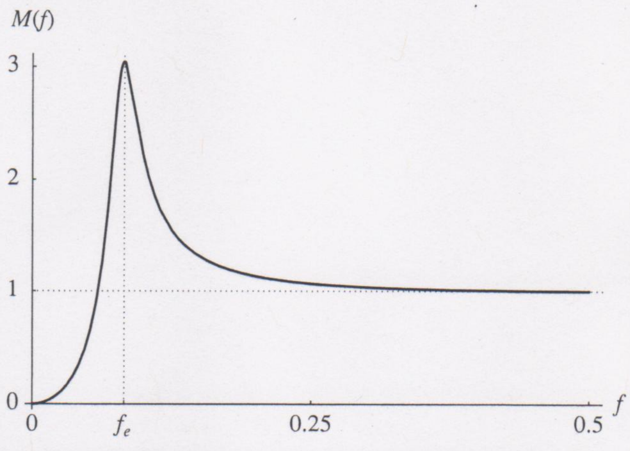

Figure 8.12 Magnitude of second-order highpass transfer function: $Q_p = 3$, $a = 0.85117$, and $b = -1.621545$.

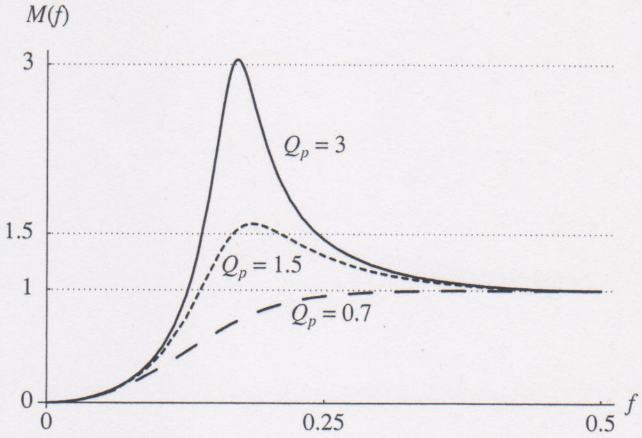

Figure 8.13 Magnitude of second-order highpass transfer functions: $Q_p = 3,\ 1.5,\ 0.7$.

The maximal value of the magnitude response, M_e, can be expressed in terms of the coefficients a and b:

$$M_e = \frac{(1+a)^2 - b^2}{2\,(1-a)\,\sqrt{4a - b^2}} \tag{8.36}$$

The magnitude of the normalized transfer function, $H_{HPn}(z)$, is shown in Fig. 8.14. This normalized transfer function

$$H_{HPn}(z) = \frac{H_{HP}(z)}{M_e} = \frac{(1-a)\,\sqrt{4a-b^2}}{2\,(1+a+b)}\;\frac{\left(z^{-1}-1\right)^2}{1+bz^{-1}+az^{-2}} \tag{8.37}$$

has the maximal magnitude, equal to 1, at the frequency f_e.

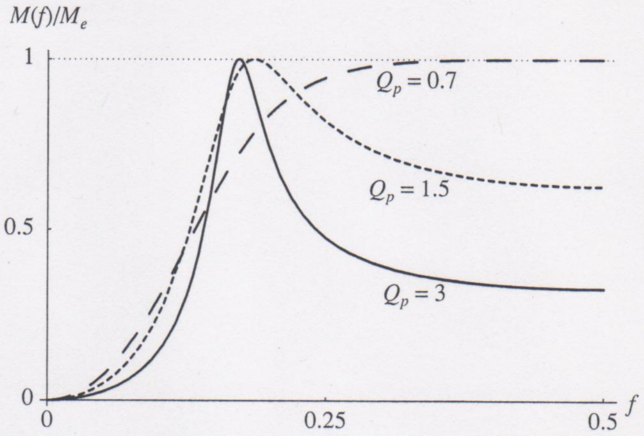

Figure 8.14 Magnitude of second-order normalized highpass transfer functions $M(f)/M_e$: $Q_p = 3,\ 1.5,\ 0.7$.

The maximal va
M_e, in terms of the c
functions shown in Fi

Bandpass Transfer F
fined as

$H_{BP}(z) =$

The key properti

$M(0) = H_{BP}(1) =$

$M(0.5) = H_{BP}(-1$

$M_e = \max_{0 \le f \le 0.5} (M(f$

where

The frequency at
times called the *resona*
Second-order ban
cies $f_{low,3dB} < f < f$
3dB), but reject sinusoi

$\frac{1}{\sqrt{2}} \le M(f) =$

$|H_{BP}(e^{j2\pi f_{low,3dB}})| =$

$f_{low,3dB} =$

$f_{high,3dB} =$

The pole Q-factor

The maximum of th
$f_{low,3dB}$, is affected by Q
(Fig. 8.16).

The maximal value of the magnitude of second-order highpass transfer functions, M_e, in terms of the coefficients is the same as for the second-order lowpass transfer functions shown in Fig. 8.11.

Bandpass Transfer Function. The second-order *bandpass transfer function* is defined as

$$H_{BP}(z) = \frac{1-a}{2}\,\frac{z^2-1}{z^2+bz+a} = \frac{1-a}{2}\,\frac{1-z^{-2}}{1+bz^{-1}+az^{-2}} \tag{8.38}$$

The key properties of the bandpass transfer function are summarized below:

$$\begin{aligned} &M(0) = H_{BP}(1) = 0, && z = 1, && f = 0 \\ &M(0.5) = H_{BP}(-1) = 0, && z = -1, && f = 0.5 \\ &M_e = \max_{0\le f\le 0.5}(M(f)) = |H_{BP}(e^{j2\pi f_e})| = 1, && z = e^{j2\pi f_e}, && f = f_e \end{aligned} \tag{8.39}$$

where

$$f_e = \frac{1}{2\pi}\cos^{-1}\frac{-b}{1+a} \tag{8.40}$$

The frequency at which the magnitude response reaches its maximum, f_e, is sometimes called the *resonant frequency* or the *central frequency*.

Second-order bandpass filters pass sinusoidal sequences from the band of frequencies $f_{low,3dB} < f < f_{high,3dB}$ with insignificant attenuation (less than $20\log_{10}\sqrt{2} \approx$ 3dB), but reject sinusoidal sequences whose frequencies are on either side of this band:

$$\begin{aligned} &\frac{1}{\sqrt{2}} \le M(f) = |H_{BP}(e^{j2\pi f})| \le 1, \quad f_{low,3dB} \le f \le f_{high,3dB} \\ &|H_{BP}(e^{j2\pi f_{low,3dB}})| = |H_{BP}(e^{j2\pi f_{high,3dB}})| = \frac{1}{\sqrt{2}} \\ &f_{low,3dB} = \frac{1}{2\pi}\cos^{-1}\frac{-b(1+a)-(1-a)\sqrt{2+2a^2-b^2}}{2\left(1+a^2\right)} \\ &f_{high,3dB} = \frac{1}{2\pi}\cos^{-1}\frac{-b(1+a)+(1-a)\sqrt{2+2a^2-b^2}}{2\left(1+a^2\right)} \end{aligned} \tag{8.41}$$

The pole Q-factor of the bandpass filter is

$$Q_p = \frac{\sqrt{(1+a)^2-b^2}}{2(1-a)} \tag{8.42}$$

The maximum of the magnitude response is 1. The 3-dB bandwidth, $f_{high,3dB} - f_{low,3dB}$, is affected by Q_p (Fig. 8.15). Higher Q-factors produce narrower bandwidths (Fig. 8.16).