# HW 2

<u>1.7</u>

11, 14

<u>1.8</u>

2, 4, 6, 10, 12

<u>2.1</u>

1, 9, 10, 12

<u>2.2</u>

31, 32

9. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

10. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ 15 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -5 \\ h \end{bmatrix}$

In Exercises 11–14, find the value(s) of $h$ for which the vectors are linearly *dependent*. Justify each answer.

11. $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$

12. $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$

Determine by inspection whether the vectors in Exercises 15–20 are linearly *independent*. Justify each answer.

15. $\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

16. $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$

17. $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$

18. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 21 and 22, mark each statement True or False. Justify each answer on the basis of a careful reading of the text.

21. a. The columns of a matrix $A$ are linearly independent if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.
    b. If $S$ is a linearly dependent set, then each vector is a linear combination of the other vectors in $S$.
    c. The columns of any $4 \times 5$ matrix are linearly dependent.
    d. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span $\{\mathbf{x}, \mathbf{y}\}$.

22. a. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and if $\mathbf{w}$ is in Span $\{\mathbf{u}, \mathbf{v}\}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.
    b. If three vectors in $\mathbb{R}^3$ lie in the same plane in $\mathbb{R}^3$, then they are linearly dependent.
    c. If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.
    d. If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than $n$ vectors.

In Exercises 23–26, describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

23. $A$ is a $2 \times 2$ matrix with linearly dependent columns.

24. $A$ is a $3 \times 3$ matrix with linearly independent columns.

25. $A$ is a $4 \times 2$ matrix, $A = [\mathbf{a}_1 \ \ \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

26. $A$ is a $4 \times 3$ matrix, $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$, such that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent and $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$.

27. How many pivot columns must a $6 \times 4$ matrix have if its columns are linearly independent? Why?

28. How many pivot columns must a $4 \times 6$ matrix have if its columns span $\mathbb{R}^4$? Why?

29. Construct $3 \times 2$ matrices $A$ and $B$ such that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, but $B\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. a. Fill in the blank in the following statement: "If $A$ is an $m \times n$ matrix, then the columns of $A$ are linearly independent if and only if $A$ has _____ pivot columns."
    b. Explain why the statement in (a) is true.

Exercises 31 and 32 should be solved *without performing row operations*. [*Hint:* Write $A\mathbf{x} = \mathbf{0}$ as a vector equation.]

31. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

32. Given $A = \begin{bmatrix} 4 & 3 & -5 \\ -2 & -2 & 4 \\ -2 & -3 & 7 \end{bmatrix}$, observe that the first column minus three times the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Each statement in Exercises 33–38 is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true. Such an example is called a *counterexample* to the statement. If a statement is true, give a justification. (One specific example cannot explain why a statement is always true. You will have to do more work here than in Exercises 21 and 22.)

33. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

34. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

35. If $\mathbf{v}_1, \ldots, \mathbf{v}_5$ are in $\mathbb{R}^5$ and $\mathbf{v}_3 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is linearly dependent.

36. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are in $\mathbb{R}^3$ and $\mathbf{v}_3$ is *not* a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

37. If $\mathbf{v}_1, \ldots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.

38. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a linearly independent set of vectors in $\mathbb{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. [*Hint:* Think about $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$.]

39. Suppose $A$ is an $m \times n$ matrix with the property that for all $\mathbf{b}$ in $\mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution. Use the

2. If $\mathbf{x}$ and $\mathbf{y}$ are production vectors, then the total cost vector associated with the combined production $\mathbf{x}+\mathbf{y}$ is precisely the sum of the cost vectors $T(\mathbf{x})$ and $T(\mathbf{y})$. ∎

## PRACTICE PROBLEMS

1. Suppose $T:\mathbb{R}^5 \to \mathbb{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix $A$ and for each $\mathbf{x}$ in $\mathbb{R}^5$. How many rows and columns does $A$ have?

2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

3. The line segment from $\mathbf{0}$ to a vector $\mathbf{u}$ is the set of points of the form $t\mathbf{u}$, where $0 \le t \le 1$. Show that a linear transformation $T$ maps this segment into the segment between $\mathbf{0}$ and $T(\mathbf{u})$.

## 1.8 EXERCISES

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T:\mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under $T$ of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T:\mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

In Exercises 3–6, with $T$ defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under $T$ is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -8 & 8 \\ 0 & 1 & 2 \\ 1 & 0 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 6 \\ 3 \\ 10 \end{bmatrix}$

7. Let $A$ be a $6 \times 5$ matrix. What must $a$ and $b$ be in order to define $T:\mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and columns must a matrix $A$ have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

For Exercises 9 and 10, find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix $A$.

9. $A = \begin{bmatrix} 1 & -3 & 5 & -5 \\ 0 & 1 & -3 & 5 \\ 2 & -4 & 4 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}$

11. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let $A$ be the matrix in Exercise 9. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

12. Let $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 4 \end{bmatrix}$, and let $A$ be the matrix in Exercise 10. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

In Exercises 13–16, use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation $T$. (Make a separate and reasonably large sketch for each exercise.) Describe geometrically what $T$ does to each vector in $\mathbb{R}^2$.

13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

14. $T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

17. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that $T$ is linear to find the images under $T$ of $2\mathbf{u}$, $3\mathbf{v}$, and $2\mathbf{u} + 3\mathbf{v}$.

**NUMERICAL NOTES**

1. The fastest way to obtain $AB$ on a computer depends on the way in which the computer stores matrices in its memory. The standard high-performance algorithms, such as in LAPACK, calculate $AB$ by columns, as in our definition of the product. (A version of LAPACK written in C++ calculates $AB$ by rows.)
2. The definition of $AB$ lends itself well to parallel processing on a computer. The columns of $B$ are assigned individually or in groups to different processors, which independently and hence simultaneously compute the corresponding columns of $AB$.

**PRACTICE PROBLEMS**

1. Since vectors in $\mathbb{R}^n$ may be regarded as $n \times 1$ matrices, the properties of transp[ose] in Theorem 3 apply to vectors, too. Let
$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$
Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?
2. Let $A$ be a $4 \times 4$ matrix and let $\mathbf{x}$ be a vector in $\mathbb{R}^4$. What is the fastest wa[y to] compute $A^2\mathbf{x}$? Count the multiplications.

## 2.1 EXERCISES

In Exercises 1 and 2, compute each matrix sum or product if it is defined. If an expression is undefined, explain why. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A$, $B - 2A$, $AC$, $CD$
2. $A + 3B$, $2C - 3E$, $DB$, $EC$

In the rest of this exercise set and in those to follow, assume that each matrix expression is defined. That is, the sizes of the matrices (and vectors) involved "match" appropriately.

3. Let $A = \begin{bmatrix} 2 & -5 \\ 3 & -2 \end{bmatrix}$. Compute $3I_2 - A$ and $(3I_2)A$.
4. Compute $A - 5I_3$ and $(5I_3)A$, where
$$A = \begin{bmatrix} 5 & -1 & 3 \\ -4 & 3 & -6 \\ -3 & 1 & 2 \end{bmatrix}.$$

In Exercises 5 and 6, compute the product $AB$ in two ways: (a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and (b) by the row–column rule for computing $AB$.

5. $A = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$
6. $A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$
7. If a matrix $A$ is $5 \times 3$ and the product $AB$ is $5 \times 7$, wh[at is] the size of $B$?
8. How many rows does $B$ have if $BC$ is a $5 \times 4$ matrix?
9. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}$. What val[ue(s)] of $k$, if any, will make $AB = BA$?
10. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.
11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Com[-]pute $AD$ and $DA$. Explain how the columns or rows o[f $A$] change when $A$ is multiplied by $D$ on the right or on the [left]. Find a $3 \times 3$ matrix $B$, not the identity matrix or the [zero] matrix, such that $AB = BA$.
12. Let $A = \begin{bmatrix} 3 & -6 \\ -2 & 4 \end{bmatrix}$. Construct a $2 \times 2$ matrix $B$ such [that] $AB$ is the zero matrix. Use two different nonzero colum[ns] for $B$.

**14.** Suppose $(B - C)D = 0$, where $B$ and $C$ are $m \times n$ matrices and $D$ is invertible. Show that $B = C$.

**15.** Let $A$ be an invertible $n \times n$ matrix, and let $B$ be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

If $[\,A \quad B\,] \sim \cdots \sim [\,I \quad X\,]$, then $X = A^{-1}B$.

If $A$ is larger than $2 \times 2$, then row reduction of $[A \quad B]$ is much faster than computing both $A^{-1}$ and $A^{-1}B$.

**16.** Suppose $A$ and $B$ are $n \times n$ matrices, $B$ is invertible, and $AB$ is invertible. Show that $A$ is invertible. [*Hint:* Let $C = AB$, and solve this equation for $A$.]

**17.** Suppose $A$, $B$, and $C$ are invertible $n \times n$ matrices. Show that $ABC$ is also invertible by producing a matrix $D$ such that $(ABC)D = I$ and $D(ABC) = I$.

**18.** Solve the equation $AB = BC$ for $A$, assuming that $A$, $B$, and $C$ are square and $B$ is invertible.

**19.** If $A$, $B$, and $C$ are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, $X$? If so, find it.

**20.** Suppose $A$, $B$, and $X$ are $n \times n$ matrices with $A$, $X$, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \tag{3}$$

a. Explain why $B$ is invertible.

b. Solve equation (3) for $X$. If a matrix needs to be inverted, explain why that matrix is invertible.

**21.** Explain why the columns of an $n \times n$ matrix $A$ are linearly independent when $A$ is invertible.

**22.** Explain why the columns of an $n \times n$ matrix $A$ span $\mathbb{R}^n$ when $A$ is invertible. [*Hint:* Review Theorem 4 in Section 1.4.]

**23.** Suppose $A$ is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why $A$ has $n$ pivot columns and $A$ is row equivalent to $I_n$. By Theorem 7, this shows that $A$ must be invertible. (This exercise and Exercise 24 will be cited in Section 2.3.)

**24.** Suppose $A$ is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why $A$ must be invertible. [*Hint:* Is $A$ row equivalent to $I_n$?]

Exercises 25 and 26 prove Theorem 4 for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

**25.** Show that if $ad - bc = 0$, then the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution. Why does this imply that $A$ is not invertible? [*Hint:* First, consider $a = b = 0$. Then, if $a$ and $b$ are not both zero, consider the vector $\mathbf{x} = \begin{bmatrix} -b \\ a \end{bmatrix}$.]

**26.** Show that if $ad - bc \neq 0$, the formula for $A^{-1}$ works.

Exercises 27 and 28 prove special cases of the facts about elementary matrices stated in the box following Example 5. Here $A$ is a $3 \times 3$ matrix and $I = I_3$. (A general proof would require slightly more notation.)

**27.** Let $A$ be a $3 \times 3$ matrix.

a. Use equation (2) from Section 2.1 to show that $\text{row}_i(A) = \text{row}_i(I) \cdot A$, for $i = 1, 2, 3$.

b. Show that if rows 1 and 2 of $A$ are interchanged, then the result may be written as $EA$, where $E$ is an elementary matrix formed by interchanging rows 1 and 2 of $I$.

c. Show that if row 3 of $A$ is multiplied by 5, then the result may be written as $EA$, where $E$ is formed by multiplying row 3 of $I$ by 5.

**28.** Suppose row 2 of $A$ is replaced by $\text{row}_2(A) - 3 \cdot \text{row}_1(A)$. Show that the result is $EA$, where $E$ is formed from $I$ by replacing $\text{row}_2(I)$ by $\text{row}_2(I) - 3 \cdot \text{row}_1(I)$.

Find the inverses of the matrices in Exercises 29–32, if they exist. Use the algorithm introduced in this section.

**29.** $\begin{bmatrix} 1 & -3 \\ 4 & -9 \end{bmatrix}$

**30.** $\begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}$

**31.** $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

**32.** $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

**33.** Use the algorithm from this section to find the inverses of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let $A$ be the corresponding $n \times n$ matrix, and let $B$ be its inverse. Guess the form of $B$, and then show that $AB = I$.

**34.** Repeat the strategy of Exercise 33 to guess the inverse $B$ of

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & & 0 \\ 3 & 3 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}.$$

Show that $AB = I$.

**35.** Let $A = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}$. Find the third column of $A^{-1}$ without computing the other columns.

**36.** **[M]** Let $A = \begin{bmatrix} -25 & -9 & -27 \\ 536 & 185 & 537 \\ 154 & 52 & 143 \end{bmatrix}$. Find the second and third columns of $A^{-1}$ without computing the first column.

**37.** Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$. Construct a $2 \times 3$ matrix $C$ (by trial and error) using only 1, $-1$, and 0 as entries, such that $CA = I_2$. Compute $AC$ and note that $AC \neq I_3$.